Dynamical DNA: a possible metric of complexity

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What do we want in a metric of complexity?

• Direct relationship of metric to dynamical action
• Hierarchical in terms of order of interaction (number of cliques, phrases or scales interacting) and size
• Invariant of coordinates (independent or dependant) and stable to small changes to the dynamics (Lipschitz invariant)
• Direct relationship to group symmetries
• Direct relationship to topological invariants (indexes)
What is wrong with Fourier?

- Invariant of coordinates
- NOT stable to small changes in the dynamics
  - at small scale, small changes in signal lead to large changes in transform
What is wrong with the wavelet transform?

- stable to small changes in the dynamics
- NOT invariant of coordinates

\[ \bar{x}_{ik} = \sum_j x_j W_{ik}(t_j) \]

where \( i \) is the scale and \( k \) is the new time index
Interesting development in machine vision

• Stephane Mallat’s Group Invariant Scattering
  • http://arxiv.org/abs/1101.2286
  • http://arxiv.org/abs/1011.3023

Define a scattering metric on:

\[ d^2(f, g) = \| S(f) - S(g) \|^2 = \sum_p \| S(p)f - S(p)g \|^2 \]

where \( S \) is a hierarical scattering operation based on iterative wavelet transform

\[ S = MWMW \ldots = \sum_p S(p) \]

where \( p \) is the path in “scale” space

\[ S(p)f = | \ldots | f \ast \psi_{s_1} | \ast \psi_{s_2} | \ldots | \ast \psi_{s_n} | \ast \psi_\infty | \]

In the limit of many MW the transform becomes invariant of independent coordinate (Lie group parameter)
Total path specified by:

\[ \begin{align*}
\lambda_1 & \quad \lambda_2 \\
\lambda_3 & \quad \lambda_4 \\
\lambda_5 & \quad \lambda_6
\end{align*} \]

scattering at scale associated with Lie Group translation of independent parameters of deformation field (e.g., k of phonon)

scattering at \( \lambda \) scale associated with \( \text{Gl}(n) \) transformation of dependant parameters (deformation field, e.g., polarization of phonon)
Practical application

unique fingerprint of texture

independent of chaotic “phase”

previous best error rate

textons 5.4%
MRF 2.7%

invariant scattering
PCA 0.29%
What do we have?

• hierarchy
• invariant of coord and stable to small changes in dynamics
• direct relationships to group symmetries (see paper)
• possible relationship to topological index
  • orientation as a function of scale, gives curvatures, gives indexes associated with singularity of system (future R&D)

BUT what does this have to do with complex dynamical systems?
Vision problem can be thought of as dynamics

define the deformation vector field

\[ \tau(\vec{x}) = [\bigtriangledown \tau(\vec{x}) - [\bigtriangledown \tau(\vec{x}_0) + \bigtriangledown \phi(t)] \cdot \tau(\vec{x}_0) \]

and the associated action

\[ S[\vec{x}(t)] = \int_0^{t_o} L(\vec{x}(t), \dot{\vec{x}}(t)) dt \]

\[ L(\vec{x}(t), \dot{\vec{x}}(t)) = [\dot{\vec{x}} - \tau(\vec{x})] \cdot g(\vec{x}) \]

image is the wave function

\[ f(\vec{x}, t) = \int e^{iS[\vec{x}(t)]/\hbar} \mathcal{D}[\vec{x}(t)] f(\vec{x}, 0) \]

which in the classical image gives the image deformation

\[ f(\vec{x}, t) = f(\vec{x} - \phi_t(\vec{x})) = f(\vec{x} - \tau(\vec{x})) \]
Mallat’s invariant scattering as an interactive renormalization

Let us consider a much simpler problem of 1D time dependant dynamics (time sliced)

\[ S_o[\{x_j\}] = \sum_j L(x_j, \dot{x}_j, t_j) \Delta t \]

change coordinate to “smoothed” wavelet basis

\[ \bar{x}_{ik} = \sum_j x_j W_{ik}(t_j) \]

useful mean field approximation in this basis

\[ \frac{\partial S_o[\{\bar{x}_{ik}\}]}{\partial \bar{x}_{ik}} = \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} + W'_{ik}(t_j) \frac{\partial L}{\partial v} \]

\[ = \langle \frac{\partial L}{\partial x} \rangle_{ik} + \langle \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \rangle_{ik} \]

\[ = \langle \nabla L \rangle_{ik} + \langle \dot{p} \rangle_{ik} \]

\[ \langle \frac{\partial L}{\partial x} \rangle_{ik} \equiv \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} \]

\[ \langle \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \rangle_{ik} \equiv \sum_j W_{ik}(t_j) \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \]
Renormalization continues and gives the invariant actions, $S_p$

define the generating function and currents

$$C[\{J_{ik}\}] = \ln \left[ \int \exp \left( -S_o[\{\bar{x}_{ik}\}] + \sum_{ik} J_{ik} \bar{x}_{ik} \right) \prod_{ik} d\bar{x}_{ik} \right]$$

