Collisional Equipartition Rate for a Magnetized Pure Electron Plasma

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Abstract*

The collisional equipartition rate between the parallel and perpendicular velocity components is calculated for a weakly correlated electron plasma that is immersed in a uniform magnetic field. Here, parallel and perpendicular refer to the direction of the magnetic field. The rate depends on the ratio $r_c/b$, where $r_c = \sqrt{T/m} / \Omega_c$ is the cyclotron radius and $b = e^2/T$ is the classical distance of closest approach. For a strongly magnetized plasma (i.e., $r_c/b << 1$), the equipartition rate is exponentially small ($\sim \exp[-2.34(r_c/b)^{-2/5}]$). For a weakly magnetized plasma (i.e., $r_c/b >> 1$), the rate is the same as for an unmagnetized plasma except that $r_c/b$ replaces $\lambda_D/b$ in the Coulomb logarithm. (It is assumed here that $r_c < \lambda_D$; otherwise the plasma is effectively unmagnetized.) This paper presents a numerical treatment that spans the intermediate regime $r_c/b \sim 1$, and connects on to asymptotic formulas in the two limits $r_c/b << 1$ and $r_c/b >> 1$. Also an improved asymptotic formula for the rate in the high field limit is given. Our results are in good agreement with experiments over eight decades in $r_c/b$.\(^1\)

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Definition of the Equipartition Rate

Consider a pure electron plasma that is immersed in a uniform magnetic field $\vec{B} = B \hat{z}$. Let the velocity distribution be of the form

$$f(\vec{v}) = \left( \frac{m}{2\pi T_{\parallel}} \right)^{1/2} \left( \frac{m}{2\pi T_{\perp}} \right) \exp \left[ -\frac{m v_{\parallel}^2}{2 T_{\parallel}} - \frac{m v_{\perp}^2}{2 T_{\perp}} \right]$$

where

$$| T_{\parallel} - T_{\perp} | \ll T_{\parallel}, T_{\perp}.$$

The collisional equipartition rate $\nu$ is defined through the equation

$$\frac{d T_{\perp}}{dt} = \nu \left( T_{\parallel} - T_{\perp} \right)$$
Weakly Correlated Plasma

\[ b \ll \lambda_D \]

where

\[ b \equiv \frac{e^2}{T} = \text{distance of closest approach} \]

\[ \lambda_D \equiv \sqrt{\frac{T}{4\pi n e^2}} = \text{Debye length} \]

Scale Length for Magnetic Field

\[ r_c \equiv \bar{v} / \Omega_c \]

where

\[ \bar{v} \equiv \sqrt{\frac{T}{m}} = \text{thermal velocity} \]

\[ \Omega_c \equiv \frac{e B}{m c} = \text{cyclotron frequency} \]
Regimes of Magnetization

1. Effectively unmagnetized plasma* 
\[ (r_c > \lambda_D) \]
\[ v = n \bar{v} b^2 \frac{8 \sqrt{\pi}}{15} \ln \left( \frac{\lambda_D}{b} \right) \]

2. Weakly magnetized plasma** 
\[ (b < r_c < \lambda_D) \]
\[ v = n \bar{v} b^2 \frac{8 \sqrt{\pi}}{15} \ln \left( \frac{r_c}{b} \right) \]

3. Strongly magnetized plasma*** 
\[ (r_c < b) \]
\[ v = n \bar{v} b^2 \left(8.81 \right)^{7/15} \frac{r_c}{b}^{7/15} \exp \left[ -2.34 \left( \frac{b}{r_c} \right)^{2/5} \right] \]


***T. M. O'Neil and P. G. Hjorth, Phys. Fluids 28, 3241 (1985); with corrections to the non-exponential part presented in this paper.
Adiabatic Invariant

\[ \tau \Omega_c >> 1 \]

\[ J_2 = \frac{m v_{1\perp}^2}{2 \Omega_c} + \frac{m v_{2\perp}^2}{2 \Omega_c} \]

is an adiabatic invariant and

\[ \Delta J_2 \sim \exp[-\Omega_c \tau] \]

Since

\[ \tau \sim \frac{\left| \vec{r}_1 - \vec{r}_2 \right|_{\text{min}}}{V_{\text{rel}}} \]

there is an exponentially small exchange of energy for

\[ \left| \vec{r}_1 - \vec{r}_2 \right|_{\text{min}} >> \frac{\bar{v}}{\Omega_c} = r_c \]
Boltzmann-like Operator

