Collisional Equipartition Rate for a Magnetized Pure Electron Plasma

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[M. E. Glinsky, T. M. O'Neil, M. N. Rosenbluth, K. Tsuruta, and S. Ichimaru, Bul. Am. Phys. Soc. **36**, 2330 (1991)]

Abstract*

The collisional equipartition rate between the parallel and perpendicular velocity components is calculated for a weakly correlated electron plasma that is immersed in a uniform magnetic field. Here, parallel and perpendicular refer to the direction of the magnetic field. The rate depends on the ratio r_c/b, where $r_c=\sqrt{T/m}/\Omega_c$ is the cyclotron radius and $b=e^2/T$ is the classical distance of closest approach. For a strongly magnetized plasma (i.e., $r_c/b \ll 1$), the equipartition rate is exponentially small ($\sim\exp[-2.34(r_c/b)^{-2/5}]$). For a weakly magnetized plasma (i.e., $r_c/b \gg 1$), the rate is the same as for an unmagnetized plasma except that r_c/b replaces λ_D/b in the Coulomb logarithm. (It is assumed here that $r_c<\lambda_D$; otherwise the plasma is effectively unmagnetized.) This paper presents a numerical treatment that spans the intermediate regime $r_c/b \sim 1$, and connects on to asymptotic formulas in the two limits $r_c/b \ll 1$ and $r_c/b \gg 1$. Also an improved asymptotic formula for the rate in the high field limit is given. Our results are in good agreement with experiments over eight decades in r_c/b.[1]

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[1] Hyatt, Driscoll and Malmberg, Phys. Rev. Lett. **59**, 2975 (1987); Beck, Fajans and Malmberg, Bul. Am. Phys. Soc. **35**, 2134 (1990).

Definition of the Equipartition Rate

Consider a pure electron plasma that is immersed in a uniform magnetic field $\vec{B} = B\,\hat{z}$. Let the velocity distribution be of the form

$$f(\vec{v}) = \left(\frac{m}{2\pi T_{\parallel}}\right)^{1/2} \left(\frac{m}{2\pi T_{\perp}}\right) \exp\left[-\frac{m v_{\parallel}^2}{2T_{\parallel}} - \frac{m v_{\perp}^2}{2T_{\perp}}\right]$$

where

$$|T_{\parallel} - T_{\perp}| << T_{\parallel}, T_{\perp} .$$

The collisional <u>equipartition rate</u> ν is defined through the equation

$$\frac{dT_{\perp}}{dt} = \nu\,(T_{\parallel} - T_{\perp}) .$$

Weakly Correlated Plasma

$$b << \lambda_D$$

where

$b \equiv e^2 / T$ = distance of closest approach

$\lambda_D \equiv \sqrt{T / 4 \pi n e^2}$ = Debye length

Scale Length for Magnetic Field

$$r_c \equiv \bar{v} / \Omega_c$$

where

$\bar{v} \equiv \sqrt{T / m}$ = thermal velocity

$\Omega_c \equiv e B / m c$ = cyclotron frequency

Regimes of Magnetization

1. Effectively unmagnetized plasma*

$(r_c > \lambda_D)$

$$\nu = n \bar{v} b^2 \frac{8\sqrt{\pi}}{15} \ln\left(\frac{\lambda_D}{b}\right)$$

2. Weakly magnetized plasma**

$(b < r_c < \lambda_D)$

$$\nu = n \bar{v} b^2 \frac{8\sqrt{\pi}}{15} \ln\left(\frac{r_c}{b}\right)$$

3. Strongly magnetized plasma***

$(r_c < b)$

$$\nu = n \bar{v} b^2 \; (8.81) \left(\frac{r_c}{b}\right)^{7/15} \exp\left[-2.34 \left(\frac{b}{r_c}\right)^{2/5}\right]$$

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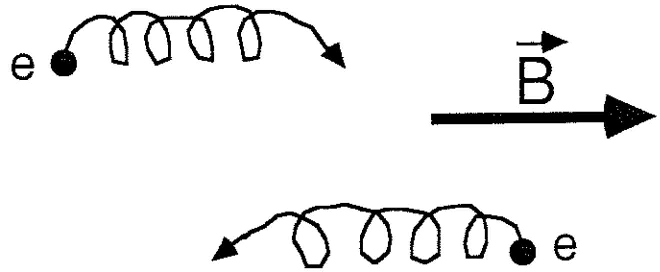
*S. Ichimaru and M.N. Rosenbluth, Phys. Fluids **13**, 2778 (1970).

