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A new perspective on renormalization: invariant actions, a dynamical DNA

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“Three Sisters” -- aboriginal womans’ place for doing business, near BHPB Yandi iron ore mine

What do we want in a metric of complexity?

- Direct relationship of metric to dynamical action
- Hierarchical in terms of order of interaction (number of cliques, phrases or scales interacting) and size
- Invariant of coordinates (independent or dependant) and stable to small changes to the dynamics (Lipschitz invariant)
- Direct relationship to group symmetries (not cemetery [sic American accent])
- Direct relationship to topological invariants (indexes)

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What is wrong with Fourier?

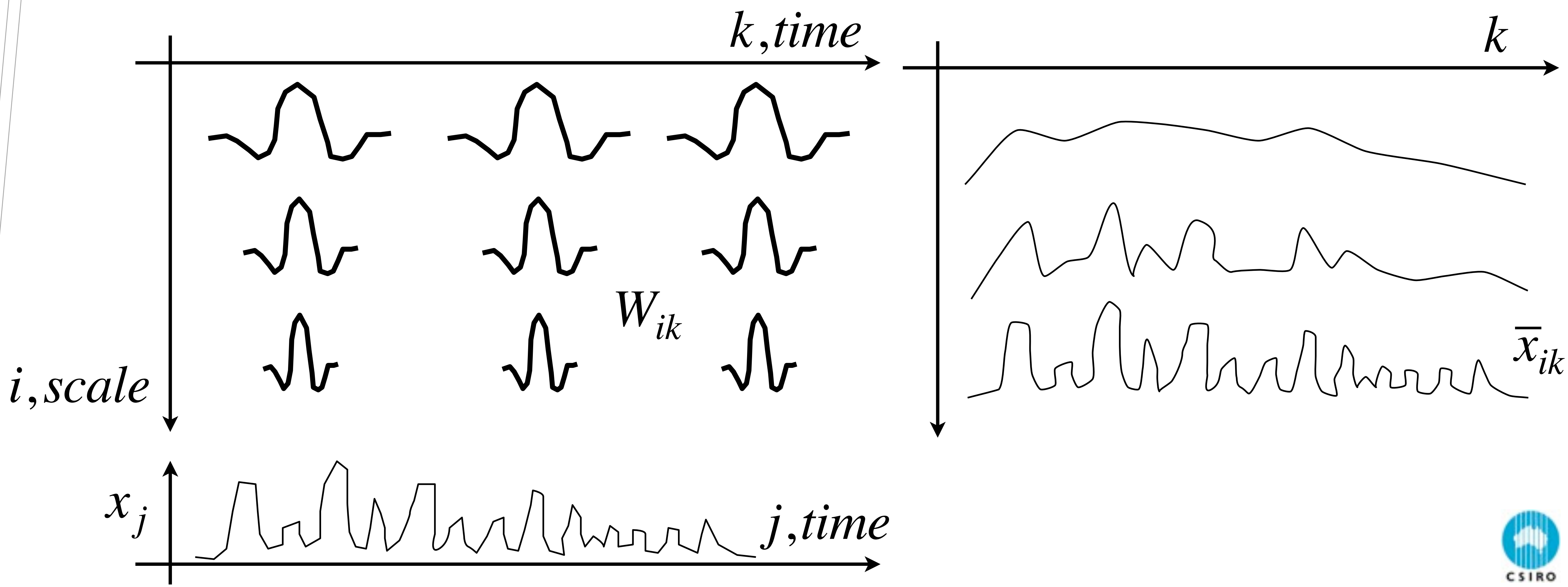
- Invariant of coordinates
- NOT stable to small changes in the dynamics
  - at small scale, small changes in signal lead to large changes in transform
  - source of ultraviolet divergence in Wilson style renormalization leading to need for regularization

What is wrong with the wavelet transform?

