Exterior Calculus

- **Manifold** - An n-dimensional (differentiable) manifold M^n is a space that is locally $\mathbb{R}^n$ in the following sense; it is covered by a family of curvilinear coordinate systems $\{U_\alpha ; \varphi_\alpha\}$ $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ such that a point $p \in U_\alpha \cap U_\beta$ that lies in two coordinate patches will have its two sets of coordinates related differentiably $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$

$$x_\beta^i(p) = f_{\beta\alpha}^i(x_\alpha^1(p), \dots, x_\alpha^n(p)) \quad \text{for } i = 1, \dots, n$$

(if the functions $f_{\beta\alpha}^i$ are C^∞, or real analytic we say that M is C^∞ or real analytic)

- **Vector** - A tangent, or contravariant vector or vector at $p_0 \in M^n$ call it v assigns to each coordinate system $(U_\alpha, \varphi_\alpha)$ holding p_0 an n-tuple (v_α^i); $v_\alpha^1, v_\alpha^2, \dots, v_\alpha^n$ such that if $p_0 \in U_\alpha \cap U_\beta$

$$\boxed{v_\beta^i = \sum_{j=1}^n \frac{\partial x_\beta^i}{\partial x_\alpha^j}(p_0)\, v_\alpha^j}$$

associated with this vector is a differential operator D_v defined by

$$D_v(f) \equiv \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p_0)\, v^i \equiv v_{p_0}(f)$$

It is easily verified that this is independant of the coordinate system.

In a specific coordinate system

$$v_{p_0} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_{p_0}$$

Tangent Space - to M^n at the point $p \in M^n$ is the vector space M^n_p consisting of all tangent vectors to M^n at p. If (U, φ) is a coordinate system for M^n holding p the n vectors

$$e_1 = \frac{\partial}{\partial x^1}\Big|_p, \dots, e_n = \frac{\partial}{\partial x^n}\Big|_p$$

form a basis for M^n_p called a coordinate basis

M^n_p is an n-dimentional vector space.

Differential of a mapping M^n and V^r are manifolds

$F: M^n \to V^r$ is a map described in terms of local coordinates x near $p \in M^n$ and y near $F(p) \in V^r$ by r functions of n variables

$$y^\alpha = F^\alpha(x^1, \dots, x^n)$$

F is a diffeomorphism provided F is 1:1 and onto and both F and F^{-1} are differentiable

The differential of the map $F_*: M^n_p \to V^r_{F(p)}$ takes tangent vectors from M^n_p to tangent vectors in $V^n_{F(p)}$ according to the following rule given above local coordinate systems.

$$\omega^\alpha = \sum_{i=1}^{n} \frac{\partial F^\alpha}{\partial x^i}(x_0)\, v^i \qquad F_*(v) = \omega$$

Definition of Dual space and covariant vector

Let E be a real vector space. If E is n-dimensional, let $e_1, \dots, e_n$ be a basis. Thus if $V \in E$ V has a unique expansion $V = e_i v^i$ (Einstien summation convension)

or $e = (e_1, \dots, e_n)$ $v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$ $\Rightarrow$ $V \equiv ev$

A real linear functional α on E is a linear transformation $\alpha : E \to \mathbb{R}$ (i.e., $\alpha(c_1 V + c_2 W) = c_1 \alpha(V) + c_2 (W)$ $V, W \in E$ $c_1, c_2 \in \mathbb{R}$,

If E is a vector space the collection of all linear functionals α on E defines a new vector space E^* the dual space to E under operations

$$(\alpha + \beta)(V) = \alpha(V) + \beta(V)$$
$$(c\alpha)(V) = c\,\alpha(V)$$

If $e_1, \dots, e_n$ is a basis for E define the dual basis $\sigma^1, \dots, \sigma^n$ of E^* by

$$\sigma^i(e_j) = \delta^i_j$$

{ note that linear functional β can be expanded in basis by $\beta(v) = \beta(e_j v^j) = \beta(e_i) v^i = \beta(e_i) \delta^i_j v^j = \beta(e_i) \sigma^i(e_j) v^j$

$$= [\beta(e_i)\sigma^i](e_j v^j)$$

$$\Rightarrow \beta = \beta(e_i)\sigma^i = b_i \sigma^i \quad \text{where} \quad b_i = \beta(e_i)$$

in above notation

$$b = (b_1, \dots, b_n) \quad \sigma = \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^n \end{pmatrix} \Rightarrow \beta \equiv b\sigma$$

now $\beta(v) = b\sigma e v = b I v = b v$ }

If $f : M^n \to \mathbb{R}$ its differential df is the linear functional on tangent spaces $df : M^n_p \to \mathbb{R}$ given by

$$df(V_p) = V_p(f)$$

In coordinates $e_1 = \frac{\partial}{\partial x^1}\Big|_p, \dots, e_n = \frac{\partial}{\partial x^n}\Big|_p$ form a basis for M^n_p

and $df(V_p) = V_p(f) = \left[v^i(p) \frac{\partial}{\partial x^i}\Big|_p\right](f) = v^i(p) \frac{\partial f}{\partial x^i}\Big|_p$

Let $f = x^i$. $dx^i(e_j) = dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$

i.e. $dx^1, \dots, dx^n$ form a dual basis to $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

in other words $dx^1, \dots, dx^n$ form a basis for $(M^n_p)^*$ the cotangent space

Now for linear functional $\alpha : M^n_p \to \mathbb{R}$ called a <u>covariant vector</u> or <u>covector</u> or <u>1-form</u> can be expanded

$$\alpha = a_i dx^i \quad \text{where} \quad a_i \equiv \alpha\left(\frac{\partial}{\partial x^i}\right)$$

The differential is now naturally expanded

$$df = df\left(\frac{\partial}{\partial x^i}\right) dx^i = \frac{\partial f}{\partial x^i} dx^i$$

Under a change of coordinates $d\bar{x}^i = d\bar{x}^i\left(\frac{\partial}{\partial x^j}\right) dx^j = \frac{\partial \bar{x}^i}{\partial x^j} dx^j$

For a general 1-form $a_i dx^i = a_i \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x}^j = \bar{a}_j d\bar{x}^j$

$\Rightarrow \bar{a}_i = \frac{\partial x^j}{\partial \bar{x}^i} a_j$ or in terms used for vectors

$$\boxed{a_i^{\beta} = \sum_{j=1}^{n} a_j^{\alpha} \frac{\partial x^j_{\alpha}}{\partial x^i_{\beta}}}$$

<u>Pull-back of a mapping</u> as for the differential of a mapping M^n and V^r are manifolds $F : M^n \to V^r$ is a map described in terms of local coordinates x near $p \in M^n$ and y near $F(p) \in V^r$ by r functions of n variables

$$y^\alpha = F^\alpha(x^1, \ldots, x^n)$$

The pull-back of the map $F^* : (V^r_y)^* \to (M^n_{F^{-1}(y)})^*$ takes covectors from $(V^n_y)^*$ to covectors in $(M^n_{F^{-1}(y)})^*$ according to the following rule

$$(F^* \beta)(v) = \beta(F_* v) \quad \text{where} \quad v \in M^n_{F^{-1}(y)} \quad \beta \in (V^r_y)^*$$
$$F_* v \in V^r_y \quad F^* \beta \in (M^n_{F^{-1}(y)})^*$$

in local coordinates

$$\alpha_i = \frac{\partial F^\alpha}{\partial x^i}(F^{-1}(y_0))\, \beta_\alpha \qquad F^*(\beta) = \alpha$$

in terms of basis vectors the transformations are

$$\boxed{F^*(dy^\alpha) = \frac{\partial y^\alpha}{\partial x^i} dx^i \qquad F_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}}$$

for the composition of two mapping $F : M^n \to V^r$ $G : V^r \to U^p$

$G \circ F : M^n \to U^p$

$$\boxed{(G \circ F)^* = F^* \circ G^* \qquad (G \circ F)_* = G_* \circ F_*}$$

<u>Tangent and Cotangent Bundles</u> - the tangent bundle to a differentiable manifold M^n is the space of <u>all</u> tangent vectors to M^n. A point in this new space consists of a point (x, v) where x is a point of M^n and v is a tangent vector to M^n at x (i.e., $v \in M^n_x$)

Denote the <u>tangent bundle</u> to M^n by TM^n.

