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Dynamical DNA: a possible metric of complexity

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“Three Sisters” -- aboriginal womans’ place for doing business, near BHPB Yandi iron ore mine

What do we want in a metric of complexity?

- Direct relationship of metric to dynamical action
- Hierarchical in terms of order of interaction (number of cliques, phrases or scales interacting) and size
- Invariant of coordinates (independent or dependant) and stable to small changes to the dynamics (Lipschitz invariant)
- Direct relationship to group symmetries
- Direct relationship to topological invariants (indexes)

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What is wrong with Fourier?

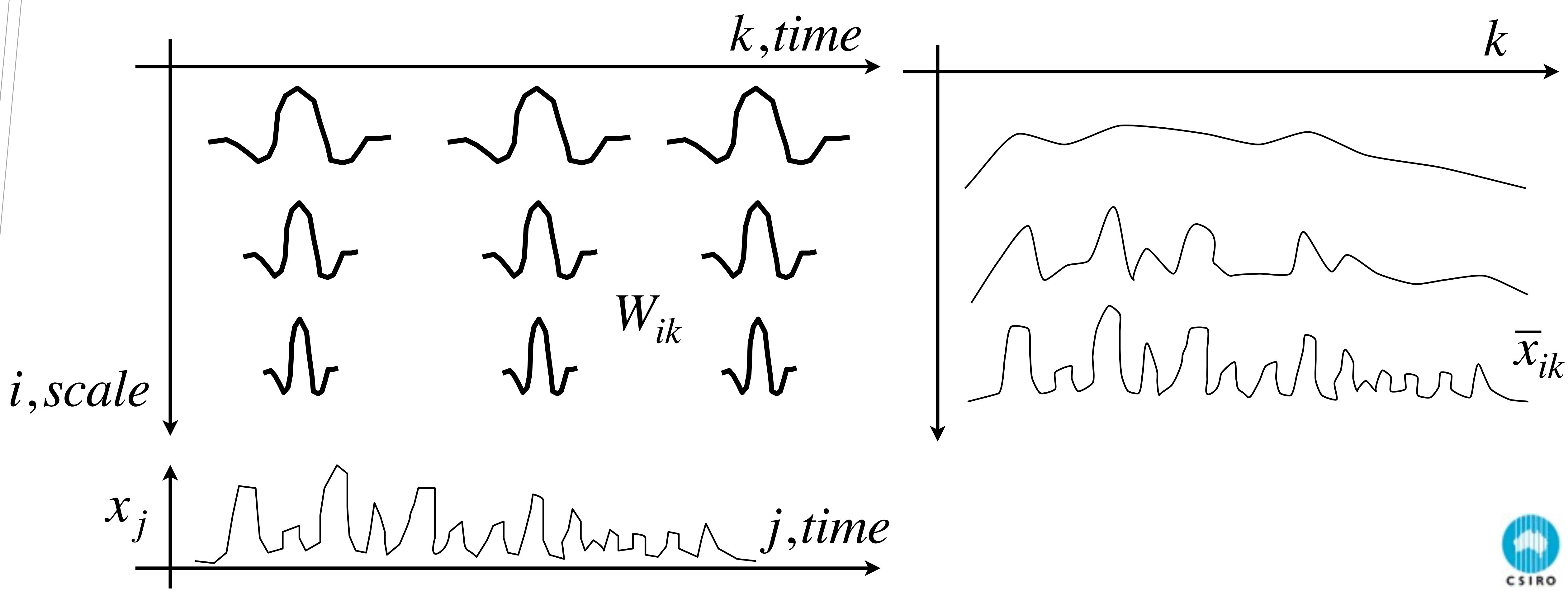
- Invariant of coordinates
- NOT stable to small changes in the dynamics
  - at small scale, small changes in signal lead to large changes in transform

What is wrong with the wavelet transform?