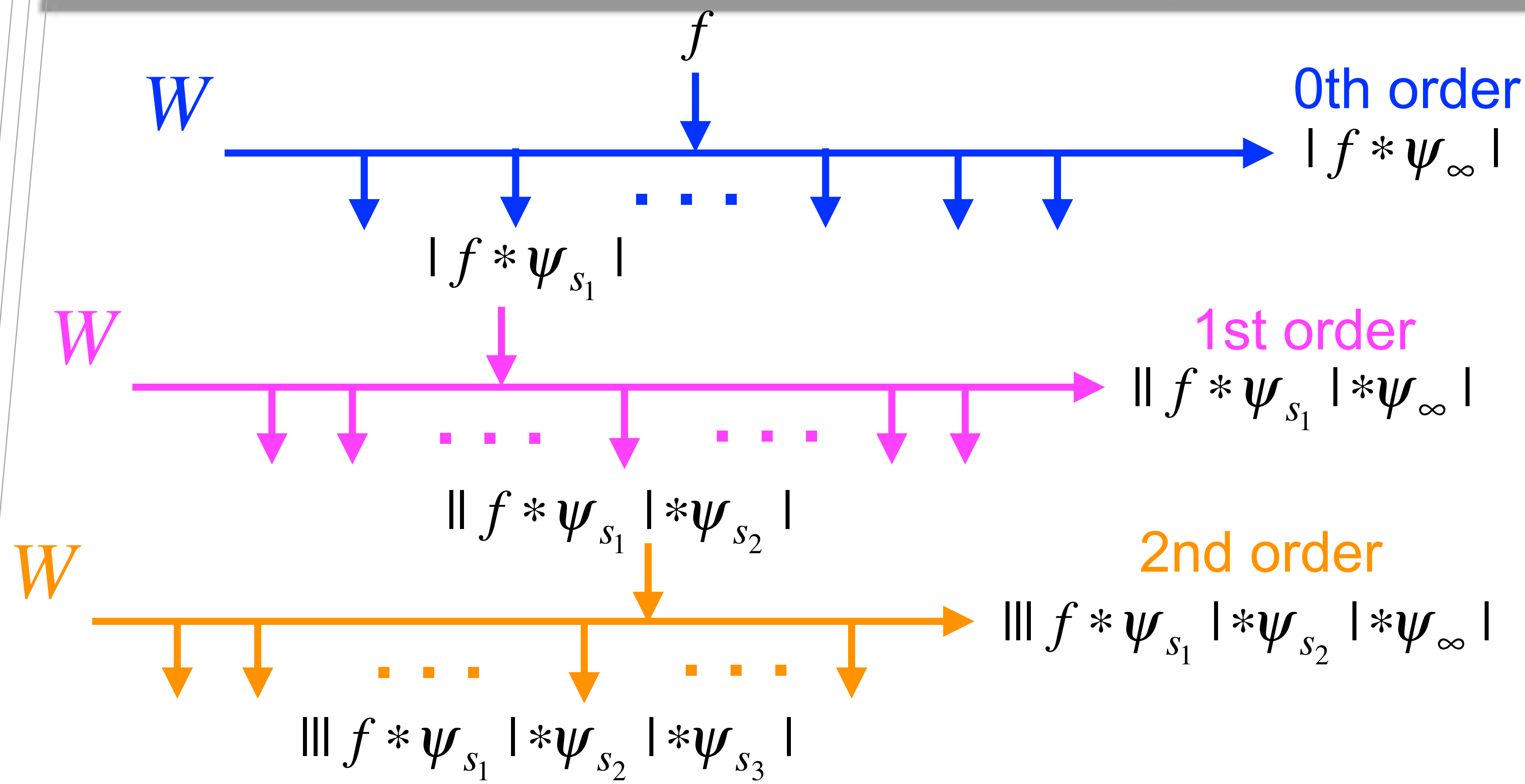
- stable to small changes in the dynamics
- NOT invariant of coordinates

