

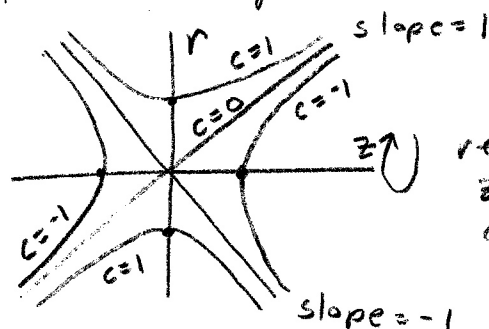
GP #1

$\frac{9}{10}$ M E Glinisky

① $x^2 + y^2 - z^2 = c = F(x, y, z)$

$$\frac{dF}{dx} = 2(x, y, -z)$$

let $r^2 = x^2 + y^2$



revolve about z-axis to get complete locus

if origin is excluded then even $F=0$ is a submanifold.

$\frac{4}{5}$

where are your conclusions?

obviously if $F > 0$ $x, y \neq 0 \forall z$
if $F < 0$ $z \neq 0 \forall x, y$

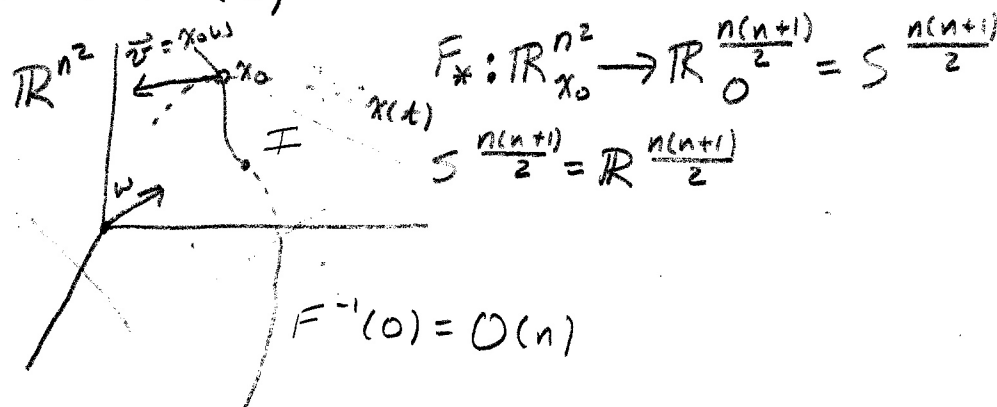
$\therefore$ if $F=c > 0$ $F=c$ can be solved for $x=f(z, y)$ since Jacobian $= \frac{dF}{dx} = 2x$ and $(0, y_0, z_0)$ is not an element of the manifold

if $F=c < 0$ $F=c$ can be solved for $z=f(x, y)$ since Jacobian $= \frac{dF}{dz} = -2z$ and $(x_0, y_0, 0)$ is not an element of manifold

if $F=c=0$ $F=c$ can not be solved for $z=f(x, y)$ $x=f(y, z)$ or $y=f(x, z)$ since origin is an element of manifold and $\frac{dF}{dx}|_0 = 0$

② Consider $SO(n)$ (i.e., matrices $X = ((x_{ij}))$ $i, j = 1, \dots, n$ $\nabla \tilde{X} = X^{-1}$ and $\det X > 0$)
and map $F = \tilde{X}X - I$ $F: \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{\frac{n(n+1)}{2}}$ $i, j = 1, \dots, n$

$$O(n) = F^{-1}(0)$$



now show F_* is onto let $\vec{w}$ be tangent to S at 0 . Need to find $\vec{v}$ a vector in $\mathbb{R}^{n^2}$ at x_0 $\nabla F_* \vec{v} = \vec{w}$. Let $X(t)$ be a curve of $n \times n$ matrices $\nabla X(0) = x_0$. Tangent vector at x_0 is $\dot{X}(0)$ the image curve $F(X(t)) = \tilde{X}(t)X(t) - I$ has tangent vector

at $t=0$ $\left. \frac{d}{dt} F(x(t)) \right|_{t=0} = \dot{\tilde{X}}(0) x_0 + \tilde{X}_0 \dot{X}(0)$ we wish this

to be ω . If $v = \dot{X}(0) = \frac{1}{2} x_0 \omega$ then $\tilde{X}_0 \dot{X}(0) = \frac{1}{2} \omega$

Since ω is symmetric $\tilde{\omega} = \omega \Rightarrow \tilde{\tilde{X}}_0 \dot{X}(0) = \tilde{X} \dot{X}(0) = \dot{\tilde{X}}(0) x_0$

$\therefore F_* v = \omega$ if $v = \frac{1}{2} x_0 \omega$. $O(n) = F^{-1}(0)$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

$\det: \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ is continuous and $\det = \pm 1$ on $O(n)$

$O(n) = SO(n) \cup \{x \in O(n) \mid \det x = -1\}$. Two subsets are not connected since $\det$ is continuous $\therefore SO(n)$ is also a submanifold.

