Differential Forms Applied to Plasma Physics

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Derivation of BBGKY Hierarchy

Consider Hamiltonian flow in the phase space of N identical particles moving on the same base manifold M

$$H^{(N)}:\quad T^*M^N \to \mathbb{R}$$

$$\mathrm{i}_{\mathrm{U}}\omega^N = -\mathrm{d}H^{(N)}$$

where

$$M^n \equiv \prod_{i=1}^{n} \times M \qquad \omega^n \equiv \sum_{i=1}^{n} dp_i \wedge dq_i \qquad \mathrm{vol}^n \equiv \prod_{i=1}^{n} \wedge dq_i \wedge dp_i$$

$$H^{(n)}\left(x^n\right) = \sum_{i=1}^{n} H_1(x_i) \quad + \quad \sum_{i<j}^{n} H_2\left(x_i, x_j\right)$$

(binary interactions)

Define the <u>"N-particle distribution" form</u>

$$\rho^N \equiv f_N\left(x^N\right) \mathrm{vol}^N$$

$$\frac{\partial \rho^N}{\partial t} + L_{\mathrm{U}}\, \rho^N = 0 \qquad \textbf{(conservation of particles)}$$

$f_N\left(x^N\right)$ **symmetric under** $x_i \leftrightarrow x_j$ **interchange** normalized so that

$$N = \frac{1}{V^N} \int_{T^*M^N} \rho^N$$

Also define the "n-particle distribution" form

$$\rho^n \quad \equiv \quad \frac{1}{V} \int_{T^*M} \rho^{n+1} \quad = \quad f_n\left(x^n\right) \mathrm{vol}^n$$

We now work towards an expression for the evolution of ρ^n in terms of ρ^{n+1}

$$0 = \frac{1}{V^{N-n}} \int_{T^*M^{N-n}} \left[\frac{\partial \rho^N}{\partial t} + L_{\mathrm{U}}\, \rho^N \right]$$

$$\Rightarrow \qquad \frac{\partial \rho^n}{\partial t} = -\frac{1}{V^{N-n}} \int_{T^*M^{N-n}} L_{\mathrm{U}}\, \rho^N$$

To modify the RHS we define the vector fields

$$\mathrm{i}_{\mathrm{U}_i} \omega^N = -\mathrm{d}H_1(x_i) \qquad \mathrm{i}_{\mathrm{U}_{ij}} \omega^N = -\mathrm{d}H_2\left(x_i, x_j\right)$$

$$\mathrm{i}_{\mathrm{U}^{(n)}} \omega^N = -\mathrm{d}H^{(n)} \qquad \mathrm{i}_{\mathrm{U}^{(n)}_{\mathrm{int}}} \omega^N = -\mathrm{d}H^{(n)}_{\mathrm{int}}$$

where

$$H^{(n)}_{\mathrm{int}} \equiv \sum_{i=1}^{n} H_2\left(x_i, x_{n+1}\right)$$

this leads to

$$-\frac{\partial \rho^n}{\partial t} = \frac{1}{V^{N-n}} \left[\begin{array}{l} \displaystyle\int_{T^*M^{N-n}} L_{U^{(n)}}\,\rho^N \quad + \quad (N-n) \int_{T^*M^{N-n}} L_{U_{n+1}}\,\rho^N \\ \displaystyle + \quad (N-n) \int_{T^*M^{N-n}} L_{U^{(n)}_{\text{int}}}\,\rho^N \\ \displaystyle + \quad \frac{(N-n)(N-n-1)}{2} \int_{T^*M^{N-n}} L_{U_{n+1,n+2}}\,\rho^N \end{array} \right]$$

$$= \quad L_{U^{(n)}}\,\rho^n \quad + \quad \frac{(N-n)}{V} \int_{T^*M} L_{U_{n+1}}\,\rho^{n+1}$$

$$+ \quad \frac{(N-n)}{V} \int_{T^*M} L_{U^{(n)}_{\text{int}}}\,\rho^{n+1}$$

$$+ \quad \frac{(N-n)(N-n-1)}{2V^2} \int_{T^*M^2} L_{U_{n+1,n+2}}\,\rho^{n+2}$$

using $L_V\,\rho^n = \mathrm{d}\left(\mathrm{i}_V\,\rho^n\right)$ **and assuming that**

$$\mathrm{i}_{U_{n+1}}\,\rho^{n+1}\Big|_{\partial(T^*M)} \quad = \quad \mathrm{i}_{U_{n+1,n+2}}\,\rho^{n+2}\Big|_{\partial(T^*M^2)} \quad = 0$$