$$\bar{x}_{ik} = \sum_j x_j W_{ik}(t_j)$$

where i is the scale and k is the new time index



Iteration makes wavelet transform invariant of coordinate

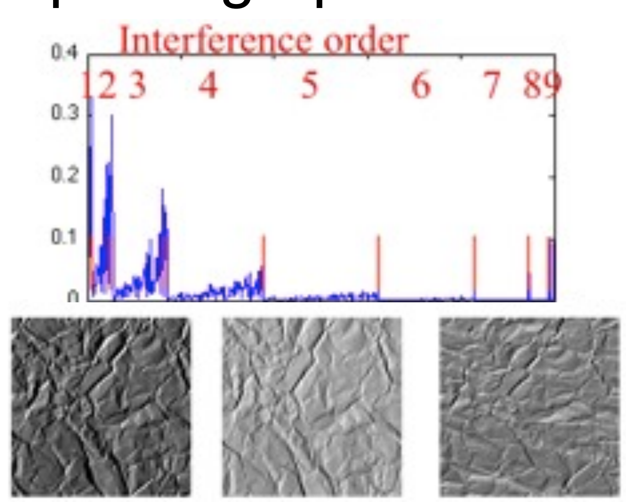


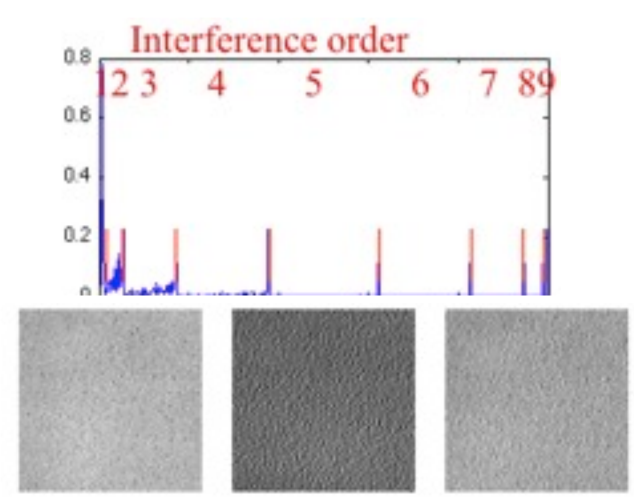
$$S^{(0)}(f)$$

$$S^{(1)}_{s_1}(f) = \begin{pmatrix} \square \\ \vdots \\ \square \end{pmatrix}$$

$$S^{(2)}_{s_1 s_2}(f) = \begin{pmatrix} \square & \cdots & \square \\ 0 & \ddots & \vdots \\ 0 & 0 & \square \end{pmatrix}$$

unique fingerprint of texture





independent of chaotic “phase”

previous best error rate
- textons 5.4%
- MRF 2.7%

invariant scattering
- PCA 0.29%

Interesting development in machine vision

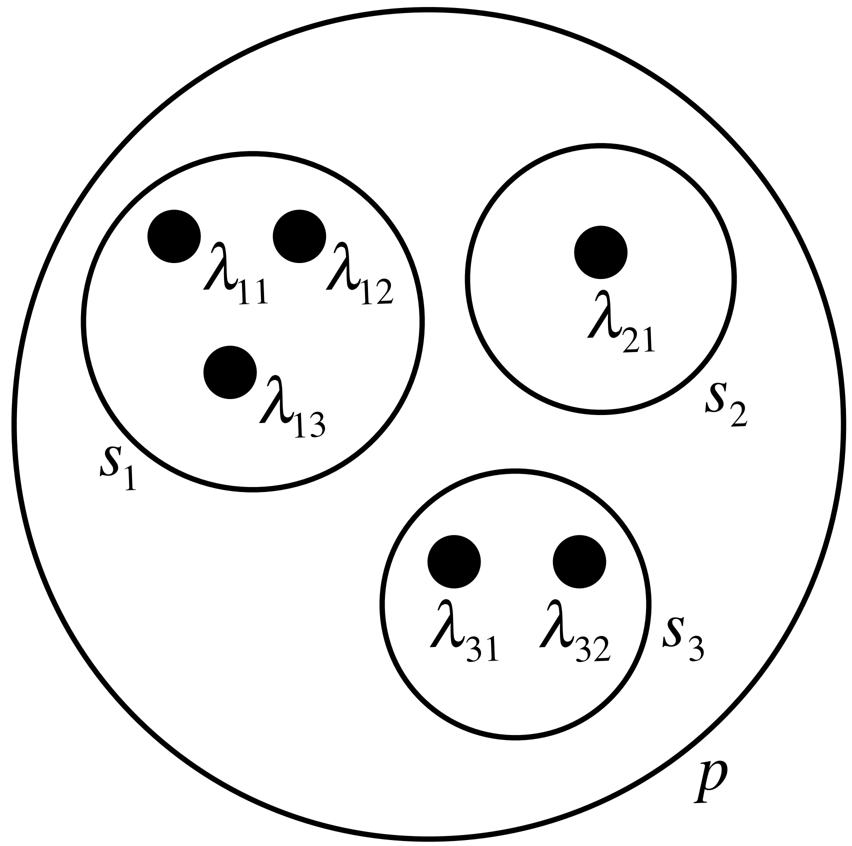
- Stephane Mallat's Group Invariant Scattering
  - arXiv:1101.2286
  - arXiv:1011.3023