We can introduce local coordinates in TM^n as follows. Let $(x, v) \in TM^n$. x lies in some local coordinate system $\mathcal{U}$ $(x^1, \ldots, x^n)$ of M^n. At x we have a basis $\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^n}$ for M^n_x. We can express $v = v^i \frac{\partial}{\partial x^i}$. Then (x, v) is completely described by the $2n$-tuple of real numbers $(x^1, \ldots, x^n, v^1(x), \ldots, v^n(x))$

If x also lies in the coordinate patch $\bar{\mathcal{U}}$ $(\bar{x}^1, \ldots, \bar{x}^n)$ the same $(x, v) \in TM^n$ would be described by the new $2n$-tuple $(\bar{x}^1, \ldots, \bar{x}^n, \bar{v}^1(x), \ldots, \bar{v}^n(x))$ where

$$\begin{cases} \bar{x}^i = \bar{x}^i(x^1, \ldots, x^n) \\ \bar{v}^i = \dfrac{\partial \bar{x}^i}{\partial x^j}(x)\, v^j(x) \end{cases}$$

It is obvious that TM^n is a $2n$-dimensional differentiable manifold.

<u>Cotangent bundle</u> to M^n is the space T^*M^n of all covectors on M^n. A point in T^*M^n is a pair (x, α). If x is in the coordinate patch $\mathcal{U}$ $(x^1, \ldots, x^n)$ of M^n we have the basis $dx^1, \ldots, dx^n$ for $(M^n_x)^*$ and α can be expressed as $\alpha = a_i dx^i$. Then $(x, \alpha) \in T^*M^n$ is completely described by the $2n$-tuple $(x^1, \ldots, x^n, a_1(x), \ldots, a_n(x))$ If x also lies in the coordinate patch $\bar{\mathcal{U}}$ $(\bar{x}^1, \ldots, \bar{x}^n)$ the same $(x, \alpha) \in T^*M^n$

would be described by the new $2n$-tuple $(\bar{x}^1, \dots, \bar{x}^n, \bar{a}_1(x), \dots, \bar{a}_n(x))$ where

$$\begin{cases} \bar{x}^i = \bar{x}^i(x^1, \dots, x^n) \\ \bar{a}_i = \dfrac{\partial x^j}{\partial \bar{x}^i}(x)\, a_j(x) \end{cases}$$

Again it is obvious that T^*M^n is a $2n$-dimensional differentiable manifold.

- Riemannian Manifold - First lets develope the idea of inner product which will allow us to associate a covector with each vector. Let E be an n-dimensional real vector space with an inner product $\langle , \rangle$. For each pair of vectors $v, w \in E$ $[\langle , \rangle : E \times E \to \mathbb{R}]$

(i) $\langle v, w \rangle$ is linear in each entry when the other is held fixed

(ii) $\langle v, w \rangle = \langle w, v \rangle$

(iii) $\langle , \rangle$ is non-singular (i.e., if $\langle v, w \rangle = 0$ $\forall w \in E \Rightarrow v = 0 \in E$)

If $e_1, \dots, e_n$ is a basis for E then the matrix

$$g_{ij} = \langle e_i, e_j \rangle$$

is called the (matrix of the) metric tensor. If $v = v^i e_i$ and $w = w^i e_i$ then

$$\boxed{\langle v, w \rangle = g_{ij} v^i w^j}$$

Now define the linear function $f_v : E \to \mathbb{R}$ defined by $f_v(w) = \langle v, w \rangle$. f_v is obviously an element of E^*. Putting $\nu = f_v = \nu_i \sigma^i$ we have

$$\nu_j = \nu(e_j) = \langle v, e_j \rangle = \langle v^i e_i, e_j \rangle = v^i \langle e_i, e_j \rangle = v^i g_{ij}$$

Thus if E has an inner product, to each vector $v \in E$ with components v^i we can associate a covector $\nu \in E^*$ with components $\nu_j = v^i g_{ij}$ having property

$$\nu(w) = \langle v, w \rangle$$

Conversly to each $\nu \in E^*$ with components ν_i we can associate a vector $v \in E$ with components $v^i = \nu_j g^{ij}$ with $((g^{ij})) =$ Inverse matrix to $((g_{ij}))$

A Riemannian metric on a manifold M^n assigns in a differentiable fashion a positive definite inner product $\langle , \rangle$ to each tangent space M^n_p. A manifold with a Riemannian metric is called a Riemannian manifold.

In terms of a coordinate basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ we have

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

In an overlap with new coordinates $\bar{x}$

$$\bar{g}_{ij} = \left\langle \frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j} \right\rangle = \frac{\partial x^\ell}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} g_{\ell m}$$

Gradiant vector - If M^n is a Riemannian metric and f is a differentiable function $f: M^n \to \mathbb{R}$ the gradiant vector ∇f is the contravariant vector associated to the covector df

$$df(w) = \langle \nabla f, w \rangle$$

In coordinates

$$(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x^j}$$

- Tensors We have defined covectors from linear functionals on vectors. We may also think of vectors as linear functionals on. Although tensors will be defined over a general finite dimensional vector space E, we will usually be concerned with the case $E = M^n_p$. Instead of using $e_1, \dots, e_n$ for a basis for E and $\sigma^1, \dots, \sigma^n$ for a basis for E^* we will use $e_i = \frac{\partial}{\partial x^i}$ and $\sigma^i = dx^i$ with the case of primary concern in mind

A covariant tensor of rank r is a multilinear real valued function $Q: \underbrace{E \times \ldots \times E}_{r \text{ times}} \to \mathbb{R}$. Thus Q is a function of r-tuples of vectors $Q(v_{(1)}, \ldots, v_{(r)})$ that is linear in each $v_{(i)}$ when remaining entries are kept fixed. In components we have from multilinearity

$$Q(v_{(1)}, \ldots, v_{(r)}) = v_{(1)}^{i_1} v_{(2)}^{i_2} \ldots v_{(r)}^{i_r} Q_{i_1 i_2 \ldots i_r} \quad \text{where}$$

$$Q_{i_1 i_2 \ldots i_r} \equiv Q\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \ldots, \frac{\partial}{\partial x^{i_r}}\right)$$

Q can now be written in components

$$Q = Q_{i_1 i_2 \ldots i_r} \, dx^{i_1} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_r}$$

where tensor product $\otimes$ is defined as

$$\alpha_{(1)} \otimes \alpha_{(2)} \otimes \ldots \otimes \alpha_{(r)} (v_{(1)}, v_{(2)}, \ldots, v_{(r)}) = \alpha_{(1)}(v_{(1)}) \, \alpha_{(2)}(v_{(2)}) \ldots \alpha_{(r)}(v_{(r)})$$

The space of all covariant tensors of rank r forms a vector space which we will write as

$$Q \in \underbrace{E^* \otimes E^* \otimes \ldots \otimes E^*}_{r\text{-times}} = \otimes^r E^*$$

It has dimension n^r.