In a weakly or strongly magnetized plasma \( r_c < \lambda_D \) dynamical shielding supersedes Debye shielding. Therefore we use a Boltzmann-like operator which omits Debye shielding.

\[
\frac{\partial f(\vec{v}_1, t)}{\partial t} = \int_0^\infty 2\pi r_\perp dr_\perp \int \left[ f(\vec{v}_2')f(\vec{v}_1') - f(\vec{v}_2)f(\vec{v}_1) \right]
\]

\[
\int d^3\vec{v}_2 \mid v_2 - v_1 \mid
\]

![Diagram](image_url)
\[
\frac{\partial T_\perp}{\partial t} = \int d^3 \vec{v}_1 \frac{m v_{1\perp}^2}{2} \frac{\partial f(\vec{v}_1, t)}{\partial t}
\]

\text{substitute from Boltzmann equation}

1. Change to center of mass and relative velocities

\[
\vec{V} = \frac{\vec{V}_1 + \vec{V}_2}{2} \quad \vec{v} = \vec{V}_1 - \vec{V}_2
\]

2. Use detailed balance

3. Use fact that center of mass dynamics and relative dynamics decouple (for uniform B)

\[
v = \frac{n}{4T^2} \int_0^\infty 2\pi r_\perp dr_\perp \int d^3\vec{v} \|v_\parallel\left[ \Delta \left( \frac{\mu v_\perp^2}{2} \right) \right]^2 f_r(\vec{v})
\]

\[
f_r(\vec{v}) = \left( \frac{\mu}{2\pi T} \right)^{3/2} \exp \left( -\frac{\mu v^2}{2T} \right)
\]

\[
T_\perp \approx T_\parallel = T \quad \mu \equiv m/2
\]
Binary Collision

\[ \Delta E_{\perp} = - \Delta E_{\parallel} = \Delta \left( \frac{\mu v_{\perp}^2}{2} \right) \]

where the decoupled equations of motion are

\[
\frac{d \vec{V}}{dt} + \Omega_c \vec{V} \times \hat{z} = 0
\]

\[
\frac{d \vec{v}}{dt} + \Omega_c \vec{v} \times \hat{z} = \frac{e^2}{\mu} \frac{\vec{r}}{|\vec{r}|^3}
\]
Monte Carlo Evaluation of the Equipartition Rate

Make a change of variables to the dimensionless

\[ \eta \equiv \frac{\vec{r}}{2b} , \quad \vec{u} \equiv \frac{\vec{v}}{\sqrt{2} v} , \quad \tau \equiv \frac{\vec{v}}{\sqrt{2} b} t \]

and write the collision rate as

\[ I \left( \frac{r_c}{b} \right) = \frac{v}{n \sqrt{b^2}} \]

\[ = 2 \sqrt{2} \pi \int_0^{\infty} d\eta_\perp \eta_\perp \int_0^{2\pi} d\psi \int_0^{\infty} d\eta_\parallel \eta_\parallel \left| u_\perp \right| u_\parallel \left[ \Delta \left( \frac{u_\perp^2}{2} \right) \right]^2 \frac{e^{-v^2/2}}{(2\pi)^{3/2}} \]

where we have used cylindrical coordinates for \( \vec{u} \).

In the expression for \( I( r_c/b ) \), \( \Delta (u_\perp^2/2) \) is a function of \(( u_\perp, u_\parallel, \psi, \eta_\perp) \) determined by integration of the equations of motion

\[ \frac{d \vec{u}}{d \tau} + \left( \frac{\sqrt{2} b}{r_c} \right) \vec{u} \times \hat{z} = \frac{1}{2} \frac{\vec{\eta}}{\eta^3} \]

\[ \frac{d \vec{\eta}}{d \tau} = \vec{u} \]

over the course of a collision.
To more efficiently do the integral for $v$, we change coordinates from $(u_\perp, u_\parallel, \psi, \eta_\perp)$ to $(x_1, x_2, x_3, x_4)$ defined by

$$
x_1 = \frac{1}{A_1} \int_0^{u_\parallel} du_\parallel \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
x_2 = \frac{1}{A_2} \int_0^{\eta_\perp} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
x_3 = \frac{1}{A_3} \int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
x_4 = \frac{1}{A_4} \int_0^{\psi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

where

$$
A_1 = \int_0^{\infty} du_\parallel \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
A_2 = \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
A_3 = \int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

$$
A_4 = \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
$$

One can easily show that the Jacobian for this transformation is

$$
\frac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_\parallel, u_\perp, \psi, \eta_\perp)} = \frac{W(u_\parallel, u_\perp, \psi, \eta_\perp)}{A_1}
$$
The equipartition rate can now be written as

\[
I \left( \frac{r_c}{b} \right) = \frac{2 A_1}{\sqrt{\pi}} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \frac{u_{\parallel} u_{\perp} \eta_{\perp} e^{-u_{\perp}^2/2}}{W(u_{\parallel}, u_{\perp}, \psi, \eta_{\perp})} \left[ \Delta \left( \frac{u_{\perp}^2}{2} \right) \right]^2
\]