Gives metric on states of system

$$d^2(f,g) = \|S(f) - S(g)\|^2 = \sum_p \|S(p)f - S(p)g\|^2$$

$$S(p)f = |\ldots| f * \psi_{s_1} | * \psi_{s_2} | \ldots | * \psi_{s_n} | * \psi_{\infty} |$$

Total path specified by:



scattering at scale associated with Lie Group translation of independent parameters of deformation field (e.g., k of phonon)

scattering at λ scale associated with Gl(n) transformation of dependant parameters (deformation field, e.g., polarization of phonon)

What do we have?

- hierarchy
- invariant of coord and stable to small changes in dynamics
- direct relationships to group symmetries
- possible relationship to topological index
  - orientation as a function of scale, gives curvatures, gives indexes associated with singularity of system (future R&D)

BUT what does this have to do with dynamical systems?

Why is the modulus taken (phase removed)?

Vision problem can be thought of as dynamics

define the deformation vector field

$$\vec{\tau}(\vec{x}) = [\overleftrightarrow{\nabla}\vec{\tau}(\vec{x}) - [\overleftrightarrow{\nabla}\vec{\tau}(\vec{x_o}) + \overleftrightarrow{I}] \cdot \vec{\tau}(\vec{x}_o)$$

and the associated action

$$S[\vec{x}(t)] = \int_0^{t_o} L(\vec{x}(t), \dot{\vec{x}}(t))dt$$

$$L(\vec{x}(t), \dot{\vec{x}}(t)) = [\dot{\vec{x}} - \vec{\tau}(\vec{x})] \cdot \vec{g}(\vec{x})$$

image is the wave function

$$f(\vec{x}, t) = \int e^{iS[\vec{x}(t)]/\hbar} \mathscr{D}[\vec{x}(t)] f(\vec{x}, 0)$$

which in the classical image gives the image deformation

$$f(\vec{x}, t) = f(\vec{x} - \phi_t(\vec{x})) = f(\vec{x} - \vec{\tau}(\vec{x}))$$

Mallat’s invariant scattering as an iterative renormaization

Let us consider a much simpler problem of 1D time dependant dynamics (time sliced)

$$S_o[\{x_j\}] = \sum_j L(x_j, \dot{x}_j, t_j)\Delta t$$

change coordinate to “smoothed” wavelet basis which respects the group symettry of the Lagrangian

$$\bar{x}_{ik} = \sum_j x_j W_{ik}(t_j)$$

useful mean field approximation in this basis

$$\begin{aligned}\frac{\partial S_o[\{\bar{x}_{ik}\}]}{\partial \bar{x}_{ik}} &= \sum_j W_{ik}(t_j)\frac{\partial L}{\partial x} + W'_{ik}(t_j)\frac{\partial L}{\partial v} \\ &= \left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} + \left\langle \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) \right\rangle_{ik} \\ &= \langle \nabla L \rangle_{ik} + \langle \dot{p} \rangle_{ik}\end{aligned}$$

$$\left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j)\frac{\partial L}{\partial x} \qquad \left\langle \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j)\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)$$