- stable to small changes in the dynamics
- NOT invariant of coordinates

$$\bar{x}_{ik} = \sum_j x_j W_{ik}(t_j)$$

where i is the scale and k is the new time index



Interesting development in machine vision

- Stephane Mallat’s Group Invariant Scattering
  - http://arxiv.org/abs/1101.2286
  - http://arxiv.org/abs/1011.3023

Define a scattering metric on:

$$d^2(f,g) = \|S(f) - S(g)\|^2 = \sum_p \|S(p)f - S(p)g\|^2$$

where S is a hierarical scattering operation based on iterative wavelet transform

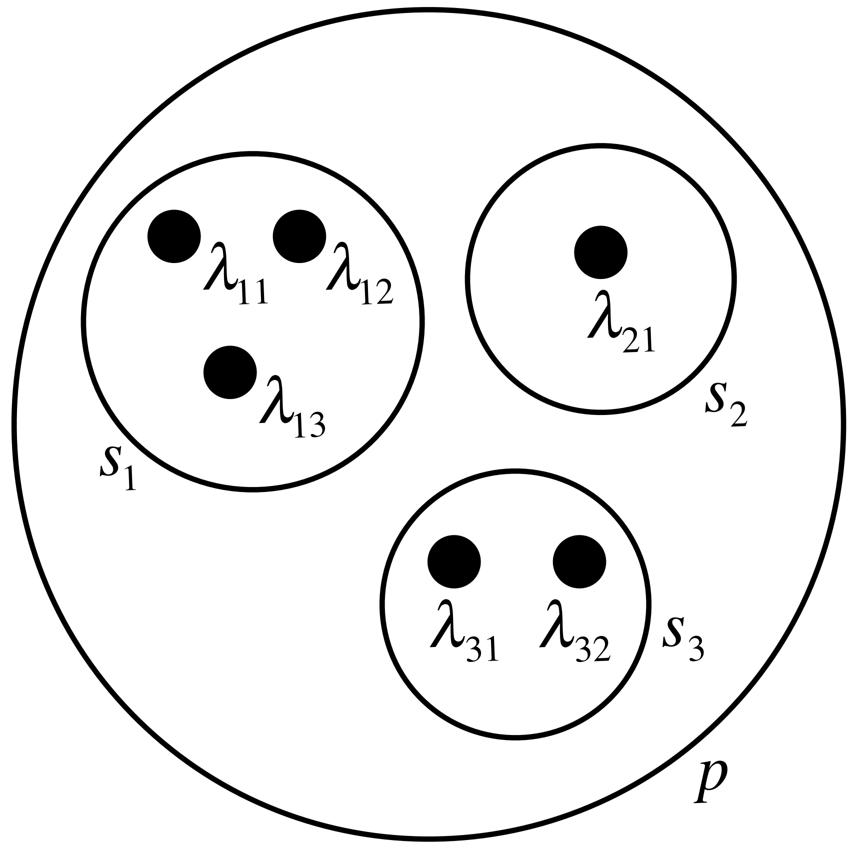
$$S = MWMW \ldots = \sum_p S(p)$$

where p is the path in “scale” space

$$S(p)f = |\ldots|f * \psi_{s_1}| * \psi_{s_2}| \ldots | * \psi_{s_n}| * \psi_\infty|$$

In the limit of many MW the transform becomes invariant of independent coordinate (Lie group parameter)