take the Legendre transform to form the effective action

$$S[\{\varphi_{ik}\}] = -C[\{J_{ik}\}] + \sum_{ik} J_{ik} \varphi_{ik}$$

expand the integrand, evaluate by stationary phase, and integrate the wavelet transformations, one gets the remarkable form for the effective action

$$S[\{\varphi_p\}] = S_o[\{\langle \varphi_p \rangle \}] + \frac{1}{2} \sum_p \frac{\partial^2 S_o[\{\langle \varphi_p \rangle \}]}{\partial \varphi_p^2} (\varphi_p - \langle \varphi_p \rangle)^2$$

where

$$\left. \frac{\partial S_o}{\partial \varphi_p} \right|_{\langle \varphi_p \rangle} = 0 \quad \quad S_p \equiv \frac{1}{2} \left. \frac{\partial^2 S_o}{\partial \varphi_p^2} \right|_{\langle \varphi_p \rangle}$$
S-matrix, stationarity, and restricted ergodicity

look at the definition of the S-matrix and that to leading asymptotic order it diagonalizes in the $|\varphi_p\rangle$ basis

$$\hat{S} \equiv \lim_{t_o \to \infty} \lim_{t_f \to \infty} U(x_f, t_f; x_o, t_o) = \lim_{t_o \to \infty} \lim_{t_f \to \infty} \int e^{S_o(x_j)} \mathcal{D}[\{x_j\}]$$

$$\hat{S} = \lim \int e^{S_o(x_j)} \mathcal{D}[\{x_j\}] = \lim \int e^{S_{eff}(\varphi_p)} \mathcal{D}[\{\varphi_p\}]$$

$$\approx \lim \int e^{S_o(|\varphi_p\rangle]} \prod_p \exp \left[ S_p (\varphi_p - \langle \varphi_p \rangle)^2 \right] d\varphi_p$$

now consider the stationary state of the wave function

$$f_\infty(x) \equiv f(x, t \to \infty)$$

by a restricted ergodicity on a $|\varphi_p\rangle$ mode basis the wavelet transformation of $f_\infty(x)$ with respect to x can be substituted for the wavelet transform of the wave function with respect to t.
Second quantization of the field and metric

\[ S = S_0 + \sum_p S_p a_p^* a_p \]

\[ |f\rangle = \prod_p \left( \frac{(a_p^*)^{N_p}}{(N_p!)^{1/2}} \right) |\varphi_o\rangle, \]

\[ |\varphi_p\rangle = \frac{(a_p^*)^{N_p}}{(N_p!)^{1/2}} |\varphi_o\rangle, \]

\[ |\langle \varphi_p | f \rangle|^2 = \frac{N_p}{\sum_p N_p}, \]

\[ \langle \varphi_p | S | \varphi_p \rangle = S_0 + N_p S_p, \]

\[ \langle f | S | f \rangle = S_0 + \sum_p N_p S_p. \]

projection of \(|f\rangle\) onto the \(|\varphi_p\rangle\) basis

\[ \sum_p |\varphi_p\rangle \langle \varphi_p | f \rangle = MW_t \ldots MW_t f(x(t), t) \]

\[ = MW_x \ldots MW_x f_\infty(x) \]

\[ = S f_\infty = \sum_p |\varphi_p\rangle S(p) f_\infty(x), \]

leads to the metric

\[ d^2(f, g) = \sum_p |\langle \varphi_p | f \rangle - \langle \varphi_p | g \rangle|^2 \]

\[ = \sum_p |S(p) f - S(p) g|^2 \]

\[ \propto \sum_p (\Delta \sqrt{N_p})^2, \]
Feynman diagram of elementary scattering

\[ S = S_{o1} + \sum_{n} S_{p_1} a_{p_1}^\dagger a_{p_1} + S_{o2} + \sum_{p_1} S_{p_2} a_{p_2}^\dagger a_{p_2} + \sum_{p} C_p a_{p_1=\mathbf{p}}^\dagger a_{p_2=\mathbf{p}} \]

\[ a_p^\dagger = a_{p_o}^\dagger \prod_{n=1}^{N_s} a_{s_{n,o}}^\dagger \prod_{m=1}^{N_{\lambda_n}} a_{s_{n,\lambda_{mn}}}^\dagger \]
Revisit of scattering and compact graphical representation

\[ |g\rangle = a_p^\dagger |f\rangle \]
Conclusions

• Mallat has come up with a transformation (group invariant scattering) that is:
  • very useful in identification of image texture
  • hierarchical
  • invariant to group transformations
  • stable to small changes to the image

• Image identification problem can be formulated as a dynamical system with an action

• Mallat’s transformation is an iterative wavelet based renormalization of the dynamics

• the renormalized coordinates are a useful basis leading to:
  • identification of fundamental excitations of system
  • second quantisation of system
  • view of changes in the system as a scattering of the fundamental excitations
  • graphical representation of the scattering in Feynman diagrams
  • definition of metric for the states of the system
Dynamical DNA

- the invariant actions, $S_p$, fully characterize the dynamics
- they provide a natural basis and metric
- the set of actions are, in analogy to biology, the DNA of the dynamics
  - a coded sequence of numbers, $S_p$, from which the character of the dynamics can be reconstructed
  - occupation numbers, $N_p$, fully characterize the state of the system