Renormalization continues and gives the invariant actions, S_p

define the generating function and currents

$$C[\{J_{ik}\}] = ln\left[\int exp\left(-S_o[\{\bar{x}_{ik}\}] + \sum_{ik} J_{ik}\bar{x}_{ik}\right)\prod_{ik} d\bar{x}_{ik}\right] = \ln(Z[\{J_{ik}\}])$$

take the Legendre transform to form the effective action

$$S[\{\varphi_{ik}\}] = -C[\{J_{ik}\}] + \sum_{ik} J_{ik}\varphi_{ik} \qquad \varphi_{ik} = \left.\frac{\partial C[\{J_{ik}\}]}{\partial J_{ik}}\right|_{J=0} = \langle \bar{x}_{ik}\rangle$$

expand the integrand, evaluate by stationary phase, and integrate the the wavelet transformations, one gets the remarkable form for the effective action

$$S[\{\varphi_p\}] = S_o[\{\langle\varphi_p\rangle\}] + \frac{1}{2}\sum_p \frac{\partial^2 S_o[\{\langle\varphi_p\rangle\}]}{\partial\varphi_p^2}(\varphi_p - \langle\varphi_p\rangle)^2$$

where

$$\left.\frac{\partial S_o}{\partial\varphi_p}\right|_{\langle\varphi_p\rangle} = 0 \qquad S_p \equiv \frac{1}{2}\left.\frac{\partial^2 S_o}{\partial\varphi_p^2}\right|_{\langle\varphi_p\rangle}$$

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Partition function decomposes by path

$$Z[\{J_p\}] = \prod_p e^{S_o[\langle\varphi_p\rangle]} \int exp\left[S_p\left(\varphi_p - \langle\varphi_p\rangle\right)^2 + J_p\varphi_p\right] d\varphi_p$$

$$= \prod_p Z_p(S_p, \langle\varphi_p\rangle, J_p)$$

$$\ln Z[\{J_p\}] = \sum_p \ln Z_p(S_p, \langle\varphi_p\rangle, J_p)$$

$$= \ln Z^{(0)} + \sum_{s_1} \ln Z^{(1)}_{s_1} + \sum_{\substack{s_1 \\ s_2 > s_1}} \ln Z^{(2)}_{s_1 s_2} + \mathscr{O}(3)$$

for dissipative system that self organises, only a finite number of p have non-zero coefficients, leading to a reduction in entropy

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In language of Jaynes (Physical Review '57)

