Collisional Equipartition Rate for a Magnetized Pure Electron Plasma

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Abstract

The collisional equipartition rate between the parallel and perpendicular velocity components of a weakly correlated electron plasma that is immersed in a uniform magnetic field is calculated. Here, parallel and perpendicular refer to the direction of the magnetic field. The rate depends on the ratio $r_c/b$, where $r_c=\sqrt{T/m}/\Omega_c$ is the cyclotron radius and $b=e^2/T$ is the classical distance of closest approach. For a strongly magnetized plasma (i.e., $r_c/b<<1$), the equipartition rate is exponentially small ($\sim \exp[-2.35(r_c/b)^{-2/5}]$).\textsuperscript{1} For a weakly magnetized plasma (i.e., $r_c/b>>1$), the rate is the same as for an unmagnetized plasma except that $r_c/b$ replaces $\lambda_D/b$ in the Coulomb logarithm.\textsuperscript{2} (It is assumed here that $r_c<\lambda_D$; for $r_c>\lambda_D$, the plasma is effectively unmagnetized.) This paper presents a numerical treatment that spans the intermediate regime $r_c/b\sim 1$, connects on to asymptotic results in the two limits $r_c/b<<1$ and $r_c/b>>1$, and is in good agreement with recent experiments (see poster by B. Beck et al.). Also an improved asymptotic expression for the rate in the high field limit is given.

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Definition of the Equipartition Rate

Consider a pure electron plasma that is immersed in a uniform magnetic field $\vec{B} = B \hat{z}$. Let the velocity distribution be of the form

$$f(\vec{v}) = \left( \frac{m}{2 \pi T_{\parallel}} \right)^{1/2} \left( \frac{m}{2 \pi T_{\perp}} \right) \exp \left[ -\frac{m v_{\parallel}^2}{2 T_{\parallel}} - \frac{m v_{\perp}^2}{2 T_{\perp}} \right]$$

where

$$| T_{\parallel} - T_{\perp} | \ll T_{\parallel}, T_{\perp}.$$

The collisional equipartition rate $\nu$ is defined through the equation

$$\frac{dT_{\perp}}{dt} = \nu (T_{\parallel} - T_{\perp}).$$
Previous Results

1. Strongly magnetized plasma

\[ r_c \ll b, \quad r_c = \frac{\bar{v}}{\Omega_c}, \quad b = \frac{e^2}{T}, \quad \bar{v} = \sqrt{\frac{T}{m}} \]

\[ \nu = n \bar{v} b^2 I \left( \frac{r_c}{b} \right) \]

\[ I \left( \frac{r_c}{b} \right) \sim \exp \left[ -2.35 \left( \frac{b}{r_c} \right)^{2/5} \right] \]

2. Weakly magnetized plasma

\[ b \ll r_c \ll \lambda_D \]

\[ I \sim \ln \left( \frac{r_c}{b} \right) \]

3. Effectively unmagnetized plasma

\[ \lambda_D < r_c \]

\[ I \sim \ln \left( \frac{\lambda_D}{b} \right) \]

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Binary Collision

\[
\frac{d \vec{v}_1}{dt} + \Omega_c \vec{v}_1 \times \hat{z} = \frac{e^2}{m} \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}
\]

\[
\frac{d \vec{v}_2}{dt} + \Omega_c \vec{v}_2 \times \hat{z} = \frac{e^2}{m} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3}
\]

Center of mass motion and relative motion decouple.

\[
\left( \vec{V} \equiv \frac{\vec{v}_1 + \vec{v}_2}{2}, \ \vec{v} \equiv \vec{v}_2 - \vec{v}_1, \ \vec{r} \equiv \vec{r}_2 - \vec{r}_1, \ \mu \equiv \frac{m}{2} \right)
\]

\[
\frac{d \vec{V}}{dt} + \Omega_c \vec{V} \times \hat{z} = 0
\]

\[
\frac{d \vec{v}}{dt} + \Omega_c \vec{v} \times \hat{z} = \frac{e^2}{\mu} \frac{\vec{r}}{|\vec{r}|^3}
\]
\[ E_{\perp} = \frac{m v_{1\perp}^2}{2} + \frac{m v_{2\perp}^2}{2} = \frac{\mu v_{\perp}^2}{2} + \frac{2m V_{\perp}^2}{2} \]

\[ E_{\parallel} = \frac{m v_{1\parallel}^2}{2} + \frac{m v_{2\parallel}^2}{2} = \frac{\mu v_{\parallel}^2}{2} + \frac{2m V_{\parallel}^2}{2} \]

Center of mass motion is unchanged during collision, therefore

\[ \Delta E_{\perp} = -\Delta E_{\parallel} = \Delta \left( \frac{\mu v_{\perp}^2}{2} \right) \]

**Relative Motion**
An adiabatic invariant exists in the limit

\[ \Omega_c \tau_c >> 1, \]

where \( \tau_c \) is the duration of a collision.

\[ \Delta E_\perp \sim \exp\left(-\Omega_c \tau_c\right) \sim \exp\left(-\Omega_c \left[ \frac{b_\parallel}{v_\parallel} g\left(\frac{r_\perp}{b_\parallel}\right)\right]\right) \]

where \( b_\parallel \equiv \frac{e^2}{\left(\frac{\mu v_\parallel^2}{2}\right)} \).