D. Montgomery, G. Joyce and L. Turner, Phys. Fluids **17, 2201 (1974).

***T. M. O'Neil and P. G. Hjorth, Phys. Fluids **28**, 3241 (1985); with corrections to the non-exponential part presented in this paper.

Adiabatic Invariant



When

$$\tau\,\Omega_c >> 1\ ,$$

$$J_2 = \frac{m\,v_{1\perp}^2}{2\,\Omega_c} + \frac{m\,v_{2\perp}^2}{2\,\Omega_c}$$

is an <u>adiabatic invariant</u> and

$$\Delta J_2 \sim \exp\left[-\Omega_c\,\tau\right]\ .$$

Since

$$\tau \sim \frac{\left|\vec{r}_1 - \vec{r}_2\right|_{min}}{v_{rel}}$$

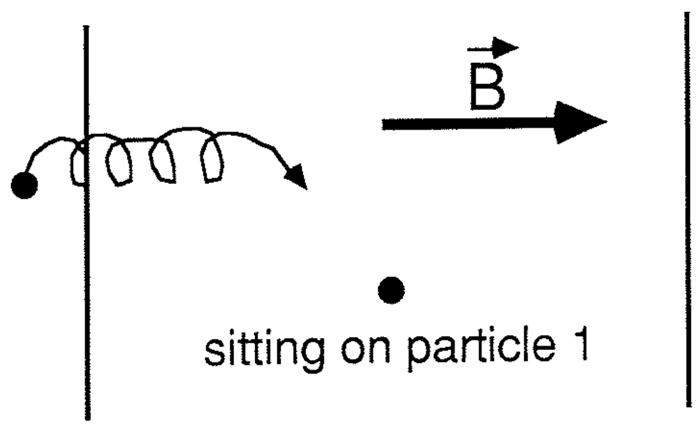
there is an exponentially small exchange of energy for

$$\left|\vec{r}_1 - \vec{r}_2\right|_{min} >> \frac{\bar{v}}{\Omega_c} = r_c$$

Boltzmann-like Operator

In a weakly or strongly magnetized plasma ($r_c < \lambda_D$) <u>dynamical shielding</u> supersedes Debye shielding. Therefore we use a Boltzmann-like operator which omits Debye shielding.

$$\frac{\partial f(\vec{v}_1, t)}{\partial t} = \int_0^\infty 2\pi r_\perp dr_\perp \int d^3 \vec{v}_2 \; | v_{2\|} - v_{1\|} | \left[f(\vec{v}_2^{\,'}) f(\vec{v}_1^{\,'}) - f(\vec{v}_2) f(\vec{v}_1) \right]$$



$$\frac{\partial T_\perp}{\partial t} = \int d^3\vec{v}_1 \quad \frac{m\, v_{1\perp}^2}{2} \; \frac{\partial f(\vec{v}_1, t)}{\partial t}$$

↑ substitute from Boltzmann equation

1. **Change to center of mass and relative velocities**

$$\vec{V} \equiv \frac{\vec{v}_1 + \vec{v}_2}{2} \qquad\qquad \vec{v} \equiv \vec{v}_1 - \vec{v}_2$$

2. **Use detailed balance**

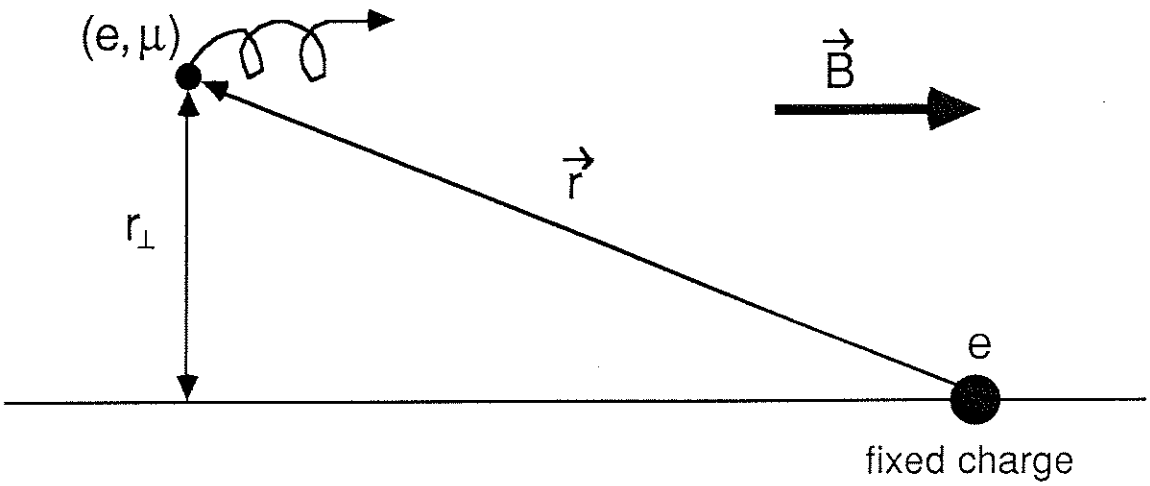
3. **Use fact that center of mass dynamics and relative dynamics decouple (for uniform B)**