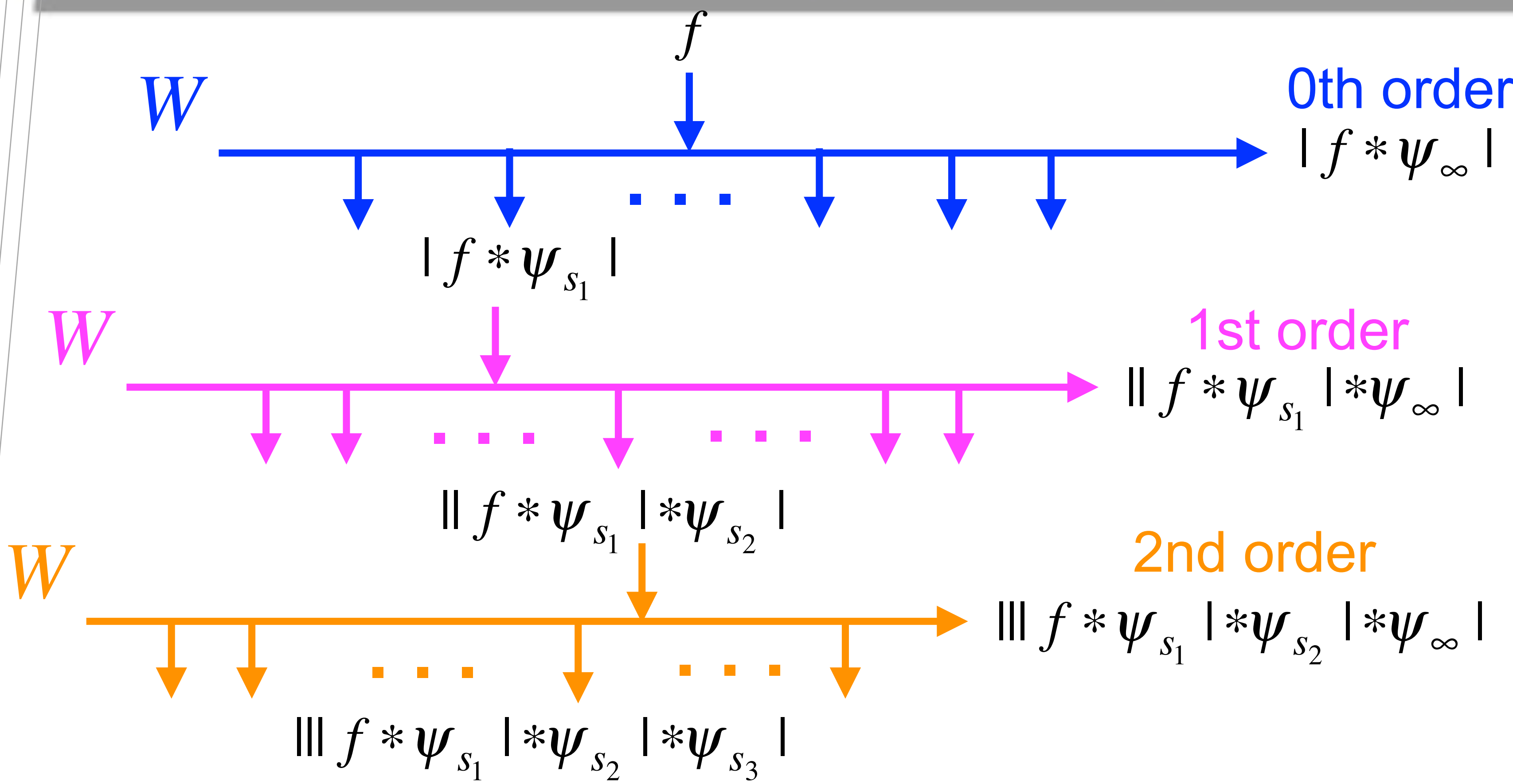
Total path specified by:



scattering at scale associated with Lie Group translation of independent parameters of deformation field (e.g., k of phonon)

scattering at λ scale associated with Gl(n) transformation of dependant parameters (deformation field, e.g., polarization of phonon)

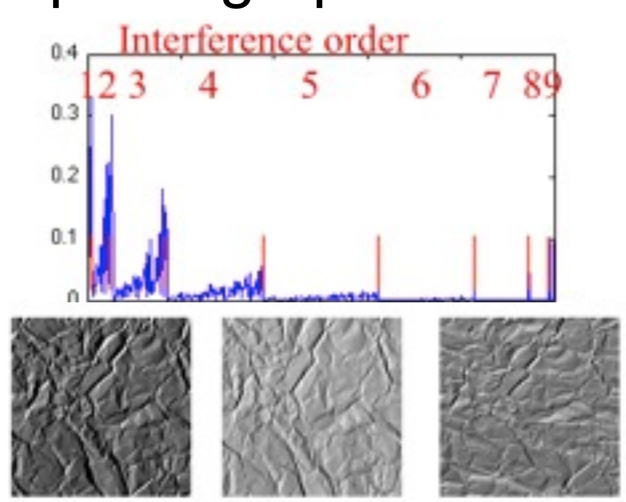
Practical application

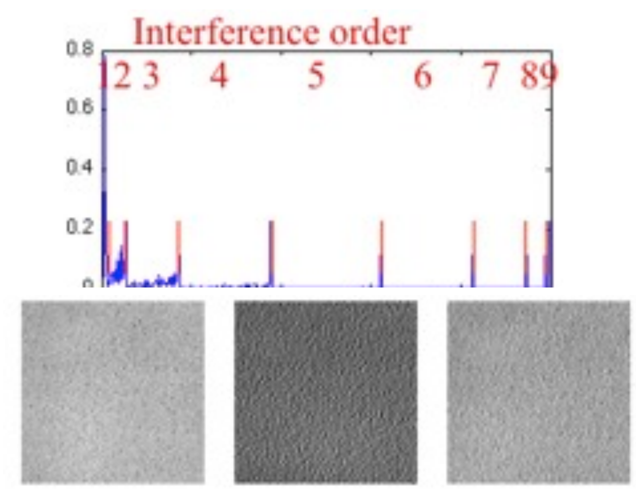


<u>previous best error rate</u>
- textons 5.4%
- MRF 2.7%

<u>invariant scattering</u>
- PCA 0.29%

unique fingerprint of texture





independent of chaotic “phase”

What do we have?