A contravariant tensor of rank s is a s-linear function on covectors $T: \underbrace{E^* \times E^* \times \ldots \times E^*}_{s\text{-times}} \to \mathbb{R}$. In components

$$T(\alpha_{(1)}, \ldots, \alpha_{(s)}) = \alpha_{(1) i_1} \alpha_{(2) i_2} \ldots \alpha_{(r) i_s} T^{i_1 i_2 \ldots i_s} \quad \text{where}$$

$$T^{i_1 i_2 \ldots i_s} \equiv T(dx^{i_1}, dx^{i_2}, \ldots, dx^{i_s})$$

T can now be written in components

$$T = T^{i_1 i_2 \ldots i_s} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_s}}$$

the tensor product of vectors is defined

$$v_{(1)} \otimes v_{(2)} \otimes \dots \otimes v_{(s)} (\alpha_{(1)}, \dots, \alpha_{(s)}) = v_{(1)}(\alpha_{(1)}) \dots v_{(s)}(\alpha_{(s)})$$

The space of all contravariant tensors of rank s forms a vector space which we will write as

$$T \in \underbrace{E \otimes E \otimes \dots \otimes E}_{s\text{-times}} = \otimes^s E$$

It has dimension n^s.

A <u>mixed tensor</u> r times covariant, s times contravariant is a multilinear function $W: \underbrace{E^* \times \dots E^*}_{s} \times \underbrace{E \times \dots \times E}_{r} \to \mathbb{R}$

In components

$$W(\alpha_{(1)}, \dots, \alpha_{(s)}, v_{(1)}, \dots, v_{(r)}) = \alpha_{(1)j_1} \dots \alpha_{(s)j_s} W^{j_1 \dots j_s}{}_{i_1 \dots i_r} v^{i_1}_{(1)} \dots v^{i_r}_{(r)}$$

where $\quad W^{j_1 \dots j_s}{}_{i_1 \dots i_r} \equiv W\left(dx^{j_1}, \dots, dx^{j_s}; \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}\right)$

W can now be written in components

$$W = W^{j_1 \dots j_s}{}_{i_1 \dots i_r} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_r}$$

Obviously $\quad W \in \otimes^r E^* \otimes^s E \quad$ with dimension n^{r+s}

<u>Transformation Properties</u> Under a change of basis

$\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial}{\partial x^k}$ and $d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^\ell} dx^\ell$ we get by multilinearity

$$\bar{W}^{j_1 \dots j_s}{}_{i_1 \dots i_r} = \frac{\partial \bar{x}^{j_1}}{\partial x^{\ell_1}} \dots \frac{\partial \bar{x}^{j_s}}{\partial x^{\ell_s}} W^{\ell_1 \dots \ell_s}{}_{k_1 \dots k_r} \frac{\partial x^{k_1}}{\partial \bar{x}^{i_1}} \dots \frac{\partial x^{k_r}}{\partial \bar{x}^{i_r}}$$

General Tensor Product If $\alpha^p \in \otimes^p E^*$ and $\beta^q \in \otimes^q E^*$ define their tensor product by

$$\alpha^p \otimes \beta^q (V_1, \dots, V_{p+q}) \equiv \alpha^p(V_1, \dots, V_p)\, \beta^q(V_{p+1}, \dots, V_{p+q})$$

Grassmann or Exterior Algebra

Let $\Lambda^p E^* \equiv \underbrace{E^* \wedge E^* \wedge \dots \wedge E^*}_{p\text{-times}} \subset \otimes^p E^*$

be the vector space of skew symmetric (alternating or antisymmetric) covariant p-tensors. Tensors such that

$$\alpha^p(V_1, \dots, V_r, \dots, V_s, \dots, V_p) = -\alpha^p(V_1, \dots, V_s, \dots, V_r, \dots, V_p)$$

The elements of $\Lambda^p E^*$ are called exterior p-forms or simply p-forms.

Note that if $\alpha^p \in \Lambda^p E^*$ and $\beta^q \in \Lambda^q E^*$ then $\alpha^p \otimes \beta^q \in \otimes^{p+q} E^*$ but $\alpha^p \otimes \beta^q$ will not necessarily be an element of $\Lambda^{p+q} E^*$. We need to define a product that is closed with respect to all forms.

It is easy to do this for $p = q = 1$

$$\alpha^1 \wedge \beta^1 \equiv \alpha^1 \otimes \beta^1 - \beta^1 \otimes \alpha^1$$

For general α and β we define the exterior or Grassmann product

$$\wedge : \Lambda^p E^* \times \Lambda^q E^* \longrightarrow \Lambda^{p+q} E^*$$

Let $\alpha^p \in \Lambda^p E^*$ and $\alpha^q \in \Lambda^q E^*$ Then define $\alpha^p \wedge \alpha^q \in \Lambda^{p+q} E^*$ by

$$\alpha^p \wedge \beta^q (V_{i_1}, \dots, V_{i_{p+q}}) = \sum_{\substack{k_1 < \dots < k_p \\ l_1 < \dots < l_q}} \delta^{k_1 \dots k_p l_1 \dots l_q}_{i_1 \dots \dots i_{p+q}}\, \alpha(V_{k_1}, \dots, V_{k_p})\, \beta(V_{l_1}, \dots V_{l_q})$$

in short hand notation

$$\alpha^p \wedge \beta^q (V_I) = \sum_{\underset{\rightarrow}{K}\,\underset{\rightarrow}{L}} \delta_I^{KL}\, \alpha(V_K)\, \beta(V_L)$$

where

$$I = (i_1, \dots, i_{p+q})$$

$$K = (k_1, \dots, k_p) \qquad \underset{\rightarrow}{K} \rightarrow k_1 < \dots < k_p$$

$$L = (l_1, \dots, l_q) \qquad \underset{\rightarrow}{L} \rightarrow l_1 < \dots < l_q$$

and

$$\delta^{j_1 \dots j_r}_{i_1 \dots i_r} = \begin{cases} 1 & \text{if } (j_1, \dots, j_r) \text{ is even permutation of } (i_1, \dots, i_r) \\ -1 & \text{if } (j_1, \dots, j_r) \text{ is odd permutation of } (i_1, \dots, i_r) \\ 0 & \text{if } j_1, \dots, j_r) \text{ is not a permutation of } (i_1, \dots, i_r) \end{cases}$$

One can now define <u>Grassman</u> or <u>Exterior algebra</u> on the space of all forms over E^*

$$\Lambda^* E \equiv (\Lambda^0 E^* = \mathbb{R}) \oplus (\Lambda^1 E^* = E^*) \oplus (\Lambda^2 E^*) \oplus \dots \oplus (\Lambda^n E^*)$$

Obviously the $\dim \Lambda^p E^* = \binom{n}{p}$ where $\dim(E) = n$.

It follows that

$$\dim(\Lambda^* E) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

It can be easily verified

$$\alpha^p \wedge \beta^q = (-)^{pq} \beta^q \wedge \alpha^p \qquad \text{"anti-commutative"}$$

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \qquad \text{associative}$$

$$\alpha \wedge \beta \in \Lambda^* E \qquad \text{closed}$$

any p-form can be expanded in terms of the a basis defined as follows: Let $\sigma^1, \dots, \sigma^n$ be the dual basis to $e_1, \dots, e_n$. $\alpha^p \in \Lambda^p E^*$ can now be expanded as

$$\alpha^p = \sum_{\vec{I}} \alpha_I \sigma^I = \sum_{i_1<\dots<i_p} \alpha_{i_1\dots i_p} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p}$$

where $\alpha_I \equiv \alpha(e_I)$

$\sigma^I(e_J) = \delta^I_J$

Exterior Differentiation

Thm: There is a unique operator (<u>exterior differentiation</u>)

$$d : \Lambda^p M^* \to \Lambda^{p+1} M^*$$

satifying

(i) d is additive $d(\alpha^p + \beta^p) = d\alpha^p + d\beta^p$

(ii) $d\alpha^0$ = differential of the function α^0

(iii) $d(\alpha^p \wedge \beta^q) = d\alpha^p \wedge \beta^q + (-)^p \alpha^p \wedge d\beta^q$

(iv) $d^2 \alpha^p = d(d\alpha^p) \equiv 0 \quad (\forall \alpha)$

If f is a 0-form (function)

$$df = \frac{\partial f}{\partial x^i} dx^i \quad (\text{from } df(v) = v(f))$$

If $a_I(x)$ are functions and $\alpha = \sum_{\vec{I}} a_I \sigma^I$

$$d\alpha = \sum_{\vec{I}} da_I \wedge \sigma^I$$

It can easily shown that the pull back of a form under a map $F: M^n \to V^r$ is again a form and this pull back commutes with both exterior differentiation and the wedge product