$$\nu = \frac{n}{4\,T^2} \int_0^\infty 2\pi\, r_\perp\, dr_\perp \int d^3\vec{v} \;\; |v_{\|}| \left[\Delta\left(\frac{\mu\, v_\perp^2}{2} \right) \right]^2 \; f_r(\vec{v})$$

$$f_r(\vec{v}) = \left(\frac{\mu}{2\pi\, T} \right)^{3/2} \exp\left(- \frac{\mu\, v^2}{2\,T} \right)$$

$$T_\perp \approx T_{\|} = T \qquad\qquad \mu \equiv m/2$$

Binary Collision



$$\Delta E_\perp = -\Delta E_\parallel = \Delta\left(\frac{\mu\, v_\perp^2}{2}\right)$$

where the decoupled equations of motion are

$$\frac{d\vec{V}}{dt} + \Omega_c \vec{V} \times \hat{z} = 0$$

$$\frac{d\vec{v}}{dt} + \Omega_c \vec{v} \times \hat{z} = \frac{e^2}{\mu} \frac{\vec{r}}{|\vec{r}|^3}$$

Monte Carlo Evaluation of the Equipartition Rate

Make a change of variables to the dimensionless

$$\vec{\eta} \equiv \frac{\vec{r}}{2\,b} \quad , \quad \vec{u} \equiv \frac{\vec{v}}{\sqrt{2}\,\bar{v}} \quad , \quad \tau \equiv \frac{\bar{v}}{\sqrt{2}\,b}\,t$$

and write the collision rate as

$$I\left(\frac{r_c}{b}\right) = \frac{\nu}{n\,\bar{v}\,b^2}$$

$$= 2\sqrt{2}\,\pi \int_0^\infty d\eta_\perp\,\eta_\perp \int_{-\infty}^\infty du_\parallel \int_0^{2\pi} d\psi \int_0^\infty du_\perp\,u_\perp\,|\,u_\parallel\,| \left[\Delta\left(\frac{u_\perp^2}{2}\right)\right]^2 \frac{e^{-u^2/2}}{(2\pi)^{3/2}}$$

where we have used cylindrical coordinates for $\vec{u}$.

In the expression for $I(r_c/b)$, $\Delta\left(u_\perp^2/2\right)$ is a function of $(u_\perp, u_\parallel, \psi, \eta_\perp)$ determined by integration of the equations of motion

$$\frac{d\vec{u}}{d\tau} + \left(\frac{\sqrt{2}\,b}{r_c}\right)\vec{u}\times\hat{z} = \frac{1}{2}\,\frac{\vec{\eta}}{\eta^3} \quad , \qquad \frac{d\vec{\eta}}{d\tau} = \vec{u}$$

over the course of a collision.

To more efficiently do the integral for ν, we change coordinates from $(u_\perp, u_\parallel, \psi, \eta_\perp)$ to (x_1, x_2, x_3, x_4) defined by

$$x_1 = \frac{1}{A_1} \int_0^{u_\parallel} du_\parallel \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_2 = \frac{1}{A_2} \int_0^{\eta_\perp} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_3 = \frac{1}{A_3} \int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_4 = \frac{1}{A_4} \int_0^{\psi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

where

$$A_1 = \int_0^{\infty} du_\parallel \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_2 = \int_0^{\infty} d\eta_\perp \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_3 = \int_0^{\infty} du_\perp \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_4 = \int_0^{2\pi} d\psi \; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