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GP #2

M.E. Glinsky

?

③
⑤

$$|A| = \sum_{\sigma} \text{sgn}(\sigma) \prod_i x_{i\sigma(i)} = F(x) \quad F: \mathbb{R}^{n^2} \rightarrow \mathbb{R}$$

$$= \sum_i x_{ij} \underbrace{x_{ij}}_{\text{cofactor}} \quad (\forall j)$$

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$$\frac{\partial F}{\partial x_{ij}} = x_{ij} \quad ((x_{ij})) ((x_{ji})) = x x^t = |x| I = I$$

$$1 = |I| = |x x^t| = |x| |x^t| = |x^t|$$

∴ given $j, \exists i \rightarrow x_{ij} \neq 0$ or else $|x| = 0$

⇒ $\frac{\partial F}{\partial x_{ij}}$ has rank 1 ⇒ $SL(n)$ is a $n^2 - 1$ dimensional submanifold of $\mathbb{R}^{n^2}$

④
⑤

Define three coordinate systems: $\{U_x(x \neq 0); u_x^1 = \frac{x}{x}, u_x^2 = \frac{y}{x}\}$
 $\{U_y(y \neq 0); u_y^1 = \frac{x}{y}, u_y^2 = \frac{z}{y}\}$
 $\{U_z(z \neq 0); u_z^1 = \frac{x}{z}, u_z^2 = \frac{y}{z}\}$

where (x, y, z) is any point on line

obviously every point (x, y, z) lies in either U_x, U_y , or U_z

∴ every point $p \in P^2(\mathbb{R})$ lies in one of the coordinate patches

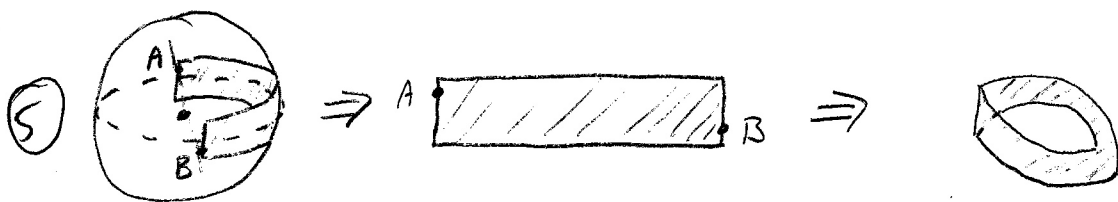
(i.e., $P^2(\mathbb{R})$ is covered by $\{U_\alpha\}$)

$$u_x^1 = \frac{1}{u_y^1} = \frac{u_z^2}{u_z^1} \quad u_y^1 = \frac{1}{u_x^1} = \frac{u_z^1}{u_z^2} \quad u_z^1 = \frac{u_y^1}{u_y^2} = \frac{1}{u_x^2}$$

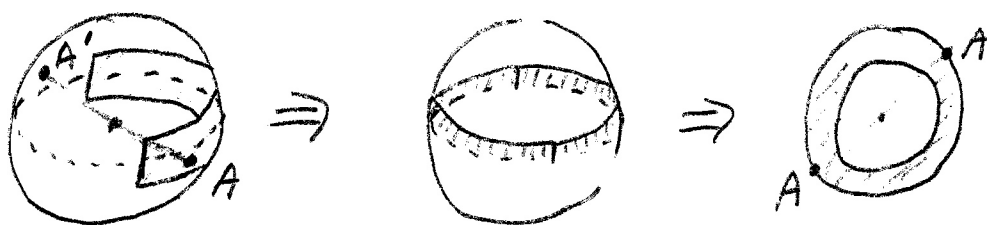
$$u_x^2 = \frac{u_y^2}{u_y^1} = \frac{1}{u_z^2} \quad u_y^2 = \frac{u_x^2}{u_x^1} = \frac{1}{u_z^1} \quad u_z^2 = \frac{1}{u_y^2} = \frac{u_x^1}{u_x^2}$$

note all $f_{j\alpha}(x_\alpha(p))$ are elements of C^∞

∴ $P^2(\mathbb{R})$ is a C^∞ 2-dimensional manifold



now identify the points in the lower hemisphere with the proper point in the upper hemisphere that lies on the line through the point in lower hemisphere and the center of the sphere.



