Differential Forms Applied to Plasma Physics

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Derivation of BBGKY Hierarchy

Consider Hamiltonian flow in the phase space of $N$ identical particles moving on the same base manifold $M$

$$H^{(N)} : T^*M^N \to \mathbb{R}$$

$$i_U \omega^N = -dH^{(N)}$$

where

$$M^n = \prod_{i=1}^n M \quad \omega^n = \sum_{i=1}^n dp_i \wedge dq_i \quad \text{vol}^n = \prod_{i=1}^n dq_i \wedge dp_i$$

$$H^{(n)}(x^n) = \sum_{i=1}^n H_1(x_i) + \sum_{i<j}^n H_2(x_i, x_j)$$

(binary interactions)

Define the "N-particle distribution" form

$$\rho^N = f_N(x^n) \text{vol}^N$$

$$\frac{\partial \rho^N}{\partial t} + L_U \rho^N = 0 \quad \text{(conservation of particles)}$$

$f_N(x^n)$ symmetric under $x_i \leftrightarrow x_j$ interchange
normalized so that

\[ N = \frac{1}{V^N} \int_{T^*_M^n} \rho^N \]

Also define the "n-particle distribution" form

\[ \rho^n = \frac{1}{V} \int_{T^*_M} \rho^{n+1} = f_n(x^n) \text{vol}^n \]

We now work towards an expression for the evolution of \( \rho^n \) in terms of \( \rho^{n+1} \)

\[ 0 = \frac{1}{V^{N-n}} \int_{T^*_M^{N-n}} \left[ \frac{\partial \rho^N}{\partial t} + \mathcal{L}_U \rho^N \right] \]

\[ \Rightarrow \quad \frac{\partial \rho^n}{\partial t} = - \frac{1}{V^{N-n}} \int_{T^*_M^{N-n}} \mathcal{L}_U \rho^N \]

To modify the RHS we define the vector fields

\[ i_{U_i} \omega^N = -dH_1(x_i) \quad i_{U_j} \omega^N = -dH_2(x_i, x_j) \]

\[ i_{U_{(n)}} \omega^N = -dH^{(n)}(x, n+1) \]

where

\[ H^{(n)}_{\text{int}} = \sum_{i=1}^n H_2(x_i, x_{n+1}) \]
this leads to

\[
- \frac{\partial \rho^n}{\partial t} = \frac{1}{V^{N-n}} \left[ \int_{T^*_{M^{N-n}}} L_{U(n)} \rho^N + (N-n) \int_{T^*_{M^{N-n}}} L_{U_{n+1}} \rho^N \right. \\
+ (N-n) \int_{T^*_{M^{N-n}}} L_{U_{n+1 \text{int}}} \rho^N \\
+ \frac{(N-n)(N-n-1)}{2} \int_{T^*_{M^{N-n}}} L_{U_{n+1,n+2}} \rho^N \left. \right]
\]

\[
= L_{U(n)} \rho^n + \frac{(N-n)}{V} \int_{T^*_{M}} L_{U_{n+1}} \rho^{n+1} \\
+ \frac{(N-n)}{V} \int_{T^*_{M \text{int}}} L_{U_{n+1}} \rho^{n+1} \\
+ \frac{(N-n)(N-n-1)}{2V^2} \int_{T^*_{M^2}} L_{U_{n+1,n+2}} \rho^{n+2}
\]

using \( L_V \rho^n = d(i_V \rho^n) \) and assuming that

\[
i_{U_{n+1}} \rho^{n+1} \big|_{\partial (T^*_{M})} = i_{U_{n+1,n+2}} \rho^{n+2} \big|_{\partial (T^*_{M^2})} = 0
\]

(no particles at boundary or no flow across boundary)
We obtain the **BBGKY Hierarchy** (Bogoliubov, Born, Green, Kirkwood and Yvon)

\[
\frac{\partial \rho^n}{\partial t} + \mathcal{I}_{U(n)} \rho^n = -n_0 \int_{T^*M} \mathcal{I}_{U_{\text{int}}} \rho^{n+1}
\]

- n-particle Hamiltonian flow
- effect of (n+1) particle on n-particle flow

**in coordinates**

Using the definitions \( \tilde{H}_i = H_1(x_i) \) and \( \tilde{H}_{ij} = H_2(x_i, x_j) \),

\[
\frac{\partial f_1}{\partial t} + \{ f_1, \tilde{H}_1 \} = -n_0 \int dp_2 dq_2 \{ f_2, \tilde{H}_{12} \}
\]

\[
\frac{\partial f_2}{\partial t} + \{ f_2, \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_{12} \} = -n_0 \int dp_3 dq_3 \{ f_3, \tilde{H}_{13} + \tilde{H}_{23} \}
\]

\[\vdots\]

\[\vdots\]

\[\vdots\]
3-Body Recombination

Consider set of all possible configurations of:

(1) fixed ion at origin
(2) N electrons undergoing classical "guiding center dynamics"

\[
A^+ + e^- + e^- \xrightarrow{R_3} A + e^-
\]

\(R_3(n_0, T)\) = 3-body recombination rate
= steady state flux of electrons towards the ion
Guiding Center Atom

\[ \Omega_c \gg \omega_z \gg \omega_D \]

\[ f_1 = f_1( J_z, \psi_z, J_D, \psi_D; t ) \]
Flux Formula

If one is given $\rho^{n+1}$ and wishes to find flow out of a region $R$ of $T^*M^n$, and this region is being convected by the n-particle flow

$$\text{Flux} = \frac{dN}{dt} = \int_{R} \frac{d\rho^n}{dt} = -n_0 \int_{\partial(R \times T^*M)} i_{U_{int}}^{(n)} \rho^{n+1}$$

**in coordinates (n=1)**

We take

$$x_1 = (J_z, \psi_z, J_D, \psi_D)$$

$$x_2 = (z, \rho_z, \theta, p_\theta)$$

Define $\partial R$ by $E = H_1(J_z, J_D)$

$$\frac{dN}{dt} = n_0 \int f_2 \left( \frac{\partial \hat{H}_{12}}{\partial \psi_z} + \frac{\omega_D}{\omega_z} \frac{\partial \hat{H}_{12}}{\partial \psi_D} \right) d\psi_z dJ_z d\psi_D d\rho_z dz d\rho_\theta d\theta$$

where $\omega_z = \frac{\partial H_1}{\partial J_z}$, $\omega_D = \frac{\partial H_1}{\partial J_D}$, $f_2 \sim \exp\left(-\frac{H^{(2)}}{k_B T}\right)$
Kinetic Bottleneck

The one-way thermal equilibrium flux is

\[ \frac{dN}{dt} \sim \left( n_0 \bar{v} b^2 \right) \left( n_0 b^3 \right) \frac{\exp(-E / k_B T)}{(-E / k_B T)^4} \]

where \( \bar{v} = \sqrt{k_B T / m} \) and \( b = e^2 / k_B T \).

The 3-body recombination rate can be estimated as

\[ R_3 \sim \min \left( \frac{dN}{dt} \right) \sim n_0^2 \bar{v} b^5 \frac{\exp(4)}{4^4} \]