the currents J_p are the LaGrange multipliers

the average renormalized paths φ_p

are known expectation values

the effective action

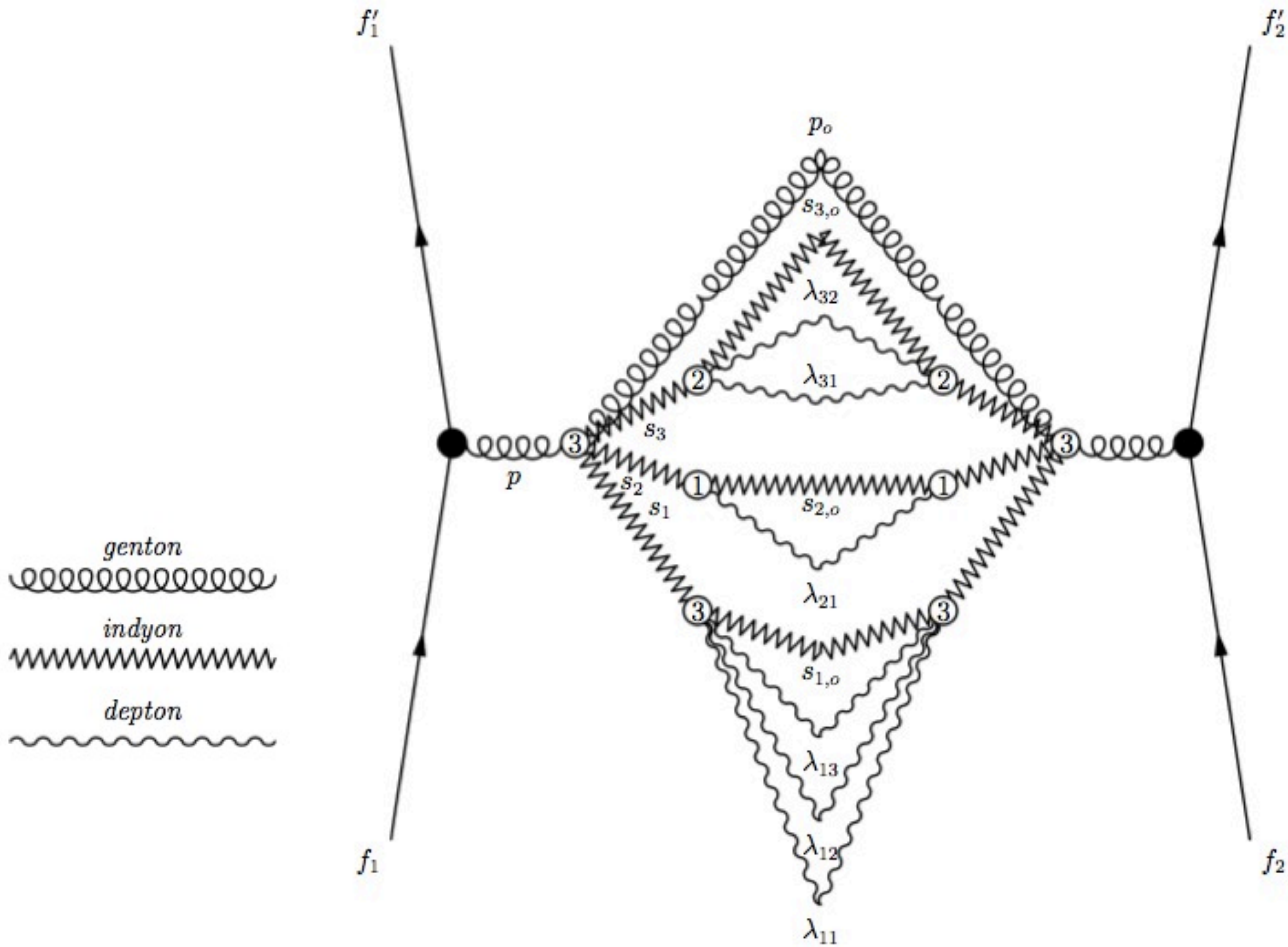
$$S[\{\varphi_p\}] = \ln Z[\{J_p\}] + \sum_p J_p \varphi_p$$ is the entropy

and the partition function is directly related to

$-\ln Z[\{J_p\}]$ the potential

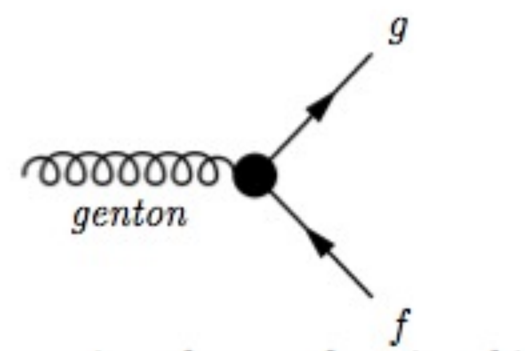
Feynman diagram of elementary excitation by J_p