(no particles at boundary <u>or</u> no flow across boundary)

We obtain the BBGKY Hierarchy (Bogoliubov, Born, Green, Kirkwood and Yvon)

$$\frac{\partial \rho^n}{\partial t} + \mathcal{L}_{U^{(n)}}\, \rho^n \quad = \quad - n_0 \int_{T^*M} \mathcal{L}_{U^{(n)}_{\mathrm{int}}}\, \rho^{n+1}$$

n-particle Hamiltonian flow — **effect of (n+1) particle on n-particle flow**

in coordinates

Using the definitions $\tilde{H}_i \equiv H_1(x_i)$ **and** $\tilde{H}_{ij} \equiv H_2(x_i, x_j)$,

$$\frac{\partial f_1}{\partial t} + \left\{ f_1, \tilde{H}_1 \right\} \quad = \quad - n_0 \int dp_2\, dq_2\, \left\{ f_2, \tilde{H}_{12} \right\}$$

$$\frac{\partial f_2}{\partial t} + \left\{ f_2, \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_{12} \right\} \quad = \quad - n_0 \int dp_3\, dq_3\, \left\{ f_3, \tilde{H}_{13} + \tilde{H}_{23} \right\}$$

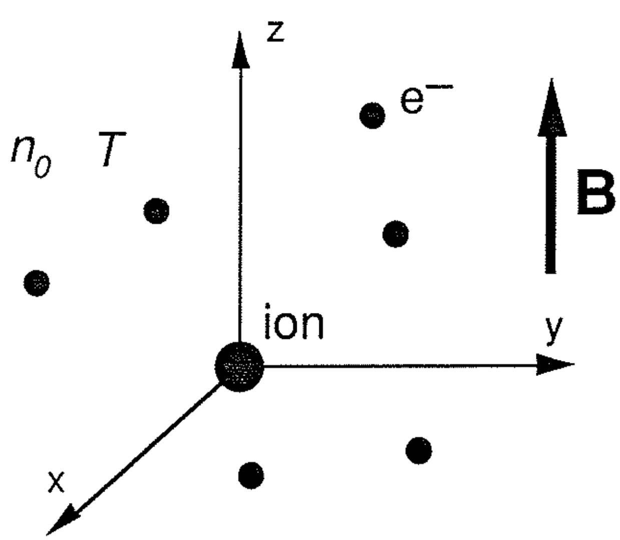
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

3-Body Recombination

Glinsky and O'Neil, Phys. Fluids B **3**, 1279 (1991)

Consider set of all possible configurations of:

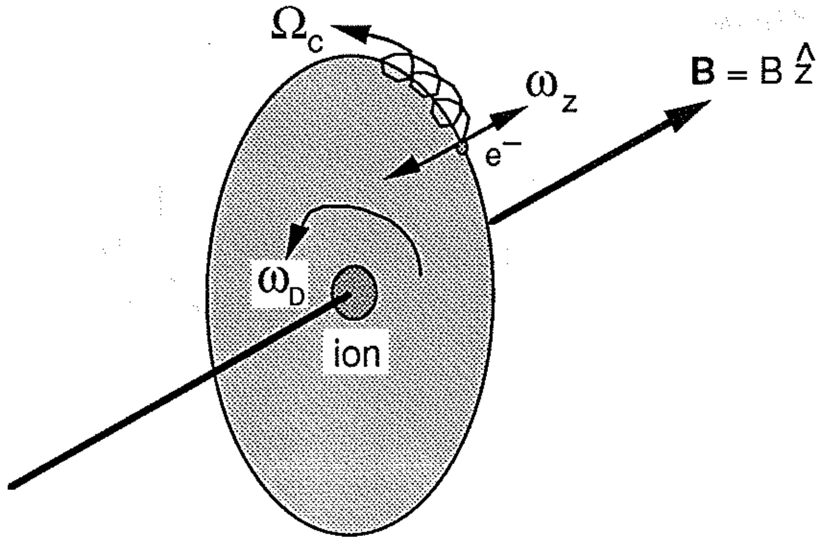
(1) fixed ion at origin
(2) N electrons undergoing classical "guiding center dynamics"



$$A^+ + e^- + e^- \xrightarrow{R_3} A + e^-$$

$R_3(n_0, T)$ = 3-body recombination rate
= steady state flux of electrons towards the ion