(i) If $\alpha^p \in \Lambda^p V^r$ $\quad F^*\alpha \in \Lambda^p M^n$

(ii) $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$

(iii) $F^*(d\alpha) = d(F^*\alpha)$

- Interior Product - If X is a vector and α^p is a p-form we define the interior product or contraction as:

$$i_X : \Lambda^p M^* \longrightarrow \Lambda^{p-1} M^*$$

$$\begin{cases} i_X \alpha^0 = 0 \\ i_X \alpha^1 = \alpha^1(X) \\ i_X \alpha^p(X_2, X_3, \ldots, X_p) = \alpha^p(X, X_2, \ldots, X_p) \end{cases}$$

In a basis, if $\alpha = a_I \sigma^I$,

$$i_X \alpha = b_J \sigma^J = X^j a_{jJ} \sigma^{jJ}$$

Also i_X can be shown to be an anti-derivation that is

$$i_X(\alpha^p \wedge \beta^q) = (i_X \alpha^p) \wedge \beta^q + (-)^p \alpha^p \wedge (i_X \beta^q)$$

Lie Derivative

We must first define the flow ϕ_t generated by a vector field X on a manifold M.

A vector field X on $U \subset M^n$ assigns in a differentiable manner a vector V_p to each $p \in U$. In coordinates any vector field on a patch (U, x) has a local expression

$$X = v^i(x) \frac{d}{dx^i}$$

A flow is a 1-parameter family of maps

$$\phi_t : M \to M$$

such that (i) $\phi_s(\phi_t(p)) = \phi_{s+t}(p) = \phi_t(\phi_s(p))$ $\left(\begin{matrix}\forall s,t \in \mathbb{R} \\ \forall p \in M\end{matrix}\right)$

(ii) $\phi_{-t}(\phi_t(p)) = p$

(iii) ϕ_t is a diffeomorphism

associated with ϕ_t is a time independent velocity vector field

$$V_p = \left[\frac{d}{dt}(\phi_t p)\right]_{t=0}$$

$$\text{or } V_p(f) = \left[\frac{d}{dt} f(\phi_t p)\right]_{t=0} \quad \text{for } f: M \to \mathbb{R}$$

Can be shown that for each differential vector field $\mathbb{X}$ one can associate a flow ϕ_t such that $\mathbb{X}$ is the velocity field of ϕ_t.

Given two vector fields $\mathbb{X}$ and $\mathbb{Y}$ on M the <u>Lie bracket</u> or <u>Lie derivative</u> of $\mathbb{Y}$ with respect to $\mathbb{X}$ is the mapping

$$\mathcal{L}_{\mathbb{X}} : X_0^*M \to X_0^*M \quad \text{where } X_0^*M = \prod_{x \in M} M_x$$

such that vector field is differentiable

$$(\mathcal{L}_{\mathbb{X}} \mathbb{Y})_x = \lim_{t \to 0} \frac{\mathbb{Y}_{\phi_t x} - (\phi_t)_* \mathbb{Y}_x}{t} \quad \text{where } \phi_t \text{ is flow generated by } \mathbb{X}$$

$x \in M$ $\mathbb{X}, \mathbb{Y} \in X_0^*M$

$$= \lim_{t \to 0} \frac{(\phi_{-t})_* \mathbb{Y}_{\phi_t x} - \mathbb{Y}_x}{t} = \frac{d}{dt}\left[(\phi_{-t})_* Y_{\phi_t x}\right]_{t=0}$$

Can be shown that

$$\mathcal{L}_{\mathbb{X}} \mathbb{Y}(f) = \mathbb{X}(\mathbb{Y}(f)) - \mathbb{Y}(\mathbb{X}(f)) = [\mathbb{X}, \mathbb{Y}](f)$$

In local coordinates when $X = X^i \frac{\partial}{\partial x^i}$ $Y = Y^i \frac{\partial}{\partial x^i}$

$$(\mathcal{L}_X Y)^i = [X, Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$$

<u>Lie derivative of forms</u> If X is a vector field (with local flow ϕ_t) and f is a function $f: M \to \mathbb{R}$ we will define the Lie derivative of f with respect to X at p by $\mathcal{L}_X : x_0^* M^* \to x_0^* M^*$

$$\mathcal{L}_X (f) \equiv X_p(f) = \frac{d}{dt} f(\phi_t p)\Big]_{t=0} = \frac{d}{dt} (\phi_t^* f)\Big]_{t=0}$$

If α^p is p-form we define

$$\mathcal{L}_X \alpha^p \equiv \frac{d}{dt} (\phi_t^* \alpha^p)\Big|_{t=0} = \lim_{t \to 0} \frac{\phi_t^* \alpha - \alpha}{t}$$

In particular if $v_1, \dots, v_p$ are invariant under the flow ϕ_t generated by X (i.e., $\mathcal{L}_X v_i = 0$) then

$$\mathcal{L}_X (\alpha^p)(v_1, \dots, v_p) = \frac{d}{dt}\left[\alpha^p_{\phi_t x}(v_1, \dots, v_p)\right]_{t=0}$$

Properties of $\mathcal{L}_X$ that are easily verified

(i) $\mathcal{L}_X(\alpha^p \wedge \beta^q) = (\mathcal{L}_X \alpha^p) \wedge \beta^q + \alpha^p \wedge (\mathcal{L}_X \beta^q)$ "derivation"

(ii) $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$

(iii) $\mathcal{L}_X = i_X \circ d + d \circ i_X$ "Cartan's Formula"

(iv) $X(\alpha^p(u_1, \dots, u_p)) = (\mathcal{L}_X \alpha^p)(u_1, \dots, u_p) + \sum_r \alpha^p(u_1, \dots, [X, u_r], \dots, u_p)$

$X, u_1, \dots, u_p$ vector field "Leibnitz rule"

(v) $d\alpha^p(u_0, \dots, u_p) = \sum_{r=0}^{p} (-)^r i_{u_r} \alpha(u_0, \dots, \hat{u}_r, \dots, u_p)$

$$+ \sum_{r<s} (-)^{r+s} \alpha^p([u_r, u_s], u_0, \dots, \hat{u}_r, \dots, \hat{u}_s, \dots, u_p)$$

(note: vectors on left side vector fields on right. $\wedge$ means omitted)

Orientation and Pseudo Forms

Let $e=(e_1,\dots,e_n)$ and $f=(f_1,\dots,f_n)$ be two bases of a vector space E. We can then uniquely write $f=eA$ (i.e., $f_i = e_j A^j{}_i$) for some non-singular matrix A. If $\det A > 0$ (resp. <0) we say that e and f have the same (resp. opposite) orientation.

We shall say that the entire manifold M^n is orientable provided we can cover M^n by coordinate patches $\{U_\alpha\}$ such that in each overlap $U_\alpha \cap U_\beta$ the Jacobian determinate

$$\frac{\partial(x^1_\beta,\dots,x^n_\beta)}{\partial(x^1_\alpha,\dots,x^n_\alpha)} > 0$$

Obviously this says the two orientations of M^n_x coming from U_α and U_β agree since

$$e_i = \frac{\partial}{\partial x^i_\alpha} = \sum_j \frac{\partial x^j_\beta}{\partial x^i_\alpha}\frac{\partial}{\partial x^j_\beta} = \sum A^j{}_i f_j$$

The forms previously discussed were independant of the notion of orientability; $\alpha^p(v_1,\dots,v_p)$ is a number which has nothing to do with the orientation of the particular tangent space in which $v_1,\dots,v_n$ live.