⑥ For $\tilde{M}$ to be a non hausdorff topological space

(i) $\tilde{M}$ is a topological space

$\Rightarrow$ (i) $\tilde{M}$ and $\emptyset$ are open

(ii) U and V are open $\Rightarrow U \cap V$ is open

(iii) union of any collection of open subsets is open

② $\exists$ a pair of distinct points that have no pairs of disjoint neighborhoods

By definition of $\tilde{M}$, $\tilde{M}$ and $\emptyset$ are open

It is also obvious that conditions (ii) and (iii) are satisfied.

Now pick points P and Q . Any open neighborhood around P must contain all of R^+ and R^- also any open neighborhood of Q must contain part of R^+ and R^- . $\therefore$ It is impossible to have disjoint neighborhoods of P and Q .

$\tilde{M}$ is $R^- \text{ --- } \frac{P}{Q} \text{ --- } R^+$

Declare as open $\tilde{M}, \emptyset, R^+ \cup R^-, R^+ \cup R^- \cup \{P\}, R^+ \cup R^- \cup \{Q\}$

examine the transformation of $\|v_{p_0}\|^2 = \sum_i (v^i)^2$

$$v_{\beta}^i = \sum_j \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^j}(p_0) v_{\alpha}^j$$

$$\begin{aligned} \|v_{p_0}\|^2 &= \sum_i (v_{\beta}^i)^2 = \sum_i \left(\sum_j \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^j} v_{\alpha}^j \right)^2 \\ &= \sum_i \left(\sum_j \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^j} v_{\alpha}^j \right) \left(\sum_k \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^k} v_{\alpha}^k \right) \\ &= \sum_{i,j,k} \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^j} \frac{\partial x_{\beta}^i}{\partial x_{\alpha}^k} v_{\alpha}^j v_{\alpha}^k \stackrel{?}{=} \sum_i (v_{\alpha}^i)^2 \end{aligned}$$

consider $x_{\beta}^i = x_{\alpha}^i$ except $x_{\beta}^1 = 2x_{\alpha}^1$

$$\sum_{i=1}^{\infty} (v_{\beta}^i)^2 = 3(v_{\alpha}^1)^2 + \sum_{i=1}^{\infty} (v_{\alpha}^i)^2$$

$$\frac{dx}{dt} = x^2 \quad x(0) = x_0$$

$$-\frac{1}{x} + C = t \quad t = -\frac{1}{x} + \frac{1}{x_0} \quad \frac{1}{x_0} - t = \frac{1}{x}$$

$$\frac{1}{x} = \frac{1 - x_0 t}{x_0}$$

$$x = \frac{x_0}{1 - x_0 t}$$

solution defined for $t \in (-\infty, \frac{1}{x_0})$

or $t \in (\frac{1}{x_0}, \infty)$

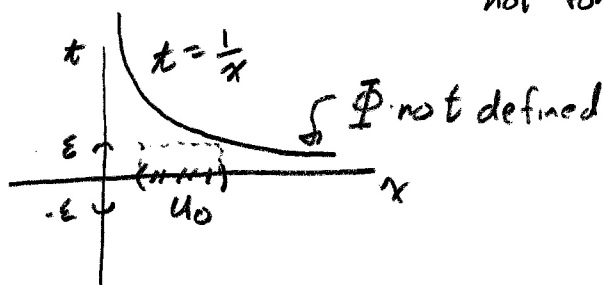
not for $t = \frac{1}{x_0}$

$$\Phi: U_0 \times (-\varepsilon, \varepsilon) \rightarrow E = \mathbb{R}$$

$$\Phi(x, t) = \frac{x}{1 - x t}$$

$$U_0 = (\frac{1}{2}, \frac{3}{2})$$

$$\varepsilon = \frac{1}{3/2} = \frac{2}{3}$$



GP #3

M E G linsky

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9 $\bar{a}_i = \sum_j a_j \frac{\partial x^j}{\partial \bar{x}^i} \quad \bar{v}^i = \sum_j v^j \frac{\partial \bar{x}^i}{\partial x^j}$