To make the Monte Carlo integration most efficient we would like to choose

\[
W(u_{\parallel}, u_{\perp}, \psi, \eta_{\perp}) \sim u_{\parallel} u_{\perp} \eta_{\perp} e^{-u_{\perp}^2/2} \left[ \Delta \left( \frac{u_{\perp}^2}{2} \right) \right]^2
\]

so that the integrand is reasonably uniform over the whole domain of integration.

We estimate the value of the integrand by picking \( N \) \((x_1, x_2, x_3, x_4)\) points from a uniform distribution for each \( x_i \) between 0 and 1. We integrate the equations of motion using a Bulirsch-Stoer technique to find \( \Delta (u_{\perp}^2/2) \). The equipartition rate is then

\[
I \left( \frac{r_c}{b} \right) \approx \frac{2 A_1}{\sqrt{\pi}} \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{u_{\parallel} u_{\perp} \eta_{\perp} e^{-u_{\perp}^2/2}}{W(u_{\parallel}, u_{\perp}, \psi, \eta_{\perp})} \left[ \Delta \left( \frac{u_{\perp}^2}{2} \right) \right]^2 \right\}_i
\]

A second Monte Carlo calculation was done using a rejection method to generate the initial configurations. The equations of motion were integrated using a 4th order Runge-Kutta scheme.
Graph of Monte Carlo Results with Asymptotic Expressions

\[ \frac{8 \sqrt{\pi}}{15} \ln \left( \frac{2.1 \, r_c}{b} \right) \]

- Integral Transform MC
- Rejection MC
- Glinsky and Rosenbluth Asymptotic Expression
New Asymptotic Expression for the Equipartition Rate in the Limit of Strong Magnetization

Start with the Hamiltonian for the relative motion in cylindrical coordinates

\[ H(r, p_r; z, p_z; \theta, p_\theta) = \left( p_\theta - \frac{\mu \Omega_c}{2} r^2 \right)^2 \frac{2}{2 \mu r^2} + \frac{p_r^2}{2 \mu} + \frac{p_z^2}{2 \mu} \frac{e^2}{\sqrt{r^2 + z^2}} \]

Since \( \theta \) is cyclic \( p_\theta \) is a constant of the motion. We can therefore write

\[ H(r, p_r; z, p_z) = \frac{p_r^2}{2 \mu} + \frac{p_z^2}{2 \mu} + V(r, z) \]

where

\[ V(r, z) = \frac{\mu \Omega_c^2}{8} \left( r^2 - 2 r_0^2 + \frac{r_0^4}{r^2} \right) + \frac{e^2}{\sqrt{r^2 + z^2}} \]

and

\[ r_0^2 = \frac{2 p_\theta}{\mu \Omega_c} \]

Expand \( V(r, z) \) as a harmonic potential in \( r \) about its guiding center (minimum), \( r_g(z) \)

\[ V(r, z) \approx V_g(z) + \frac{\mu \Omega_c^2(z)}{2} [r - r_g(z)]^2 \]
Change the independent variable from $t$ to $z$. This leads to the $z$-dependant Hamiltonian

$$H'(r, p_r; z) = \pm \sqrt{2\mu} \left[ H - V_g(z) - \frac{\mu \Omega^2(z)}{2} \left[ r - r_g(z) \right]^2 - \frac{p_r^2}{2\mu} \right]$$

Use the action angle variables $(P, \psi)$ associated with the $r$-degree of freedom and obtain the new Hamiltonian

$$H''(P, \psi; z) = \pm \sqrt{2\mu} \left( H - V_g(z) - \Omega P \right) - \sqrt{2\mu \Omega P} \frac{d r_g}{d z} \cos \psi + \frac{P}{2} \sin 2\psi \frac{d}{dz}(\ln \Omega)$$

Make a pertabative expansion

$$P = P^{(0)} + P^{(1)} + \ldots \quad \psi = \psi^{(0)} + \psi^{(1)} + \ldots$$

where the expansion parameter is $1/z$.