There is a different notion of exterior form which is dependant on the orientation

A <u>pseudo form</u> of degree p on a manifold M^n assigns to each orientation, o, of the tangent space M^n_x a exterior p-form in such a way that

$$\alpha^p_{-o} = -\alpha^p_o$$

Note: an example of a commonly encountered pseudo-form is the volume form

$$vol^3_o = o(x)\sqrt{\det g_{ij}(x)}\, dx^1 \wedge dx^2 \wedge dx^3 \quad (\text{Riemannian Manifold}) \; n=3$$

where

$$o(x) = \begin{cases} +1 & \text{if } \frac{\partial}{\partial x^1}\, \frac{\partial}{\partial x^2}\, \frac{\partial}{\partial x^3} \text{ is positively oriented wrt } o \\ -1 & \text{if } \frac{\partial}{\partial x^1}\, \frac{\partial}{\partial x^2}\, \frac{\partial}{\partial x^3} \text{ is negatively oriented wrt } o \end{cases}$$

The reason for all this nit picking about orientation and pseudo forms will become apparent with the development of integration of forms over manifolds.

Integration of Forms over Oriented Manifolds

<u>Partition of Unity</u> - Given a finite coordinate covering $\{U_\alpha, \varphi_\alpha\}$ $(\alpha = 1, \ldots, N)$ a partition of unity will exibit N real valued functions $f_\alpha : V^p \to \mathbb{R}$ that are differentiable and have the following properties

(i) $f_\alpha(x) \geq 0 \quad (\alpha = 1, \ldots, N)$

(ii) The support of f_α is a closed subset $\subset U_\alpha$ in particular $f_\alpha(p) = 0$ for $p \notin U_\alpha$

(iii) $\sum_\alpha f_\alpha(p) = 1 \quad \forall p \in V^p$

the <u>support</u> of a function is the closure of a set of points at which the function is not zero

It can be shown that a partition of unity can always be constructed for every coordinate cover.

<u>Integration of forms</u> - Let M^n be an manifold. Let V^p be an oriented submanifold (perhaps with boundary) of M^n. We can think of V^p as a manifold in its own right. We then have the inclusion map

$$i : V^p \to M^n$$

where $i(x)$ says think of the point $x \in V^p$ as a point in M^n. If $u^1, \dots, u^p$ are local coordinates for V^p and $x^1, \dots, x^n$ those for M^n then i is simply given by

$$x^i = x^i(u^1, \dots, u^p)$$

Let ω^p be a p-form on M^n. We may restrict the form to V^p by taking the pull back $i^*\omega^p$. Now let $\{U_\alpha, \varphi_\alpha\}$ be a finite covering of V^p by coordinate charts that are positively oriented for the orientation of V^p given. We then define

$$\int_{V^p \subset M^n} \omega^p \equiv \int_{V^p} i^*\omega^p = \int_{V^p} (i^*\omega^p) \sum_\alpha f_\alpha = \sum_\alpha \int_{V_p} f_\alpha i^* \omega^p$$

$$= \sum_\alpha \int_{U_\alpha} f_\alpha i^*\omega^p \equiv \sum_\alpha \int_{U_\alpha} \omega^p_\alpha \qquad \text{where } \omega^p_\alpha = f_\alpha i^*\omega^p$$

$$= \sum_\alpha \int_{\varphi_\alpha(U_\alpha)_0} \varphi_\alpha^{*} \omega^p_\alpha = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)_0 \subset \mathbb{R}^p_0} f_\alpha(\varphi_\alpha^{-1}(u))(i \circ \varphi_\alpha^{-1})^* \omega^p$$

$$= \sum \int_{\varphi_\alpha(U_\alpha)_0} f_\alpha(u)\,(i \circ \varphi_\alpha^{-1})^* \omega^p\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^p}\right) du^1 \wedge \dots \wedge du^p$$

$$\equiv \sum_\alpha o(u) \int_{|\varphi_\alpha(U_\alpha)|} f_\alpha(u)\, \omega^p\left((i \circ \varphi_\alpha^{-1})_* \frac{\partial}{\partial u^1}, \dots, (i \circ \varphi_\alpha^{-1})_* \frac{\partial}{\partial u^p}\right) du^1 \dots du^p$$

(note: this last integral is the normal multiple Riemann integral) where $i \circ \varphi_\alpha^{-1} : \mathbb{R}^p \to M^n$

$f_\alpha(p) : U_\alpha \subset V^p \to \mathbb{R}$

$f_\alpha(u) = f_\alpha(\varphi_\alpha^{-1}(u)) : \mathbb{R}^p \to \mathbb{R}$

$|\varphi_\alpha(U_\alpha)|$ will be $\varphi_\alpha(U_\alpha)$ without orientation

orientation for $\varphi_\alpha(U_\alpha)$ will be induced by the orientation for U_α which is induced by the orientation for V^p

Notice that this integration does not depend upon the orientation of M^n, only on the orientation of V^p. If the orientation of V^p is reversed so is the value of the integral.

Integration of pseudo-forms

For pseudo-forms either an orientation must be given for M^n (i.e., M^n must be orientable) or some orientation must be given for each coordinate patch. Any orientation can be given to each coordinate patch because of the fact that <u>the value of the integral of a pseudo-form is independent of the orientation of M^n or V^p</u>

$$\int_{V_o^p} \omega_o^p = -\int_{V_{-o}^p} -\omega_{-o}^p = \int_{V_{-o}^p} \omega_{-o}^p$$

This adds some technical details with regard to defining an orientation for each map but leads to the advantage that pseudo forms can be integrated over V^p that are not orientable where true forms must be integrated over orientable V^p since the orientation matters for them.

☆ Stoke's Thm

A manifold with boundary is the same as a manifold except that charts (U, φ) of the following form are allowed

$$\varphi : U \to \{x \in \mathbb{R}^n : x^n \geq 0\}$$

the boundary of M, ∂M, is the set of points

$$\partial M = \{x \in M : \varphi_\alpha(x) = 0 \text{ for some } \alpha\}$$

Now for one of the main results of exterior calculus.

> Thm (Stoke's): Let $V^p \subset M^n$ be a compact submanifold-with-boundary. Let ω^{p-1} be a continuously differentiable true (V^p must be oriatable) or pseudo $p-1$ form on M^n. Then
>
> $$\int_{V^p} d\omega^{p-1} = \int_{\partial V^p} \omega^{p-1}$$

Differentiation of Integrals

Time-independent: Let α^p be a p-form (time-independent), C_p a p dimensional submanifold (perhaps with boundary ∂C_p) of a manifold M^n. We shall concider a "variation" of C_p arising as follows. We suppose that there is a flow $\phi_t : M^n \to M^n$. $C_p(t) = \varphi_t C_p$. We are interested in the variation of the integral $I(t) \equiv \int_{C_p(t)} \alpha^p = \int_{\phi_t C_p} \alpha^p = \int_{C_p} \phi_t^* \alpha^p$

Differentiating,

$$I'(t) = \lim_{h\to 0} \frac{1}{h}\left[I(t+h) - I(t) \right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_{C_p} \phi^*_{t+h}\alpha^p - \int_{C_p} \phi_t^*\alpha^p \right] = \lim_{h\to 0} \frac{1}{h} \int_{C_p} \phi_t^*(\phi_h^*\alpha^p - \alpha^p)$$

$$= \lim_{h\to 0} \frac{1}{h} \int_{\phi_t C_p} \phi_h^*\alpha^p - \alpha^p = \int_{\phi_t C_p} \lim_{h\to 0} \frac{1}{h}(\phi_h^*\alpha^p - \alpha^p)$$

Cartan's formula

$$\Rightarrow \quad \boxed{\frac{d}{dt}\int_{C_p(t)} \alpha^p = \int_{C_p(t)} \mathcal{L}_{\vec{v}}\,\alpha^p = \int_{C_p(t)} i_{\vec{v}}\, d\alpha^p + \int_{\partial C_p(t)} i_{\vec{v}}\,\alpha^p}$$

<u>Time-dependent</u> Let α^p be a time dependant p-form on M^n $\alpha^p = a_I(x,t)\,\sigma^I$. Suppose we have a time dependant "flow" of fluid in M^n with velocity flow $\vec{v}(t,x) = v^i(t,x)\frac{\partial}{\partial x^i}$. We wish to calculate

$$\frac{d}{dt}\int_{C_p(t)} \alpha^p$$

We pass to $\mathbb{R}\times M^n$, $\alpha^p = \sum_{i_1<\dots<i_p} a_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$ (no dt terms)

and $X = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i}$ generating the flow ψ_t.