- hierarchy
- invariant of coord and stable to small changes in dynamics
- direct relationships to group symmetries (see paper)
- possible relationship to topological index
  - orientation as a function of scale, gives curvatures, gives indexes associated with singularity of system (future R&D)

BUT what does this have to do with complex dynamical systems?

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Vision problem can be thought of as dynamics

define the deformation vector field

$$\vec{\tau}(\vec{x}) = [\overleftrightarrow{\nabla}\vec{\tau}(\vec{x}) - [\overleftrightarrow{\nabla}\vec{\tau}(\vec{x_o}) + \overleftrightarrow{I}] \cdot \vec{\tau}(\vec{x}_o)$$

and the associated action

$$S[\vec{x}(t)] = \int_0^{t_o} L(\vec{x}(t), \dot{\vec{x}}(t))dt$$

$$L(\vec{x}(t), \dot{\vec{x}}(t)) = [\dot{\vec{x}} - \vec{\tau}(\vec{x})] \cdot \vec{g}(\vec{x})$$

image is the wave function

$$f(\vec{x}, t) = \int e^{iS[\vec{x}(t)]/\hbar} \mathscr{D}[\vec{x}(t)] f(\vec{x}, 0)$$

which in the classical image gives the image deformation

$$f(\vec{x}, t) = f(\vec{x} - \phi_t(\vec{x})) = f(\vec{x} - \vec{\tau}(\vec{x}))$$

Mallat's invariant scattering as an interative renormaization

Let us consider a much simpler problem of 1D time dependant dynamics (time sliced)

$$S_o[\{x_j\}] = \sum_j L(x_j, \dot{x}_j, t_j)\Delta t$$

change coordinate to "smoothed" wavelet basis

$$\bar{x}_{ik} = \sum_j x_j W_{ik}(t_j)$$

useful mean field approximation in this basis

$$\begin{aligned} \frac{\partial S_o[\{\bar{x}_{ik}\}]}{\partial \bar{x}_{ik}} &= \sum_j W_{ik}(t_j)\frac{\partial L}{\partial x} + W'_{ik}(t_j)\frac{\partial L}{\partial v} \\ &= \left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} + \left\langle \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) \right\rangle_{ik} \\ &= \langle \nabla L \rangle_{ik} + \langle \dot{p} \rangle_{ik} \end{aligned}$$

$$\left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j)\frac{\partial L}{\partial x} \qquad \left\langle \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j)\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)$$

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Renormalization continues and gives the invariant actions, Sp

define the generating function and currents

$$C[\{J_{ik}\}] = ln\left[\int exp\left(-S_o[\{\bar{x}_{ik}\}] + \sum_{ik} J_{ik}\bar{x}_{ik}\right)\prod_{ik} d\bar{x}_{ik}\right]$$

take the Legendre transform to form the effective action

$$S[\{\varphi_{ik}\}] = -C[\{J_{ik}\}] + \sum_{ik} J_{ik}\varphi_{ik}$$

expand the integrand, evaluate by stationary phase, and integrate the the wavelet transformations, one gets the remarkable form for the effective action

$$S[\{\varphi_p\}] = S_o[\{\langle\varphi_p\rangle\}] + \frac{1}{2}\sum_p \frac{\partial^2 S_o[\{\langle\varphi_p\rangle\}]}{\partial\varphi_p^2}\left(\varphi_p - \langle\varphi_p\rangle\right)^2$$

where

$$\left.\frac{\partial S_o}{\partial\varphi_p}\right|_{\langle\varphi_p\rangle} = 0 \qquad S_p \equiv \frac{1}{2}\left.\frac{\partial^2 S_o}{\partial\varphi_p^2}\right|_{\langle\varphi_p\rangle}$$