One can easily show that the Jacobian for this transformation is

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_\parallel, u_\perp, \psi, \eta_\perp)} = \frac{W(u_\parallel, u_\perp, \psi, \eta_\perp)}{A_1}$$

The equipartition rate can now be written as

$$I\left(\frac{r_c}{b}\right) = \frac{2A_1}{\sqrt{\pi}} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \; \frac{u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2}}{W(u_{\|}, u_{\perp}, \psi, \eta_{\perp})} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2$$

To make the Monte Carlo integration most efficient we would like to choose

$$W(u_{\|}, u_{\perp}, \psi, \eta_{\perp}) \sim u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2$$

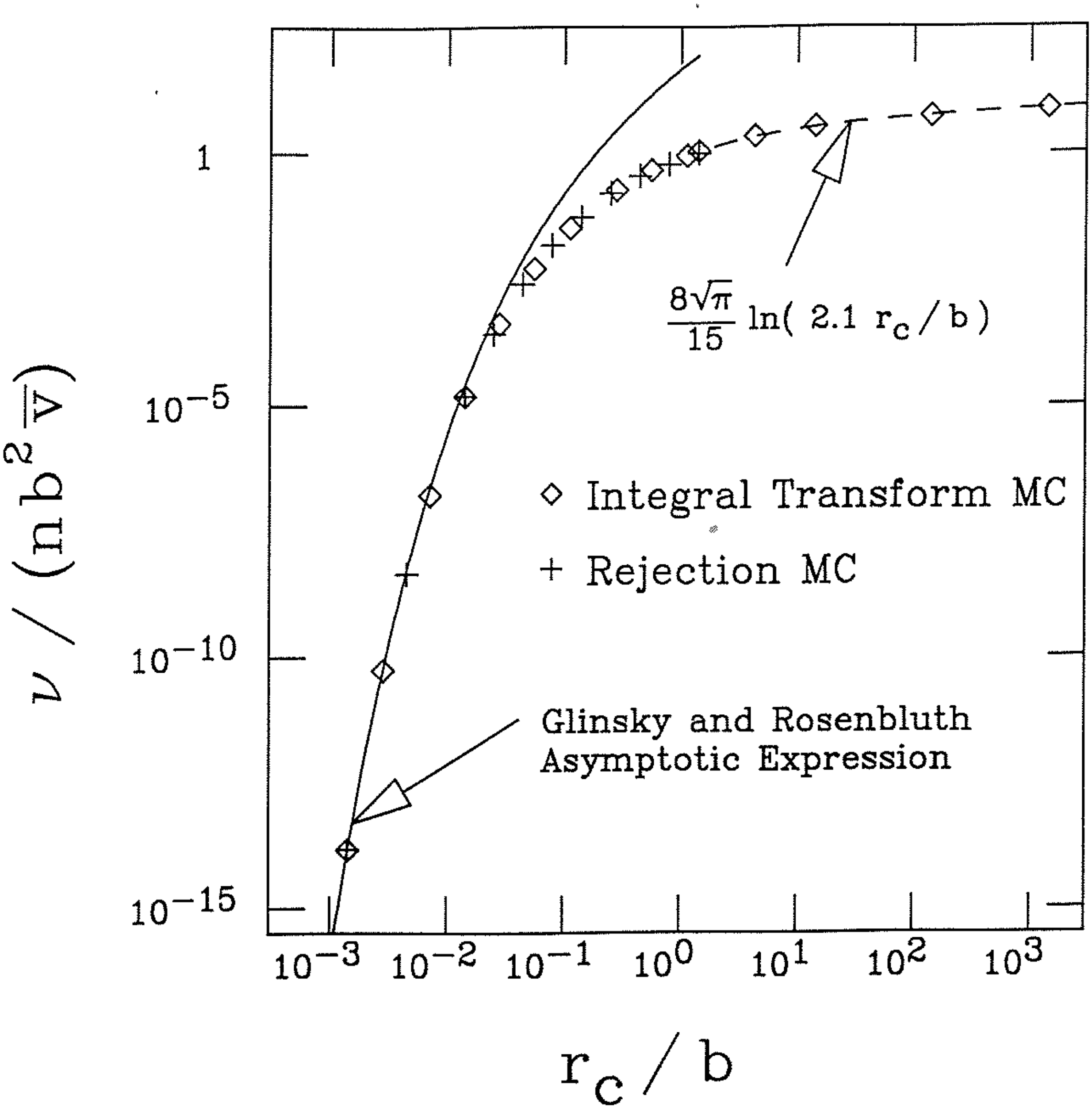
so that the integrand is reasonably uniform over the whole domain of integration.