Let $d = dt\wedge\frac{\partial}{\partial t} + \vec{d}$ with $\vec{d} = dx^i\frac{\partial}{\partial x^i}$. Using the formula for the time independant case yields the formula

$$\boxed{\begin{aligned}\frac{d}{dt}\int_{C_p(t)} \alpha^p &= \int_{C_p(t)} \frac{\partial\alpha^p}{\partial t} + i_{\vec{v}}\vec{d}\alpha^p + \vec{d}\, i_{\vec{v}}\alpha \\ &= \int_{C_p(t)} \frac{\partial\alpha}{\partial t} + i_{\vec{v}}\, d\alpha + \int_{\partial C_p(t)} i_{\vec{v}}\,\alpha \quad \text{(space only)}\end{aligned}}$$

Vector Algebra and Calculus

First lets associate a true vector $\vec{A}$ with a true 1-form, α^1, and a pseudo 2-form, α_0^2, the following way:

$$\alpha^1 = \langle\ ,\vec{A}\rangle$$
$$\alpha_0^2 = i_{\vec{A}}\, vol_0^3 = o(u)\sqrt{g(u)}\,(a^1 du^1\wedge du^3 + a^2 du^3\wedge du^1 + a^3 du^1\wedge du^2)$$

for a Riemannian manifold and vector $\vec{A} = a^i \frac{\partial}{\partial x^i}$

Conversly, one can associate a pseudo-vector $\vec{B}_0$ with a pseudo 1-form and a true 2 form, the following way:

$$\beta_0^1 = \langle\ ,\vec{B}_0\rangle$$
$$\beta_0^2 = i_{\vec{B}_0}\, vol_0^2$$

Given a pair of true vectors $\vec{A}$ and $\vec{B}$ one can express there dot product as a 0-form or a 3-form in $\mathbb{R}^3$

$$i_{\vec{B}}\alpha^1 = \vec{A}\cdot\vec{B} \qquad \text{true 0-form}$$
$$\alpha^1\wedge\beta^2 = (\vec{A}\cdot\vec{B})\, vol_0^3 \qquad \text{pseudo 3-form}$$

The pseudo vector $\vec{A}\times\vec{B}$ can be associated with a 2-form and 1-form in $\mathbb{R}^3$

$$\alpha^1\wedge\beta^1 = i_{\vec{A}\times\vec{B}}\, vol_0^3 \qquad \text{true 2-form}$$
$$-i_{\vec{A}}\beta^2 = \langle\ ,\vec{A}\times\vec{B}\rangle \qquad \text{pseudo 1-form}$$

The vector triple product can be associated with a 0-form or 3-form in $\mathbb{R}^3$

$$\alpha^1\wedge\beta^1\wedge\gamma^1 = (\vec{A}\times\vec{B})\cdot\vec{C}\, vol_0^3 \qquad \text{true 3-form}$$
$$-i_{\vec{A}}\beta^2(\vec{C}) = (A\times\vec{B})\cdot\vec{C} \qquad \text{pseudo 0-form}$$

Vector identities can now be easily proved via forms, the only decision being whether to use 1-forms or 2-forms.

$$\langle\ ,\vec{A}\times(\vec{B}\times\vec{C})\rangle = -i_{\vec{A}}(\beta^1\wedge\gamma^1) = (-i_{\vec{A}}\beta^1)\gamma^1 + \beta^1 i_{\vec{A}}\gamma^1$$

$$= (-\vec{A}\cdot\vec{B})\langle\ ,\vec{C}\rangle + \langle\ ,\vec{B}\rangle(\vec{A}\cdot\vec{C})$$

$$= \langle\ ,\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\rangle$$

$$\Rightarrow\quad \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Now for calculus. By definition and problem #

$$df = \langle\ ,\nabla f\rangle \qquad \text{true 1-form (f true-scalar)}$$

$$d\alpha^1 = i_{\nabla\times\vec{A}}\, vol_0^3 \qquad \text{true 2-form}$$

$$d\beta^2 = (\nabla\cdot\vec{B})\, vol_0^3 \qquad \text{pseudo 3-form}$$

To compute div $\vec{B}$

$$(\vec{\nabla}\cdot\vec{B})\, vol_0 = d(i_{\vec{B}}\, vol_0) = \left\{\frac{1}{\sqrt{g}}\sum_i \frac{\partial}{\partial u^i}(\sqrt{g}\, b^i)\right\} vol_0$$

$$\Rightarrow\quad \vec{\nabla}\cdot\vec{B} = \frac{1}{\sqrt{g}}\sum_i \frac{\partial}{\partial u^i}(\sqrt{g}\, b^i)$$

To compute a vector identity

$$[\nabla\cdot(\vec{A}\times\vec{B})]\, vol_0 = d(\alpha^1\wedge\beta^1) = d\alpha^1\wedge\beta^1 - \alpha^1\wedge d\beta^1$$

$$= \underbrace{(i_{\nabla\times\vec{A}}\, vol_0)}_{\substack{\text{true 2-form}\\ \text{for } (\nabla\times\vec{A})_0}}\wedge\beta^1 - \alpha^1\wedge \underbrace{i_{\nabla\times\vec{B}}\, vol_0}_{\substack{\text{true 2-form}\\ \text{for } (\nabla\times\vec{B})_0}}$$

$$= [\vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}(\nabla\times\vec{B})]\, vol_0$$

$$\Rightarrow\quad \nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot B - (\nabla\times\vec{B})\cdot\vec{A}$$

One last thing to note, by Cartan's formula

$$\mathcal{L}_X vol^3 = i_X \underbrace{d\,vol^3}_{=0} + d\, i_X vol^3 = d(i_X vol^3)$$
$$= (\nabla\cdot X)\, vol_0^3$$

To compute $\nabla\times(\vec A\times\vec B)$ do the following

$$i_{\nabla\times(\vec A\times\vec B)}\, vol_0^3 = d(i_{\vec B}\alpha^2) \overset{\text{Cartan's formula}}{=} \mathcal{L}_{\vec B}\alpha^2 - i_B\, d\alpha^2$$
$$= \mathcal{L}_{\vec B}\alpha^2 - i_{\vec B}\, d(i_{\vec A} vol^3)$$
$$= \mathcal{L}_{\vec B}\alpha^2 - i_{\vec B}(\vec\nabla\cdot\vec A\, vol^3)$$
$$= \mathcal{L}_{\vec B}\alpha^2 - (\vec\nabla\cdot\vec A)\, i_{\vec B} vol^3$$
$$= \mathcal{L}_{\vec B} i_{\vec A} vol^3 - (\nabla\cdot\vec A)\, i_{\vec B} vol^3$$

prob 17

$$= i_{\vec A}\mathcal{L}_{\vec B} vol^3 + i_{[\vec B,\vec A]} vol^3 - (\nabla\cdot\vec A)\, i_{\vec B} vol^3$$
$$= (\vec\nabla\cdot\vec B)\, i_{\vec A} vol^3 + i_{[\vec B,\vec A]} vol^3 - (\nabla\cdot\vec A)\, i_{\vec B} vol^3$$
$$= i_{(\nabla\cdot\vec B)\vec A + [\vec B,\vec A] - (\nabla\cdot\vec A)\vec B}\; vol^3$$

$$\Rightarrow \quad \nabla\times(\vec A\times\vec B) = (\nabla\cdot\vec B)\vec A - (\nabla\cdot\vec A)\vec B + [\vec B,\vec A]$$