S-matrix, stationarity, and restricted ergodicity

look at the definition of the S-matrix and that to leading asymototic order it diagnalizes in the $|\varphi_p\rangle$ basis

$$\hat{S} \equiv \lim_{\substack{t_o \to -\infty \\ t_f \to \infty}} U(x_f, t_f; x_o, t_o) = \lim_{\substack{t_o \to -\infty \\ t_f \to \infty}} \int e^{S_o[\{x_j\}]} \mathscr{D}[\{x_j\}].$$

$$\hat{S} = \lim \int e^{S_o[\{x_j\}]} \mathscr{D}[\{x_j]\} = \lim \int e^{S_{eff}[\{\varphi_p\}]} \mathscr{D}[\{\varphi_p\}]$$

$$\approx \lim \int e^{S_o[\langle \varphi_p \rangle]} \prod_p exp\left[S_p\left(\varphi_p - \langle \varphi_p \rangle\right)^2\right] d\varphi_p$$

now consider the stationary state of the wave function

$$f_\infty(x) \equiv f(x, t \to \infty)$$

by a restricted ergodicity on a $|\varphi_p\rangle$ mode basis the wavelet transformation of $f_\infty(x)$ with respect to x can be substituted for the wavelet transform of the wave function with respect to t.

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Second quantizaiton of the field and metric

$$S = S_o + \sum_p S_p a_p^\dagger a_p$$

$$|f\rangle = \prod_p \left(\frac{(a_p^\dagger)^{N_p}}{(N_p!)^{1/2}} \right) |\varphi_o\rangle\,,$$

$$|\varphi_p\rangle = \frac{(a_p^\dagger)^{N_p}}{(N_p!)^{1/2}} |\varphi_o\rangle\,,$$

$$|\langle \varphi_p | f \rangle|^2 = \frac{N_p}{\sum_p N_p},$$

$$\langle \varphi_p | S | \varphi_p \rangle = S_o + N_p S_p,$$

$$\langle f | S | f \rangle = S_o + \sum_p N_p S_p.$$

projection of $|f\rangle$ onto the $|\varphi_p\rangle$ basis

$$\sum_p |\varphi_p\rangle \langle \varphi_p | f \rangle = MW_t \ldots MW_t f(x(t), t)$$

$$= MW_x \ldots MW_x f_\infty(x)$$

$$= S f_\infty = \sum_p |\varphi_p\rangle S(p) f_\infty(x),$$

leads to the metric

$$d^2(f,g) = \sum_p |\langle \varphi_p | f \rangle - \langle \varphi_p | g \rangle|^2$$