In a plasma, the adiabatic invariant produces a **dynamical shielding** (for the exchange of parallel and perpendicular kinetic energy).

\[ |\vec{r}_1 - \vec{r}_2|_{\min} > r_c = \frac{\bar{v}}{\Omega_c} \quad \Rightarrow \quad \Omega_c \tau_c > 1 \]

\[ \Rightarrow \quad \text{dynamical shielding} \]
Therefore, we can use a Boltzmann-like collision operator, provided that \( r_c < \lambda_D \). The Boltzmann operator omits Debye shielding (recall Landau's derivation of the Fokker-Planck operator from the Boltzmann operator; Debye shielding was included in an \textit{ad hoc} fashion), but nothing is lost if the dynamical screening length is shorter than the Debye length.

A Boltzmann-like operator leads to an integral expression for the rate  

\[
\frac{\partial f(\vec{v}_1, t)}{\partial t} = \int_0^\infty 2\pi r_\perp dr_\perp \int d^3 v_2 | v_{2\parallel} - v_{1\parallel} | \left[ f(\vec{v}'_2)f(\vec{v}'_1) - f(\vec{v}_2)f(\vec{v}_1) \right]
\]

\[
\frac{\partial T_\perp}{\partial t} = \int d^3 \vec{v}_1 \frac{m v_{1\perp}^2}{2} \frac{\partial f(\vec{v}_1, t)}{\partial t}
\]

(\text{use center of mass, relative velocities, and detailed balance})

\[
v = \frac{n}{4\pi T^2} \int_0^\infty 2\pi r_\perp dr_\perp \int d^3 \vec{v} | v_\parallel | \left[ \Delta \left( \frac{\mu v_\perp^2}{2} \right) \right]^2 f_r(\vec{v})
\]

\[
f_r(\vec{v}) = \left( \frac{\mu}{2\pi T} \right)^{3/2} \exp \left( -\frac{\mu v^2}{2T} \right)
\]

\[
T_\perp \approx T_\parallel = T
\]
Monte Carlo Evaluation of the Equipartition Rate

Make a change of variables to the dimensionless

\[ \bar{\eta} = \frac{r}{2b}, \quad \bar{u} = \frac{v}{\sqrt{2} \bar{v}}, \quad \tau = \frac{v}{\sqrt{2} b} t \]

and write the collision rate as

\[ I\left( \frac{r_c}{b} \right) = \frac{v}{n \bar{v} b^2} \]

\[ = 2 \sqrt{2} \pi \int_0^\infty d\eta_\perp \eta_\perp \int_0^\infty du_\parallel \int_0^{2\pi} dv \int_0^\infty du_\perp u_\perp \mid u_\parallel \mid \Delta \left( \frac{u_\perp^2}{2} \right)^2 \frac{e^{-u^2/2}}{(2\pi)^{3/2}} \]

where we have used cylindrical coordinates for \( \bar{u} \).

In the expression for \( I\left( \frac{r_c}{b} \right), \quad \Delta \left( \frac{u_\perp^2}{2} \right) \) is a function of \( (u_\perp, u_\parallel, \psi, \eta_\perp) \) determined by integration of the equations of motion

\[ \frac{d \bar{u}}{d \tau} + \left( \frac{\sqrt{2} b}{r_c} \right) \bar{u} \times \hat{z} = \frac{1}{2} \bar{\eta}^3 \]

\[ \frac{d \bar{\eta}}{d \tau} = \bar{u} \]

over the course of a collision.
To more efficiently do the integral for $v$, we change coordinates from $(u_\perp, u_\parallel, \psi, \eta_\perp)$ to $(x_1, x_2, x_3, x_4)$ defined by

\[
\begin{align*}
    x_1 &= \frac{1}{A_1} \int_0^{u_\parallel} du_\parallel \int_0^\infty d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    x_2 &= \frac{1}{A_2} \int_0^{\eta_\perp} d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    x_3 &= \frac{1}{A_3} \int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    x_4 &= \frac{1}{A_4} \int_0^{\psi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
\end{align*}
\]

where

\[
\begin{align*}
    A_1 &= \int_0^\infty du_\parallel \int_0^\infty d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    A_2 &= \int_0^{\eta_\perp} d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    A_3 &= \int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp) \\
    A_4 &= \int_0^{\psi} d\psi \ W(u_\parallel, u_\perp, \psi, \eta_\perp)
\end{align*}
\]