We estimate the value of the integrand by picking N (x_1, x_2, x_3, x_4) points from a uniform distribution for each x_i between 0 and 1. We integrate the equations of motion using a Bulirsch-Stoer technique to find $\Delta(u_{\perp}^2/2)$. The equipartition rate is then

$$I\left(\frac{r_c}{b}\right) \approx \frac{2A_1}{\sqrt{\pi}} \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2}}{W(u_{\|}, u_{\perp}, \psi, \eta_{\perp})} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2 \right\}_i$$

A second Monte Carlo calculation was done using a rejection method to generate the initial configurations. The equations of motion were integrated using a 4th order Runge-Kutta scheme.

Graph of Monte Carlo Results with Asymptotic Expressions



New Asymptotic Expression for the Equipartition Rate in the Limit of Strong Magnetization

Start with the Hamiltonian for the relative motion in cylindrical coordinates

$$H(r, p_r; z, p_z; \theta, p_\theta) = \frac{\left(p_\theta - \frac{\mu\,\Omega_c}{2}\, r^2\right)^2}{2\,\mu\, r^2} + \frac{p_r^2}{2\,\mu} + \frac{p_z^2}{2\,\mu} + \frac{e^2}{\sqrt{r^2 + z^2}}$$

Since θ is cyclic p_θ is a constant of the motion. We can therefore write

$$H(r, p_r; z, p_z) = \frac{p_r^2}{2\,\mu} + \frac{p_z^2}{2\,\mu} + V(r, z)$$

where

$$V(r, z) \equiv \frac{\mu\,\Omega_c^2}{8}\left(r^2 - 2\,r_0^2 + \frac{r_0^4}{r^2}\right) + \frac{e^2}{\sqrt{r^2 + z^2}}$$

and

$$r_0^2 \equiv 2\,p_\theta / \mu\,\Omega_c\,.$$

Expand $V(r,z)$ as a harmonic potential in r about its guiding center (minimum), $r_g(z)$

$$V(r, z) \approx V_g(z) + \frac{\mu\;\Omega^2(z)}{2}\left[r - r_g(z)\right]^2$$

Change the independent variable from t **to** z. **This leads to the** z**-dependant Hamiltonian**

$$H'(r, p_r; z) = \pm\sqrt{2\mu\left\{H - V_g(z) - \frac{\mu\,\Omega^2(z)}{2}[r - r_g(z)]^2 - \frac{p_r^2}{2\mu}\right\}}$$

Use the action angle variables (P,ψ) **associated with the** r**-degree of freedom and obtain the new Hamiltonian**

$$H''(P, \psi; z) = \pm\sqrt{2\mu\left(H - V_g - \Omega P\right)} - \sqrt{2\mu\Omega P}\,\frac{d\,r_g}{d\,z}\cos\psi + \frac{P}{2}\sin 2\psi\,\frac{d}{dz}(\ln\Omega)$$

Make a pertabative expansion

$$P = P^{(0)} + P^{(1)} + \ldots \qquad\qquad \psi = \psi^{(0)} + \psi^{(1)} + \ldots$$

where the expansion parameter is $1/z$.

Since dr_g/dz **and** $d\Omega/dz$ **are both of** 4^{th} **order in** $1/z$ **we find the equations of interest are**

$$\frac{d\,P^{(0)}}{d\,z} = 0$$

$$\frac{d\,\psi^{(0)}}{d\,z} = \pm\frac{\mu\,\Omega}{\sqrt{2\mu\left(H - V_g - \Omega P^{(0)}\right)}}$$

$$\frac{d\,P^{(4)}}{d\,z} = -\sqrt{2\mu\,\Omega\,P^{(0)}}\,\frac{d\,r_g}{d\,z}\sin\psi^{(0)} - P^{(0)}\cos 2\psi^{(0)}\,\frac{d}{d\,z}(\ln\Omega)$$

