A new perspective on renormalization: invariant actions, a dynamical DNA

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"Three Sisters" -- aboriginal womans' place for doing business, near BHPB Yandi iron ore mine
What do we want in a metric of complexity?

- Direct relationship of metric to dynamical action
- Hierarchical in terms of order of interaction (number of cliques, phrases or scales interacting) and size
- Invariant of coordinates (independent or dependant) and stable to small changes to the dynamics (Lipschitz invariant)
- Direct relationship to group symmetries (not cemetery [sic American accent])
- Direct relationship to topological invariants (indexes)
What is wrong with Fourier?

- Invariant of coordinates
- NOT stable to small changes in the dynamics
  - at small scale, small changes in signal lead to large changes in transform
  - source of ultraviolet divergence in Wilson style renormalization leading to need for regularization
What is wrong with the wavelet transform?

• stable to small changes in the dynamics
• NOT invariant of coordinates

\[ \bar{x}_{ik} = \sum_j x_j W_{ik}(t_j) \]

where \( i \) is the scale and \( k \) is the new time index
Iteration makes wavelet transform invariant of coordinate.

(unique fingerprint of texture)

previous best error rate

textons 5.4%
MRF 2.7%

invariant scattering

PCA 0.29%

independent of chaotic “phase”
Interesting development in machine vision

• Stephane Mallat’s Group Invariant Scattering
  • arXiv:1101.2286
  • arXiv:1011.3023

Gives metric on states of system

\[ d^2(f, g) = \| S(f) - S(g) \|^2 = \sum_p \| S(p)f - S(p)g \|^2 \]

\[ S(p)f = |\cdots | f \ast \psi_{s_1} | \ast \psi_{s_2} | \cdots | \ast \psi_{s_n} | \ast \psi_\infty | \]
Total path specified by:

scattering at scale associated with Lie Group translation of independent parameters of deformation field (e.g., k of phonon)

scattering at \( \lambda \) scale associated with GL(n) transformation of dependant parameters (deformation field, e.g., polarization of phonon)
What do we have?

- hierarchy
- invariant of coord and stable to small changes in dynamics
- direct relationships to group symmetries
- possible relationship to topological index
  - orientation as a function of scale, gives curvatures, gives indexes associated with singularity of system (future R&D)

BUT what does this have to do with dynamical systems?

Why is the modulus taken (phase removed)?
Vision problem can be thought of as dynamics

define the deformation vector field

\[ \bar{\tau}(\bar{x}) = \left[ \bar{\nabla} \tau(\bar{x}) - \left[ \bar{\nabla} \tau(\bar{x}_o) + \left\uparrow \bar{I} \right\downarrow \right] \cdot \bar{\tau}(\bar{x}_o) \]

and the associated action

\[ S[\bar{x}(t)] = \int_0^{t_o} L(\bar{x}(t), \dot{\bar{x}}(t)) dt \]

\[ L(\bar{x}(t), \dot{\bar{x}}(t)) = [\dot{x} - \bar{\tau}(\bar{x})] \cdot \bar{g}(\bar{x}) \]

image is the wave function

\[ f(\bar{x}, t) = \int e^{iS[\bar{x}(t)]/\hbar} \mathcal{D}[\bar{x}(t)] f(\bar{x}, 0) \]

which in the classical image gives the image deformation

\[ f(\bar{x}, t) = f(\bar{x} - \phi_t(\bar{x})) = f(\bar{x} - \bar{\tau}(\bar{x}))) \]
Mallat’s invariant scattering as an iterative renormalization

Let us consider a much simpler problem of 1D time dependant dynamics (time sliced)

\[ S_o[\{x_j\}] = \sum_j L(x_j, \dot{x}_j, t_j) \Delta t \]

change coordinate to “smoothed” wavelet basis which respects the group symettry of the Lagrangian

\[ \bar{x}_{ik} = \sum_j x_j W_{ik}(t_j) \]

useful mean field approximation in this basis

\[
\frac{\partial S_o[\{\bar{x}_{ik}\}]}{\partial \bar{x}_{ik}} = \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} + W'_{ik}(t_j) \frac{\partial L}{\partial v} = \left< \frac{\partial L}{\partial x} \right>_{ik} + \left< \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \right>_{ik} = \left< \nabla L \right>_{ik} + \left< \dot{p} \right>_{ik}
\]