One can easily show that the Jacobian for this transformation is

\[
\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_\parallel, u_\perp, \psi, \eta_\perp)} = \frac{W(u_\parallel, u_\perp, \psi, \eta_\perp)}{A_1}
\]
The equipartition rate can now be written as

\[
I \left( \frac{r_c}{b} \right) = \frac{2 A_1}{\sqrt{\pi}} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \frac{u_{||} u_{\perp} \eta_{\perp} e^{-u^2/2}}{W(u_{||}, u_{\perp}, \psi, \eta_{\perp})} \left[ \Delta \left( \frac{u_{\perp}}{2} \right) \right]^2
\]

To make the Monte Carlo integration most efficient we would like to choose

\[
W(u_{||}, u_{\perp}, \psi, \eta_{\perp}) \sim u_{||} u_{\perp} \eta_{\perp} e^{-u^2/2} \left[ \Delta \left( \frac{u_{\perp}}{2} \right) \right]^2
\]

so that the integrand is reasonably uniform over the whole domain of integration.

We estimate the value of the integrand by picking \( N (x_1, x_2, x_3, x_4) \) points from a uniform distribution for each \( x_i \) between 0 and 1. We integrate the equations of motion using a Bulirsch-Stoer technique to find \( \Delta \left( \frac{u_{\perp}^2}{2} \right) \).

The equipartition rate is then

\[
I \left( \frac{r_c}{b} \right) \approx \frac{2 A_1}{\sqrt{\pi}} \frac{1}{N} \sum_{i=1}^N \left\{ \frac{u_{||} u_{\perp} \eta_{\perp} e^{-u^2/2}}{W(u_{||}, u_{\perp}, \psi, \eta_{\perp})} \left[ \Delta \left( \frac{u_{\perp}}{2} \right) \right]^2 \right\}_i
\]

A second Monte Carlo calculation was done using a rejection method to generate the initial configurations. The equations of motion were integrated using a 4th order Runge-Kutta scheme.
Graph of Monte Carlo Results

\[ \frac{\nu}{(nb^2 v)} \]

\[ \frac{8\sqrt{\pi}}{15} \ln\left(2.1 \frac{r_c}{b}\right) \]

- Glinsky
- Tsuruta

O'Neil and Hjorth
Asymptotic Expression

\[ 10^{-3} \quad 10^{-2} \quad 10^{-1} \quad 10^{0} \quad 10^{1} \quad 10^{2} \quad 10^{3} \]
Monte Carlo Results Compared to Experiment

Monte Carlo Integral
Glinsky et. al.

Hyatt, Driscoll
and Malmberg

B = 280 G
T = 3,000 to
30,000 °K

Beck, Fajans
and Malmberg

B = 30 to 60 kG
T = 20 to 10^4 °K

\( \nu / (n b^2 \overline{\nu}) \)

\( r_c / b \)
Analytic Expression for the Equipartition Rate

One can write the change in perpendicular energy during a collision as an asymptotic series for \( \Delta \left( \frac{u_r^2}{2} \right) \) in the limit \( u_\parallel^3 \frac{r_c}{b} \ll 1 \). When the series is substituted into the integral for \( \nu \), the following expression was obtained by O'Neil and Hjorth

\[
I(\bar{\varepsilon}) \approx (2.48) \bar{\varepsilon}^{1/5} \exp(-E\bar{\varepsilon}^{-2/5}) \quad \text{for} \quad \bar{\varepsilon} \ll 1
\]

where

\[
\bar{\varepsilon} \equiv \frac{r_c}{b} \quad \text{and} \quad E \equiv \frac{5}{6} (3\pi)^{2/5} 2^{1/5} \approx 2.35
\]

A more detailed evaluation by Rosenbluth gives

\[
I(\bar{\varepsilon}) \approx \left[ (10.2) \bar{\varepsilon}^{7/15} + (86.0) \bar{\varepsilon}^{11/15} \right] \exp(-E\bar{\varepsilon}^{-2/5})
\]
Monte Carlo Results Compared to Asymptotic Expressions for $r_c/b \ll 1$

$$\exp\left(\frac{F}{\epsilon - 2/5}\right) I(\epsilon)$$

$\bar{\epsilon} = \frac{r_c}{b}$