Guiding Center Atom



<u>**Ordering of Frequencies**</u>

$$\Omega_c >> \omega_z >> \omega_D$$

$$f_1 = f_1(\, J_z\, ,\ \psi_z\, ,\, J_D\, ,\ \psi_D\, ;\, t\,)$$

Flux Formula

If one is given ρ^{n+1} and wishes to find flow out of a region R of T^*M^n, and this region is being convected by the n-particle flow

$$\text{Flux} = \frac{dN}{dt} = \int_R \frac{d\rho^n}{dt} = -n_0 \int_{\partial(R \times T^*M)} \mathrm{i}_{U_{\mathrm{int}}^{(n)}} \rho^{n+1}$$

in coordinates (n=1)

We take $$x_1 = (J_z, \psi_z, J_D, \psi_D)$$ $$x_2 = (z, p_z, \theta, p_\theta)$$

Define ∂R **by** $E = H_1(J_z, J_D)$

$$\frac{dN}{dt} = n_0 \int f_2 \left(\frac{\partial \tilde{H}_{12}}{\partial \psi_z} + \frac{\omega_D}{\omega_z} \frac{\partial \tilde{H}_{12}}{\partial \psi_D} \right) d\psi_z \, dJ_D \, d\psi_D \, dp_z \, dz \, dp_\theta \, d\theta$$

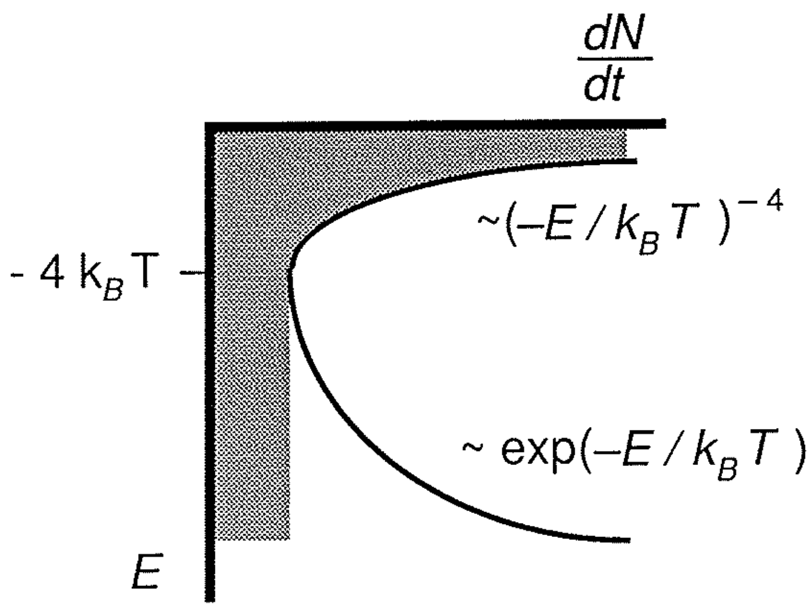
where $\omega_z \equiv \dfrac{\partial H_1}{\partial J_z}$, $\omega_D \equiv \dfrac{\partial H_1}{\partial J_D}$, $f_2 \sim \exp\left(-\mathrm{H}^{(2)} / k_B T\right)$

Kinetic Bottleneck

The one-way thermal equilibrium flux is

$$\frac{dN}{dt} \sim \left(n_0 \, \bar{v} \, b^2 \right) \left(n_0 \, b^3 \right) \frac{\exp(-E / k_B T)}{(-E / k_B T)^4}$$

where $\bar{v} \equiv \sqrt{k_B T / m}$ **and** $b \equiv e^2 / k_B T$ **.**



The 3-body recombination rate can be estimated as

$$R_3 \sim \min\left(\frac{dN}{dt}\right) \sim n_0^2 \, \bar{v} \, b^5 \, \frac{\exp(4)}{4^4}$$