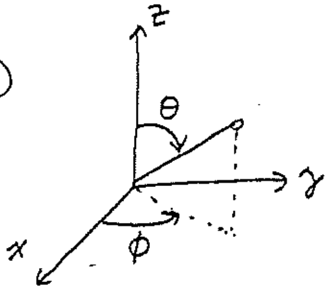
to relate to conventional formulas

$$[\vec B,\vec A]^i = B^j\frac{\partial A^i}{\partial x^j} - A^j\frac{\partial B^i}{\partial x^j} = \vec B\cdot\vec\nabla A^i - \vec A\cdot\vec\nabla B^i$$

$$[\vec B,\vec A] \equiv \vec B\cdot\vec\nabla\vec A - \vec A\cdot\vec\nabla\vec B$$

$$\Rightarrow \quad \nabla\times(\vec A\times\vec B) = (\nabla\cdot\vec B)\vec A - (\nabla\cdot\vec A)\vec B + \vec B\cdot\nabla\vec A - \vec A\cdot\nabla\vec B$$

Problems on Exterior Calculus



① In $\mathbb{R}^3$ let $x^1 = x \quad x^2 = y \quad x^3 = z$

let $\bar{x}^1 = r \quad \bar{x}^2 = \theta \quad \bar{x}^3 = \phi$

(a) compute $\bar{g}_{ij}$ in coordinates $\bar{x}$

$$\bar{g}_{r\theta} \equiv \bar{g}_{12} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle$$

(use: chain rule $\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \dots$)

(b) Compute coefficients $\overline{(\nabla f)}^i$ in

$$\nabla f = \overline{(\nabla f)}^r \frac{\partial}{\partial r} + \overline{(\nabla f)}^\theta \frac{\partial}{\partial \theta} + \overline{(\nabla f)}^\phi \frac{\partial}{\partial \phi}$$

(c) Verify that $\frac{\partial}{\partial r} \; \frac{\partial}{\partial \theta} \; \frac{\partial}{\partial \phi}$ are orthogonal but not unit vectors

(d) Put $\hat{e}_r = \frac{\partial}{\partial r} / \| \frac{\partial}{\partial r} \| \quad \hat{e}_\theta = \frac{\partial}{\partial \theta} / \| \partial/\partial\theta \| \quad \hat{e}_\varphi = \frac{\partial}{\partial \varphi} / \| \partial/\partial\varphi \|$

and write $\nabla f = (\nabla f)^r \hat{e}_r + (\nabla f)^\theta \hat{e}_\theta + (\nabla f)^\varphi \hat{e}_\varphi$

(? Why do we need ∇f when df is so easy ?)

② Compute in terms of a basis:

(a) $\alpha^1 \wedge \beta^1$

(b) $\alpha^1 \wedge \gamma^2$

(c) $\alpha^1 \wedge \beta^1 \wedge \eta^1$

③ Compute $(\alpha \wedge \beta)_I$ (the components of $\alpha \wedge \beta$ in terms of a basis)

④ Compute in terms of $dx \; dy \; dz$ in $\mathbb{R}^3$

(a) $d\alpha^0$

(b) $d\alpha^1$

(c) $d\alpha^2$

(d) $d(d\alpha^1)$

5. Show that

(a) $$F^*(dx \wedge dy) = \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv$$

for $F: \mathbb{R}^2 \to \mathbb{R}^2$
$(x,y) \to (u,v)$

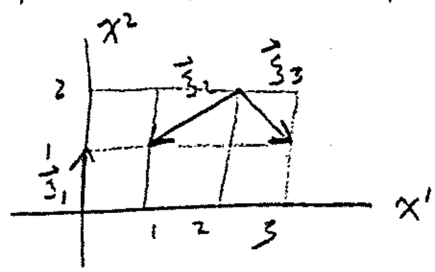
(b) $$F^*(dx \wedge dy \wedge dz) = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du \wedge dv \wedge dw$$

$F: \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \to (u,v,w)$

6. Calculate the value of the forms

$$\omega_1 = dx^1$$
$$\omega_2 = x^1 dx^2$$
$$\omega_3 = dr^2 \quad (r^2 = (x^1)^2 + (x^2)^2)$$

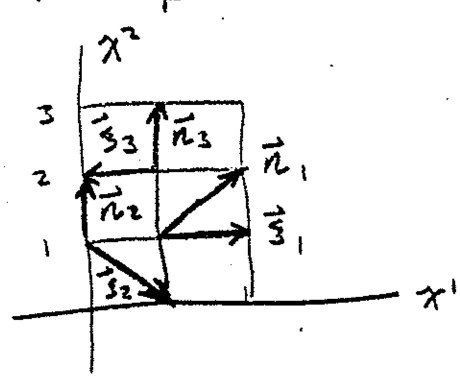
on the vectors $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$



7. Calculate the values of the forms

$$\omega_1 = dx^1 \wedge dx^2$$
$$\omega_2 = x^1 dx^1 \wedge dx^2 - x^2 dx^2 \wedge dx^1$$
$$\omega_3 = r\, dr \wedge d\varphi \quad (x^1 = r\cos\varphi \quad x^2 = r\sin\varphi)$$

on the pairs of vectors $(\vec{\xi}_1, \vec{\eta}_1)$ $(\vec{\xi}_2, \vec{\eta}_2)$ $(\vec{\xi}_3, \vec{\eta}_3)$



(8) Calculate the value of the forms

$$\omega_1 = dx^2 \wedge dx^3$$
$$\omega_2 = x^1\, dx^3 \wedge dx^2$$
$$\omega_3 = dx^3 \wedge dr^2 \qquad r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

on $\vec{\xi} = (1,1,1)$, $\vec{\eta} = (1,2,3)$ at point $\vec{x} = (2,0,0)$

(9) Use problem #1 to write down 'vol^3 for $\mathbb{R}^3$ in spherical coodinates r, θ, φ

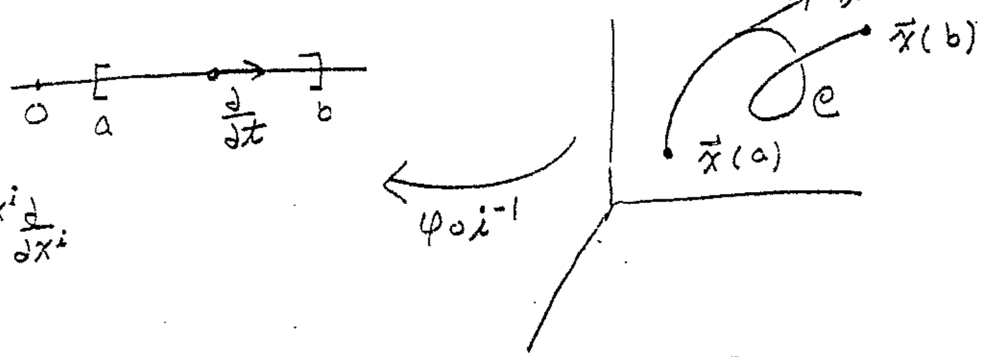
simple atlas $\{(\mathcal{C} = U, \varphi)\}$

(10) For the case $M^n = \mathbb{R}^3$

(a) line intergral $\quad \varphi(\mathcal{C}) = [a,b] \subset \mathbb{R} \qquad i : \mathcal{C} \to \mathbb{R}^3$, $\quad \varphi : \mathcal{C} \to \mathbb{R}$

$i \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}^3$

$t \to \vec{x}(t) \qquad \frac{d\vec{x}}{dt} = (i \circ \varphi^{-1})_*\left(\frac{d}{dt}\right)$



$$\frac{d\vec{x}}{dt} = \frac{dx^i}{dt}\frac{d}{dx^i}$$

With a 1-form $\alpha^1 = \alpha_i dx^i$ on $\mathbb{R}^3$, one can associate a contravariant vector $\vec{a} = a^i \frac{d}{dx^i}$ since $\mathbb{R}^3$ is Riemannian. $\alpha_i = g_{ij} a^j$. Therefore given a vector field $\vec{a}$ on $\mathbb{R}^3$ one can construct α^1.