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$$\begin{aligned} \sum_i \bar{a}_i \bar{v}^i &= \sum_i \left(\sum_j a_j \frac{\partial x^j}{\partial \bar{x}^i} \right) \left(\sum_k v^k \frac{\partial \bar{x}^i}{\partial x^k} \right) \\ &= \sum_{ijk} a_j v^k \left(\frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^k} \right) = \sum_{jk} a_j v^k \left(\sum_i \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial x^k} \right) \\ &= \sum_{jk} a_j v^k \frac{\partial x^j}{\partial x^k} = \sum_{jk} a_j v^k \delta_k^j = \sum_j a_j v^j \quad \text{QED} \end{aligned}$$

$$\begin{aligned} \sum_i \bar{v}^i \bar{w}^i &= \sum_i \left(\sum_j v^j \frac{\partial \bar{x}^i}{\partial x^j} \right) \left(\sum_k w^k \frac{\partial \bar{x}^i}{\partial x^k} \right) \\ &= \sum_{ijk} v^j w^k \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial \bar{x}^i}{\partial x^k} = \sum_{jk} v^j w^k \underbrace{\left(\sum_i \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial \bar{x}^i}{\partial x^k} \right)}_{\text{does not equal } \delta_k^j} \end{aligned}$$

1. $\sum v^i w^i$ is not independent of coordinate system

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$$\begin{array}{lll} x^1 = x & \bar{x}^1 = r & x = r \sin \theta \cos \varphi \\ x^2 = y & \bar{x}^2 = \theta & y = r \sin \theta \sin \varphi \\ x^3 = z & \bar{x}^3 = \varphi & z = r \cos \theta \end{array}$$

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$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} \\ &= (\sin \theta \cos \varphi) \frac{\partial}{\partial x} + (\sin \theta \sin \varphi) \frac{\partial}{\partial y} + (\cos \theta) \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= (r \cos \theta \cos \varphi) \frac{\partial}{\partial x} + (r \cos \theta \sin \varphi) \frac{\partial}{\partial y} + (-r \sin \theta) \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \varphi} &= (-r \sin \theta \sin \varphi) \frac{\partial}{\partial x} + (r \sin \theta \cos \varphi) \frac{\partial}{\partial y} + (0) \frac{\partial}{\partial z} \end{aligned}$$

$$(\bar{g}_{ij}) = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} & \bar{g}_{13} \\ \bar{g}_{21} & \bar{g}_{22} & \bar{g}_{23} \\ \bar{g}_{31} & \bar{g}_{32} & \bar{g}_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$(\bar{\nabla} f)^i = \sum_j \bar{g}^{ij} \frac{\partial f}{\partial \bar{x}^j} \quad (\bar{g}^{ij}) = (\bar{g}_{ij})^{-1} \quad (\bar{g}^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2 \theta) \end{pmatrix}$$

$$f = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \varphi} \frac{\partial}{\partial \varphi}$$

$$(\langle \frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j} \rangle) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\bar{x}^1)^2 & 0 \\ 0 & 0 & (\bar{x}^1)^2 \sin^2 \bar{x}^2 \end{pmatrix}$$

from this it is obvious that $\{\frac{\partial}{\partial \bar{x}^i}\}$ form an orthogonal basis

$$\text{but } (\langle \frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^i} \rangle) \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. it is not an ON basis

$$\hat{e}_r = \frac{\partial}{\partial r} \quad \hat{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \quad \hat{e}_\varphi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \hat{e}_\varphi$$

(11) $\frac{dx}{dt} = \lambda(x) \nabla f \Rightarrow \frac{dx^i}{dt} = \lambda(x) g^{ij} \frac{\partial f}{\partial x^j}$ (E. Summation Notation)

(5) $\frac{df}{dt} \underset{\text{chain rule}}{=} \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} = \frac{\partial f}{\partial x^i} \lambda(x) g^{ij} \frac{\partial f}{\partial x^j} = 1$

$$\text{if } \lambda(x) = [g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}]^{-1}$$

g^{ij} is positive definite i.e. $g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \neq 0$