$$= \sum_p |S(p) f - S(p) g|^2$$

$$\propto \sum_p \left(\Delta \sqrt{N_p} \right)^2,$$

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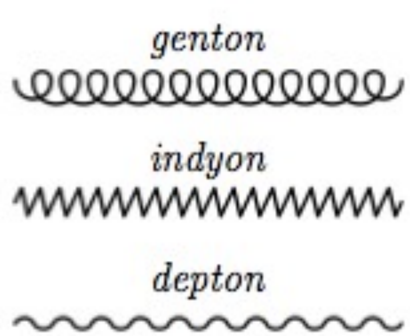
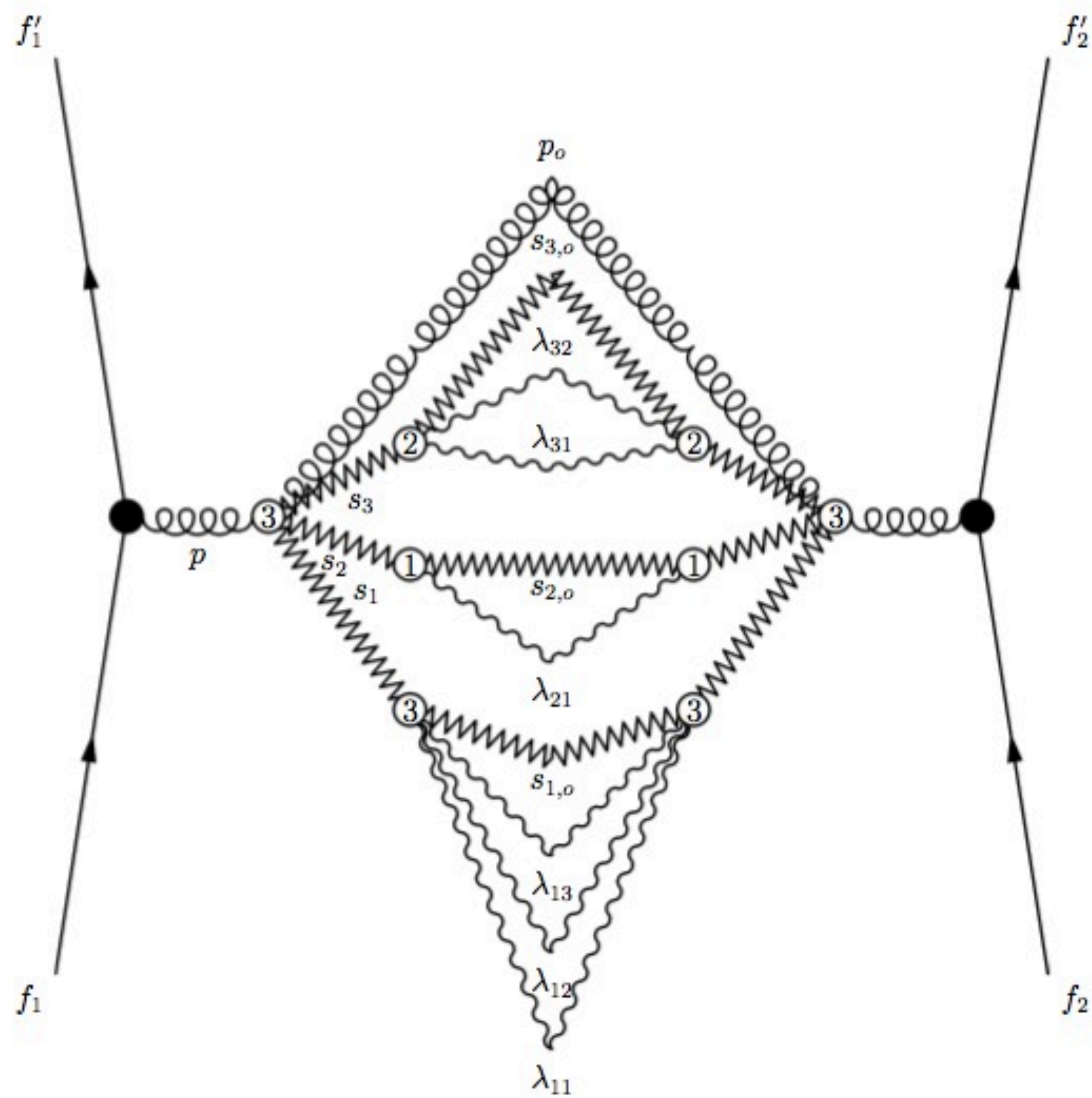
Feynman diagram of elementary scattering

$$S = S_{o1} + \sum_{p_1} S_{p_1} a^{\dagger}_{p_1} a_{p_1} + S_{o2} + \sum_{p_1} S_{p_2} a^{\dagger}_{p_2} a_{p_2} + \sum_{p} C_p a^{\dagger}_{p_1=p} a_{p_2=p}$$

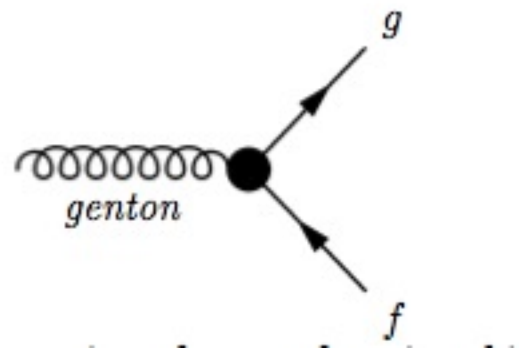
$$a^{\dagger}_p = a^{\dagger}_{p_o} \prod_{n=1}^{N_s} a^{\dagger}_{s_n,o} \prod_{m=1}^{N_{\lambda n}} a^{\dagger}_{s_n,\lambda_{mn}}$$