Hence

$$P^{(0)} = P_0 = \text{pre-collision value}$$

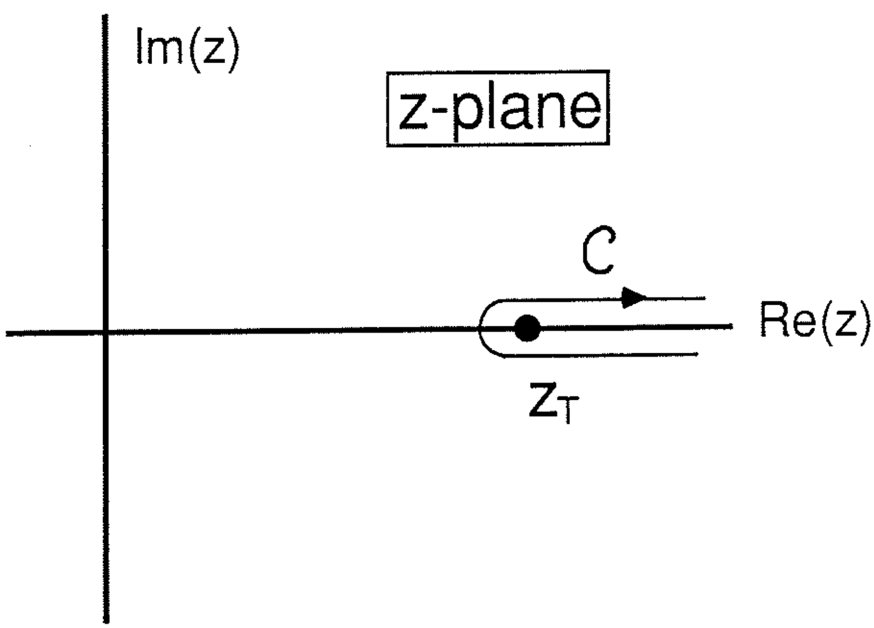
$$\Delta\psi^{(0)} = \pm \int_{z_T}^{z} \frac{\mu\ \Omega(z')\,dz'}{\sqrt{2\mu\left[H - V_g(z') - P_0\,\Omega(z')\right]}}$$

$$\Delta P^{(4)} = 2\sqrt{P_0}\ \cos\psi_0 \left| \int_C \exp\left[i\,\Delta\psi^{(0)}\right] \sqrt{\frac{\mu\,\Omega(z)}{2}}\ \frac{d\,r_g(z)}{d\,z}\ dz \right|$$

where

$$(\Delta E_\perp)^2 \approx \Omega_c^2 \left[\Delta P^{(4)}\right]^2 ,$$

z_T **is the turning point where** $H = V_g(z_T) + P_0\ \Omega(z_T)$,
and



We find $\exp[\, i \Delta \psi^{(0)}]$ as a power series expansion in $(v_{\|0} / \Omega_c b)^{2/3}$ and $(v_{\perp 0} / v_{\| 0})^2$ whose coefficients are functions of r_0 and $1/z$. We substitute this result into the expression for $(\Delta E_{\perp})^2$ and do the contour integration. The resulting expression for $(\Delta E_{\perp})^2$ is used to obtain the large magnetic field $(r_c << b)$ asymptotic expression