(12) $(G \circ F)_* \left(\frac{\partial}{\partial x^i} \right) = \sum_{\alpha} \frac{\partial z^\alpha}{\partial x^i} \frac{\partial}{\partial z^\alpha} = \sum_{\alpha} \frac{\partial z^\alpha}{\partial y^\beta} \frac{\partial y^\beta}{\partial x^i} \frac{\partial}{\partial z^\alpha}$

(5) $F_* \left(\frac{\partial}{\partial x^i} \right) = \sum_{\alpha} \frac{\partial z^\alpha}{\partial x^i} \frac{\partial}{\partial z^\alpha}$

$$G_* [F_* \left(\frac{\partial}{\partial x^i} \right)] = \sum_{\alpha} \frac{\partial z^\alpha}{\partial x^i} G_* \left(\frac{\partial}{\partial z^\alpha} \right) = \sum_{\alpha} \frac{\partial z^\alpha}{\partial x^i} \sum_{\beta} \frac{\partial z^\beta}{\partial z^\alpha} \frac{\partial}{\partial z^\beta} = (G \circ F)_* \left(\frac{\partial}{\partial x^i} \right)$$

$$G^*(dz^\alpha) = \sum_{\beta} \frac{\partial z^\alpha}{\partial z^\beta} dz^\beta$$

$$F^*[G^*(dz^\alpha)] = \sum_{\beta} \frac{\partial z^\alpha}{\partial z^\beta} F^*(dz^\beta) = \sum_{\beta} \frac{\partial z^\alpha}{\partial z^\beta} \sum_i \frac{\partial y^\beta}{\partial x^i} dx^i$$

$$= \sum_i \left(\sum_{\beta} \frac{\partial z^\alpha}{\partial z^\beta} \frac{\partial y^\beta}{\partial x^i} \right) dx^i = \sum_i \frac{\partial z^\alpha}{\partial x^i} dx^i = (G \circ F)^*(dz^\alpha)$$

$$\bar{v}^i \frac{d}{d\bar{x}^i} = \left(\frac{\partial \bar{x}^i}{\partial x^j} v^j \right) \left(\frac{\partial x^k}{\partial \bar{x}^i} \frac{d}{dx^k} + \frac{\partial v^k}{\partial \bar{x}^i} \frac{d}{dv^k} \right) \quad \bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j$$

$$= \left(\frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^i} \right) \left(v^j \frac{d}{dx^k} \right) = \frac{\partial x^k}{\partial \bar{x}^i} \left(v^j \frac{d}{dx^k} \right) = v^k \frac{d}{dx^k} \quad \frac{\partial \bar{v}^i}{\partial v^j} = \frac{\partial \bar{x}^i}{\partial x^j}$$

well defined

$$\bar{v}^i \frac{d}{d\bar{v}^i} = \left(\frac{\partial \bar{x}^i}{\partial x^j} v^j \right) \left(\frac{\partial x^k}{\partial \bar{v}^i} \frac{d}{dx^k} + \frac{\partial v^k}{\partial \bar{v}^i} \frac{d}{dv^k} \right) = v^k \frac{d}{dv^k}$$

$$\frac{\partial \bar{v}^i}{\partial \bar{x}^k} = \frac{d}{d\bar{x}^k} \left(\frac{\partial \bar{x}^i}{\partial \bar{x}^k} \bar{v}^l \right)$$

$$= \frac{\partial \bar{v}^l}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial \bar{x}^k} + \bar{v}^l \frac{d}{d\bar{x}^k} \left(\frac{\partial \bar{x}^i}{\partial \bar{x}^k} \right) = \bar{v}^l \frac{\partial^2 \bar{x}^i}{\partial \bar{x}^k \partial \bar{x}^k} \frac{\partial \bar{x}^r}{\partial \bar{x}^k}$$

$$= v^m \frac{\partial^2 x^i}{\partial x^r \partial x^k} \frac{\partial \bar{x}^l}{\partial x^m} \frac{\partial \bar{x}^r}{\partial x^k}$$

$$= v^k \frac{d}{dx^k} + v^j v^m \frac{\partial^2 x^k}{\partial \bar{x}^r \partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^m} \frac{\partial \bar{x}^r}{\partial x^i} \frac{\partial \bar{x}^i}{\partial x^j} \frac{d}{dv^k}$$

??

not well defined

$$\left(v^i \frac{d}{dv^i} \right) (a_i dx^i + b_i dv^i) = v^i b_i$$

assigns to a point in the target space
M.p., a vector field with magnitude and
direction of the point in the vector space