Show that:

$$\int_{\mathcal{C}} \alpha^1 = \int_a^b \alpha^1\left(\frac{d\vec{x}}{dt}\right) dt = \int_a^b \left\langle \vec{a}, \frac{d\vec{x}}{dt} \right\rangle dt \equiv \int_{\mathcal{C}} \vec{a} \cdot d\vec{x}$$

(b) <u>surface</u> <u>integral</u> atlas $\{(S=\mathcal{U},\varphi)\}$

$\varphi(S)\subset \mathbb{R}^2$

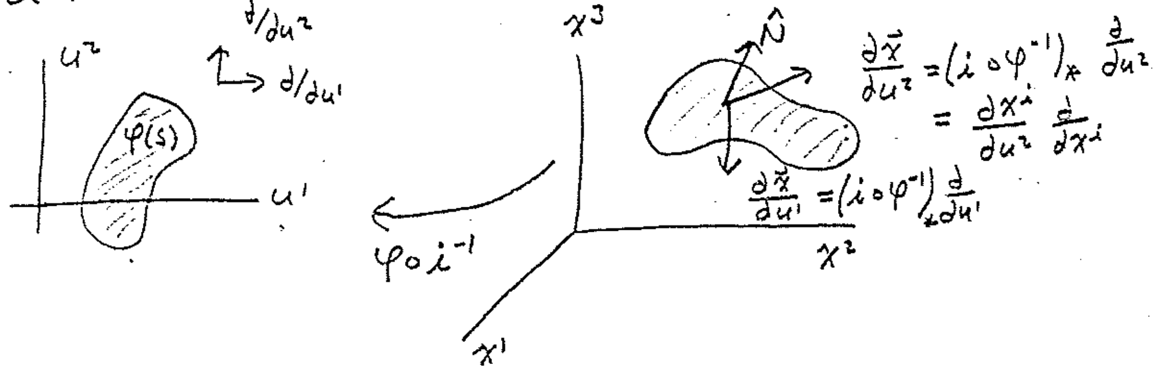
$i: S\to\mathbb{R}^3$

$\varphi: S\to\mathbb{R}^2$

$i\circ\varphi^{-1}: \mathbb{R}^2\to\mathbb{R}^3$

$(u^1,u^2)\to \vec{x}(u^1,u^2)$

suppose that coordinates u^1,u^2 are chosen so that $o(u)=+1$



Let a 2-form $\beta^2 = b_1\, dx^2\wedge dx^3 + b_2\, dx^3\wedge dx^1 + b_3\, dx^1\wedge dx^2$ be given on $\mathbb{R}^3$. One can associate a unique pseudo-vector $\vec{B}_0$ with β^2 by $\beta^2 = i_{\vec{B}_0}\, vol_0^3$ for Riemannian $\mathbb{R}^3$ (where $vol_0^3 = o(x)\sqrt{g(x)}\; dx^1\wedge dx^2\wedge dx^3 = o(x)\, dx^1\wedge dx^2\wedge dx^3$) pseudo-vector field on $\mathbb{R}^3$, $\vec{B}_0$,

Therefore given a pseudo-vector field on $\mathbb{R}^3$, $\vec{B}_0$, one can construct β^2.

Show that:

$$\int_S \beta^2 = \iint_{\varphi(S)}\left[b_1\frac{\partial(x^2,x^3)}{\partial(u^1,u^2)} + b_2\frac{\partial(x^3,x^1)}{\partial(u^1,u^2)} + b_3\frac{\partial(x^1,x^2)}{\partial(u^1,u^2)}\right]du_1\,du_2$$

$$=\int_{\varphi(S)} \langle \vec{B}_0,\hat{N}\rangle\, vol_0^3\left(\hat{N},\frac{\partial\vec{x}}{\partial u^1},\frac{\partial\vec{x}}{\partial u^2}\right)du^1\,du^2$$

for $o\left(\hat{N},\frac{\partial\vec{x}}{\partial u^1},\frac{\partial\vec{x}}{\partial u^2}\right)=+1$

$$\equiv \int \langle\vec{B},\hat{N}\rangle\, dS$$

(11) As a follow up to 10(b), we have seen that the "area form" for a parameterized surface in $\mathbb{R}^3$ is the pull-back $(i\circ\varphi^{-1})^*$ of the two form

$$i_{\hat{N}}\, vol^3 = dA$$

For $i \circ \varphi^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$

$$(u^1, u^2) \to (u^1, u^2, f(u^1, u^2))$$

one recovers the classical expression for the area

$$dS = (i \circ \varphi^{-1})_* dA\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right) du^1 du^2$$

$$= \sqrt{\left(\frac{\partial f}{\partial u^1}\right)^2 + \left(\frac{\partial f}{\partial u^2}\right)^2 + 1}\; du^1 du^2 \qquad o\left(\hat{N}, \frac{\partial \vec{x}}{\partial u^1}, \frac{\partial \vec{x}}{\partial u^2}\right) = +1$$

(note: $\hat{N} = \vec{n} / \|\vec{n}\|$ where $\vec{n} = \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2}$)

(12) Apply Stoke's thm to:

(a) $\int_{\mathcal{C} \subset \mathbb{R}^3} d\alpha^0$ to derive Fundamental thm of Calculus

(b) $\int_{S \subset \mathbb{R}^3} d\alpha^1$ to derive classical Stoke's Thm

(c) $\int_{V \subset \mathbb{R}^3} d\alpha^3$ to derive divergence thm

(13) Show $\frac{d}{dt}\int_{U^n} vol^n = \int_{U^n} (\nabla \cdot \vec{v})\, vol^n = \int_{\partial U^n} \langle \vec{v}, \hat{N} \rangle\, vol^{n-1}_{\partial U}$

where U^n is a compact region with boundary on a Riemannian M^n, $vol^{n-1}_{\partial U} = i_{\hat{N}}\, vol^3$, and $\vec{v}$ is the velocity of the flow of U^n

(Note: This is true even if M^n is not orientable since vol^3 is a pseudo form and $\vec{v}$ is a true vector)

(14) Let $\vec{A}$ and $\vec{B}$ be time dependant vector fields on $\mathbb{R}^3$ and $\rho(t, \vec{x})$ be continuous. Show that

(a) $\frac{d}{dt}\int_{\mathcal{C}} \vec{A} \cdot d\vec{r} = \int_{\mathcal{C}} \left[\frac{\partial \vec{A}}{\partial t} - \vec{v} \times \operatorname{curl} \vec{A} + \operatorname{grad}(\vec{v} \cdot \vec{A})\right] \cdot d\vec{r}$

(b) $\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} = \int_S \left[\frac{\partial \vec{B}}{\partial t} + (\nabla\cdot\vec{B})\vec{v} - \nabla\times(\vec{v}\times\vec{B})\right]\cdot d\vec{S}$

(c) $\frac{d}{dt}\int_V \rho\, dV = \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right] dV$.

(15) Calculate for Riemannian manifold M^n with curvilinear coordinates (metric g_{ij}):

(a) $\Delta f = \nabla^2 f = \vec{\nabla}\cdot(\vec{\nabla} f)$

(b) $\nabla\times\vec{A}$

using exterior methods

(16) Derive the following using vector identities

(a) $\vec{\nabla}(fg) = f\,\vec{\nabla} g + g\,\vec{\nabla} f$

(b) $\nabla\cdot(f\vec{A}) = f\,\nabla\cdot\vec{A} + \vec{A}\cdot\vec{\nabla} f$

(c) $\nabla\times(f\vec{A}) = f\,\nabla\times\vec{A} + (\vec{\nabla} f)\times\vec{A}$

(d) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = ???$

(17) Show that

$$\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}$$