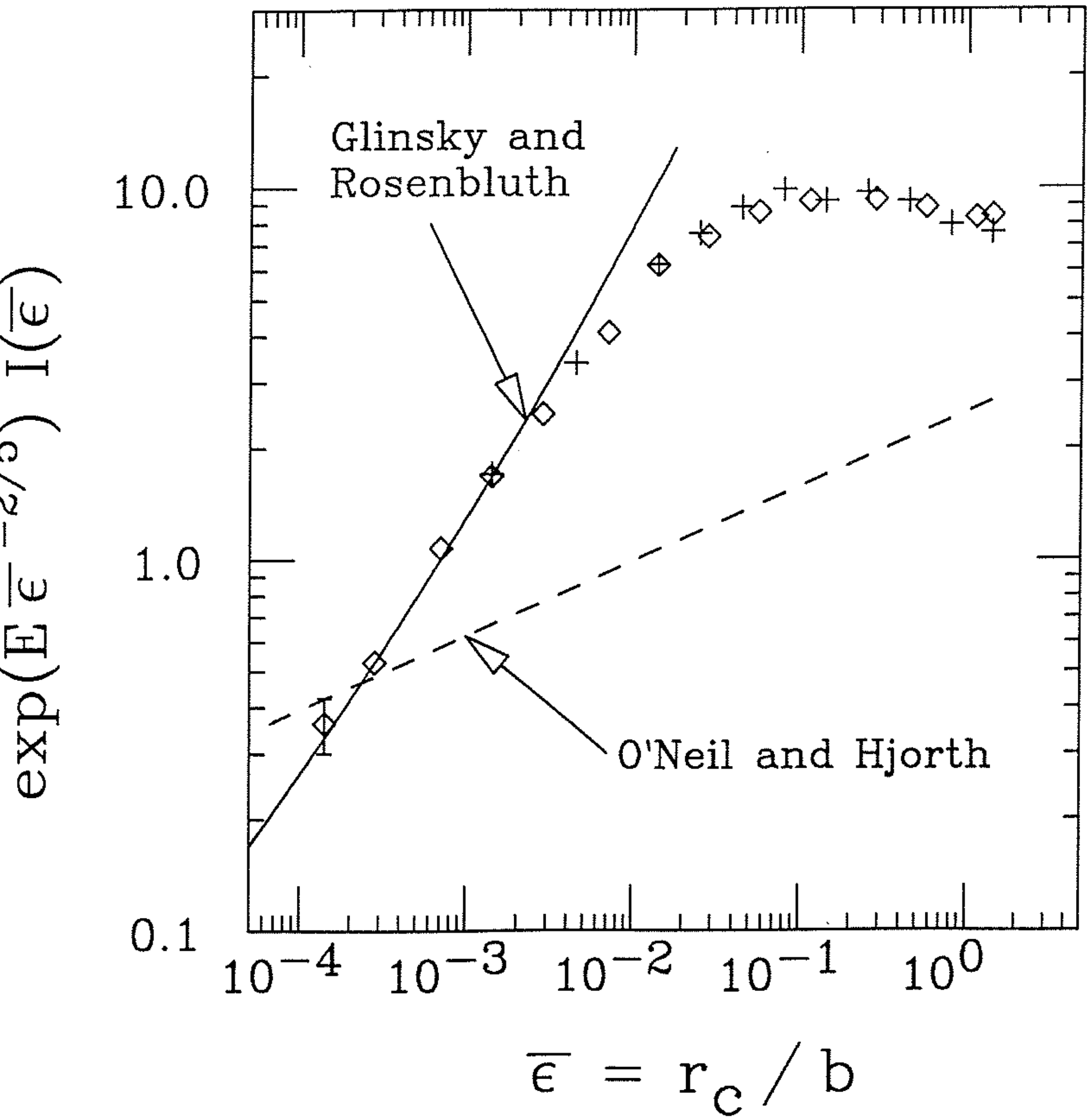
$$I(\bar{\varepsilon}) \approx \exp(-E\,\bar{\varepsilon}^{-2/5}) \begin{bmatrix} (8.81)\,\bar{\varepsilon}^{7/15} + (91.7)\,\bar{\varepsilon}^{11/15} \\ +(1.45)\,\bar{\varepsilon}^{13/15} + (351)\,\bar{\varepsilon}^{15/15} \\ +(25.5)\,\bar{\varepsilon}^{17/15} + O\left(\bar{\varepsilon}^{19/15}\right) \end{bmatrix}$$

where

$$\bar{\varepsilon} \equiv \frac{r_c}{b} \quad \text{and} \quad E \equiv \frac{5}{6}(3\pi)^{2/5}\, 2^{1/5} \approx 2.34 .$$

Note that the 2nd and 4th terms have surprisingly large coefficients. The first term dominates when $r_c / b < 10^{-5}$.

Monte Carlo Results Compared to Asymptotic Expression for $r_c/b<<1$



Asymptotic Expression for the Equipartition Rate in the Limit of Weak Magnetization

Montgomery, Joyce and Turner found that

$$I\left(\frac{r_c}{b}\right) \approx \frac{8\sqrt{\pi}}{15} \ln\left(A\,\frac{r_c}{b}\right)$$

for $r_c << b$. This comes from an integral of the form

$$I\left(\frac{r_c}{b}\right) \sim \frac{8\sqrt{\pi}}{15} \int_b^{r_c} \frac{d\,r_\perp}{r_\perp}$$

<u>MJT</u> uses integration along unperturbed orbits:

1. Upper cutoff arises naturally from dynamical shielding
2. Lower cutoff is *ad hoc* (orbits are not unperturbed)

Both arise naturally in <u>Monte Carlo</u> evolution so that the constant is determined to be:

$$A \approx 2.12(41)$$

Monte Carlo Results Compared to Experiment