Since $dr_g/dz$ and $d\Omega/dz$ are both of 4th order in $1/z$ we find the equations of interest are

$$\frac{d}{dz} P^{(0)} = 0$$

$$\frac{d}{dz} \psi^{(0)} = \pm \frac{\mu \Omega}{\sqrt{2\mu} \left( H - V_g(z) - \Omega P^{(0)} \right)}$$

$$\frac{d}{dz} P^{(4)} = -\sqrt{2\mu \Omega P^{(0)}} \frac{d r_g}{d z} \sin \psi^{(0)} - P^{(0)} \cos 2\psi^{(0)} \frac{d}{dz}(\ln \Omega)$$
Hence

\[ P^{(0)} = P_0 = \text{pre-collision value} \]

\[ \Delta \psi^{(0)} = \pm \int_{z_T}^{z} \frac{\mu \Omega(z')}{\sqrt{2 \mu [H - V_g(z') - P_0 \Omega(z')]} } \, dz' \]

\[ \Delta P^{(4)} = 2 \sqrt{P_0} \cos \psi_0 \left| \int_{C} \exp \left[ i \Delta \psi^{(0)} \right] \sqrt{\frac{\mu \Omega(z)}{2}} \frac{d r_g(z)}{d z} \, dz \right| \]

where

\[ (\Delta E_{\perp})^2 = \Omega_c^2 [\Delta P^{(4)}]^2 \]

\[ z_T \text{ is the turning point where } H = V_g(z_T) + P_0 \Omega(z_T) , \]

and

\[ \text{Im}(z) \]

\[ \text{Re}(z) \]

\[ z\text{-plane} \]

\[ C \]

\[ z_T \]
We find $\exp \left[ i \Delta \psi^{(0)} \right]$ as a power series expansion in \((v_{\parallel 0} / \Omega_c b)^{2/3}\) and \((v_{\perp 0} / v_{\parallel 0})^2\) whose coefficients are functions of \(r_0\) and \(1/z\). We substitute this result into the expression for \((\Delta E_{\perp})^2\) and do the contour integration. The resulting expression for \((\Delta E_{\perp})^2\) is used to obtain the large magnetic field \((r_c \ll b)\) asymptotic expression

\[
I(\bar{\varepsilon}) \approx \exp \left( -E \bar{\varepsilon}^{-2/5} \right) \left[ \left(8.81\right) \bar{\varepsilon}^{7/15} + \left(91.7\right) \bar{\varepsilon}^{11/15} \right] + \left(1.45\right) \bar{\varepsilon}^{13/15} + \left(351\right) \bar{\varepsilon}^{15/15} + \left(25.5\right) \bar{\varepsilon}^{17/15} + \mathcal{O} \left( \bar{\varepsilon}^{19/15} \right) \right]
\]

where

\[
\bar{\varepsilon} \equiv \frac{r_c}{b} \quad \text{and} \quad E \equiv \frac{5}{6} \left(3\pi\right)^{2/5} 2^{1/5} \approx 2.34
\]

Note that the 2\textsuperscript{nd} and 4\textsuperscript{th} terms have surprisingly large coefficients. The first term dominates when \(r_c / b < 10^{-5}\).
Monte Carlo Results Compared to Asymptotic Expression for $r_c/b << 1$

\[ \exp(E_{\overline{e}^{-2/5}}) \cdot I(\overline{e}) \]

\[ \overline{e} = \frac{r_c}{b} \]

Glinsky and Rosenbluth

O'Neil and Hjorth
Asymptotic Expression for the Equipartition Rate in the Limit of Weak Magnetization

Montgomery, Joyce and Turner found that

\[
I\left(\frac{r_c}{b}\right) \approx \frac{8\sqrt{\pi}}{15} \ln\left(\frac{r_c}{A\frac{r_c}{b}}\right)
\]

for \( r_c \ll b \). This comes from an integral of the form

\[
I\left(\frac{r_c}{b}\right) \sim \frac{8\sqrt{\pi}}{15} \int_b^{r_c} \frac{d r_\perp}{r_\perp}
\]

**MJT** uses integration along unperturbed orbits:

1. Upper cutoff arises naturally from dynamical shielding

2. Lower cutoff is *ad hoc* (orbits are not unperturbed)

Both arise naturally in *Monte Carlo* evolution so that the constant is determined to be:

\[
A \approx 2.12(41)
\]
Monte Carlo Results Compared to Experiment

Monte Carlo Integral
Glinsky et. al.

Beck, Fajans and Malmberg

B = 20 to 60 kG
T = 30 to 350,000 °K

Hyatt, Driscoll and Malmberg

B = 280 G
T = 3,000 to 30,000 °K