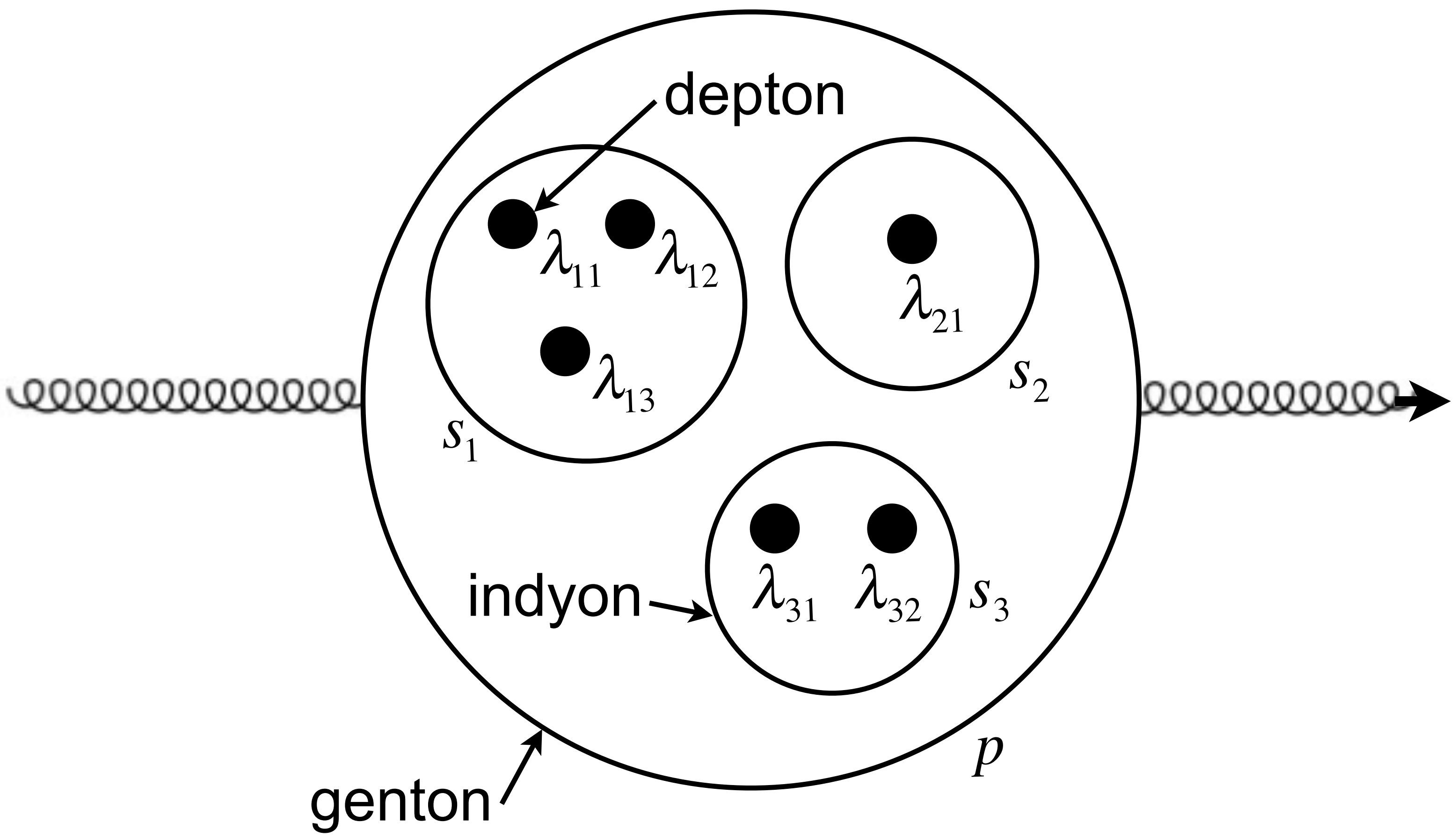
$$\left| N^{(0)}, \{N^{(1)}_{s_1}\}, \{N^{(2)}_{s_1 s_2}\}, \ldots \right\rangle$$



Revisit of scattering and compact graphical representation

$$|g\rangle = a_p^{\dagger} |f\rangle$$





Conclusions

- Mallat has come up with a transformation (group invariant scattering) that is:
  - very useful in identification of image texture
  - hierarchical
  - invariant to group transformations
  - stable to small changes to the image
- Image identification problem can be formulated as a dynamical system with an action
- Mallat's transformation is an iterative wavelet based renormalization of the dynamics
- the renormalized coordinates are a useful basis leading to:
  - identification of fundamental excitations of system
  - factorization of entropy
  - view of changes in the system as a scattering of the fundamental excitations
  - graphical representation of the scattering in Feynman diagrams
  - definition of metric for the states of the system

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Dynamical DNA

- the invariant actions, S_p, fully characterize the dynamics
- they provide a natural basis and metric
- the set of actions are, in analogy to biology, the DNA of the dynamics
  - a coded sequence of numbers, S_p, from which the character of the dynamics can be reconstructed
  - occupation numbers, N_p, fully characterize the state of the system

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