\[
\left< \frac{\partial L}{\partial x} \right>_{ik} \equiv \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} \quad \left< \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) \right>_{ik} \equiv \sum_j W_{ik}(t_j) \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right)
\]
Renormalization continues and gives the invariant actions, $S_p$

define the generating function and currents

$$C[\{J_{ik}\}] = \ln \left[ \int \exp \left( -S_0[\{\bar{x}_{ik}\}] + \sum_{ik} J_{ik} \bar{x}_{ik} \right) \prod_{ik} d\bar{x}_{ik} \right] = \ln(Z[\{J_{ik}\}])$$

take the Legendre transform to form the effective action

$$S[\{\varphi_{ik}\}] = -C[\{J_{ik}\}] + \sum_{ik} J_{ik} \varphi_{ik}$$

$$\varphi_{ik} = \frac{\partial C[\{J_{ik}\}]}{\partial J_{ik}} \bigg|_{J=0} = \langle \bar{x}_{ik} \rangle$$

expand the integrand, evaluate by stationary phase, and integrate the wavelet transformations, one gets the remarkable form for the effective action

$$S[\{\varphi_p\}] = S_0[\langle \varphi_p \rangle] + \frac{1}{2} \sum_p \frac{\partial^2 S_0[\langle \varphi_p \rangle]}{\partial \varphi_p^2} (\varphi_p - \langle \varphi_p \rangle)^2$$

where

$$\frac{\partial S_0}{\partial \varphi_p} \bigg|_{\langle \varphi_p \rangle} = 0$$

$$S_p = \frac{1}{2} \frac{\partial^2 S_0}{\partial \varphi_p^2} \bigg|_{\langle \varphi_p \rangle}$$
Partition function decomposes by path

\[ Z[\{J_p\}] = \prod_p e^{S_o(\{\varphi_p\})} \int e^{S_p (\varphi_p - \langle \varphi_p \rangle)^2 + J_p \varphi_p} \, d\varphi_p \]

\[ = \prod_p Z_p(S_p, \langle \varphi_p \rangle , J_p) \]

\[ \ln Z[\{J_p\}] = \sum_p \ln Z_p(S_p, \langle \varphi_p \rangle , J_p) \]

\[ = \ln Z^{(0)} + \sum_{s_1} \ln Z^{(1)}_{s_1} + \sum_{s_1} \sum_{s_2 > s_1} \ln Z^{(2)}_{s_1 s_2} + \mathcal{O}(3) \]

for dissipative system that self organises, only a finite number of \( p \) have non-zero coefficients, leading to a reduction in entropy
In language of Jaynes (Physical Review ’57)

the currents \( J_p \) are the LaGrange multipliers

the average renormalized paths \( \varphi_p \) are known expectation values

the effective action

\[
S[\{\varphi_p\}] = \ln Z[\{J_p\}] + \sum_p J_p \varphi_p
\]

is the entropy

and the partition function is directly related to

\[
- \ln Z[\{J_p\}]
\]

the potential
Feynman diagram of elementary excitation by $J_p$

$$\left| N^{(0)}, \{ N_{s_1}^{(1)} \}, \{ N_{s_1,s_2}^{(2)} \}, \ldots \right|$$
Revisit of scattering and compact graphical representation

\[ |g\rangle = a_p^\dagger |f\rangle \]
Conclusions

• Mallat has come up with a transformation (group invariant scattering) that is:
  • very useful in identification of image texture
  • hierarchical
  • invariant to group transformations
  • stable to small changes to the image

• Image identification problem can be formulated as a dynamical system with an action

• Mallat’s transformation is an iterative wavelet based renormalization of the dynamics

• the renormalized coordinates are a useful basis leading to:
  • identification of fundamental excitations of system
  • factorization of entropy
  • view of changes in the system as a scattering of the fundamental excitations
  • graphical representation of the scattering in Feynman diagrams
  • definition of metric for the states of the system
Dynamical DNA

• the invariant actions, $S_p$, fully characterize the dynamics
• they provide a natural basis and metric
• the set of actions are, in analogy to biology, the DNA of the dynamics
  • a coded sequence of numbers, $S_p$, from which the character of the dynamics can be reconstructed
  • occupation numbers, $N_p$, fully characterize the state of the system