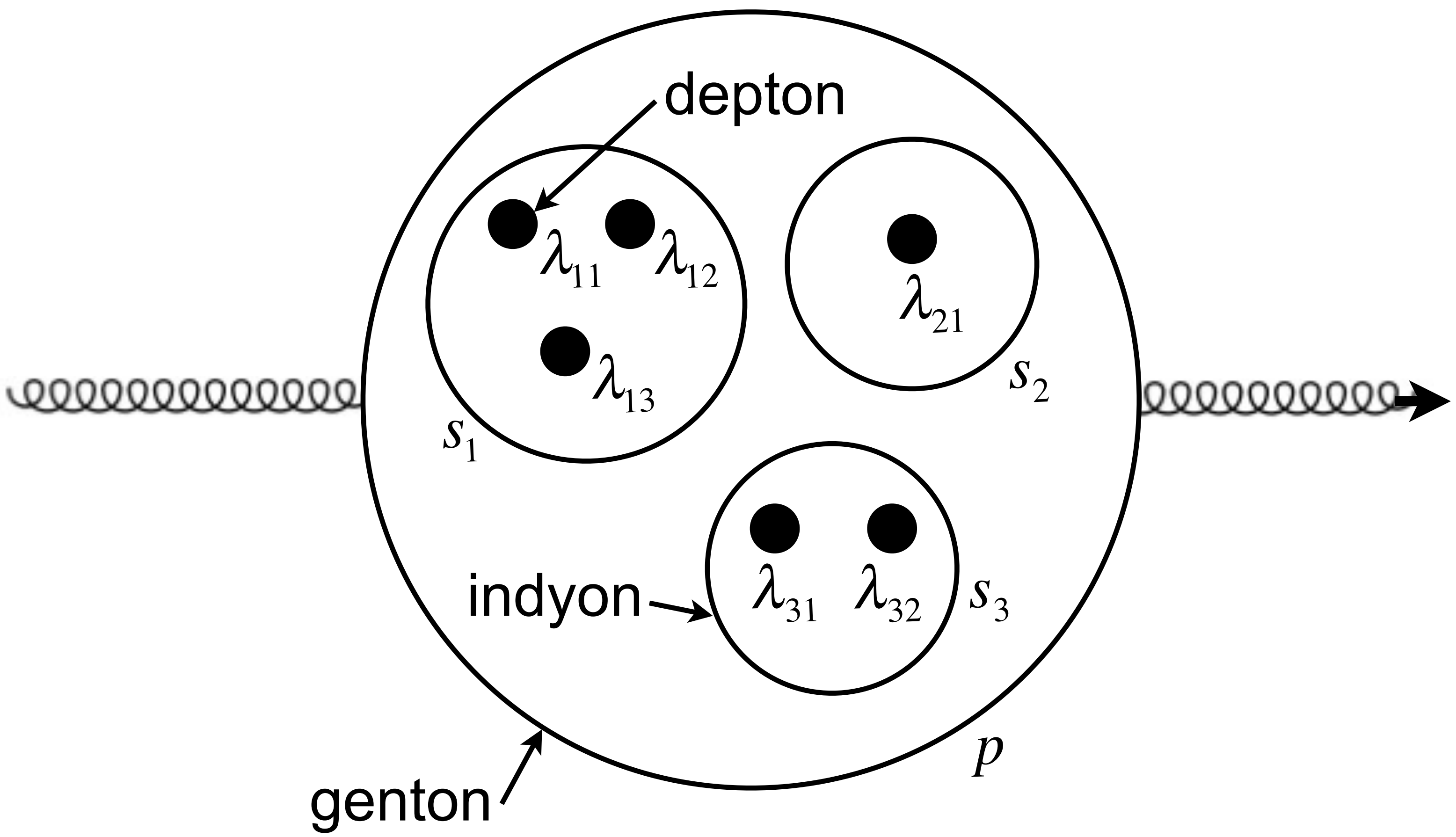




Revisit of scattering and compact graphical representation

$$|g\rangle = a_p^\dagger |f\rangle$$





Conclusions

- Mallat has come up with a transformation (group invariant scattering) that is:
  - very useful in identification of image texture
  - hierarchical
  - invariant to group transformations
  - stable to small changes to the image
- Image identification problem can be formulated as a dynamical system with an action
- Mallat's transformation is an iterative wavelet based renormalization of the dynamics
- the renormalized coordinates are a useful basis leading to:
  - identification of fundamental excitations of system
  - second quantisation of system
  - view of changes in the system as a scattering of the fundamental excitations
  - graphical representation of the scattering in Feynman diagrams
  - definition of metric for the states of the system

Dynamical DNA

- the invariant actions, Sp, fully characterize the dynamics
- they provide a natural basis and metric
- the set of actions are, in analogy to biology, the DNA of the dynamics
    - a coded sequence of numbers, Sp, from which the character of the dynamics can be reconstructed
    - occupation numbers, Np, fully characterize the state of the system

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