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$$d \otimes \beta(v, w) \equiv d(v) \beta(w) = \left[(a_i dx^i) \left(v^j \frac{d}{dx^j} \right) \right] \left[(b_k dx^k) \left(w^l \frac{d}{dx^l} \right) \right]$$

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$$= [a_i v^j \delta_j^i] [b_k w^l \delta_l^k] = (a_i v^i) (b_k w^k)$$

$$= a_i b_j v^i w^j = a_i b_j dx^i \left(v^k \frac{d}{dx^k} \right) dx^j \left(w^l \frac{d}{dx^l} \right)$$

$$= [a_i b_j (dx^i \otimes dx^j)](v, w)$$

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$$\bar{v}_j = \bar{g}_{ij} v^i = g_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} v^m \frac{\partial \bar{x}^i}{\partial x^m} = g_{kl} v^m \underbrace{\left(\frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial x^k}{\partial \bar{x}^i} \right)}_{\delta_m^k} \frac{\partial x^l}{\partial \bar{x}^j}$$

$$= g_{kl} v^k \frac{\partial x^l}{\partial \bar{x}^j} = v^k \frac{\partial x^l}{\partial \bar{x}^j}$$

2

$$\frac{\partial \bar{v}^i}{\partial \bar{x}^j} = \frac{d}{d\bar{x}^j} \left(\frac{\partial \bar{x}^i}{\partial x^k} v^k \right) = \frac{\partial v^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} + \frac{\partial^2 \bar{x}^i}{\partial \bar{x}^j \partial x^k} v^k$$

$$= \frac{\partial v^k}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k}$$

is a mixed tensor

$$\begin{aligned}
 \frac{\partial \bar{x}^i}{\partial \bar{x}^j} - \frac{\partial \bar{x}^j}{\partial \bar{x}^i} &= \frac{\partial v^k}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} + \frac{\partial^2 \bar{x}^i}{\partial \bar{x}^j \partial x^k} v^k \\
 &\quad - \frac{\partial v^k}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^k} - \frac{\partial^2 \bar{x}^j}{\partial \bar{x}^i \partial x^k} v^k \\
 &= \frac{\partial v^k}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} - \frac{\partial v^k}{\partial x^p} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^k}
 \end{aligned}$$

is not a mixed tensor

GP #4

MEGLinsky 10/20

(16)

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$$\left[\sum_I \alpha(e_I) \sigma^I \right] (e_J) = \sum_I \alpha(e_I) \delta_J^I = \alpha(e_J) \sum_{\substack{I \\ I=J}} \delta_J^I$$

Eg 2.9

obvious since I must be a permutation of J and since I must be increasing only one ordering can exist

$$\Rightarrow \alpha^I = \sum_I \alpha(e_I) \sigma^I$$

definition of p-form (ie above equation)

(17)

in $\mathbb{R}^n$ look at $\det g^{n-1}$

$$(\alpha \wedge \beta)_m = (\alpha \wedge \beta)(e_m) = \sum_{I, J} \delta_m^{IJ} ((e_I) \wedge (e_J))$$

Eq 8.5

$$= \sum_I \sum_J \delta_m^{IJ} \alpha_I \beta_J = \sum_k \alpha_{m_k} \sum_J \delta_m^{m_k J} \beta_J = \sum_k \alpha_{m_k} \sum_J \delta_{m_k m_k}^J \beta_J (-1)^{k-1}$$

too complicated

$i = m_k$ or $\delta_m^{IJ} = 0$

$$I = \{m_1, m_2, \dots, m_k, \dots, m_n\}$$

$$\bar{m}_k = \{m_{k+1}, m_{k+2}, \dots, m_n, m_1, \dots, m_{k-1}\}$$

$$M_k = \{m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_n\}$$

$\bar{m}_k =$ ordered M_k

$(n-k)(k-1)$ permutations

$$= \sum_k \alpha_{m_k} \delta_{m_k m_k}^{m_k} \beta_{m_k} (-1)^{k-1} = \sum_k \alpha_{m_k} \underbrace{\delta_{m_k m_k}^{m_k}}_1 \beta_{m_k} (-1)^{k-1} = \sum_k \alpha_{m_k} \beta_{m_k} (-1)^{k-1}$$

$$= \sum_k \alpha_{m_k} \beta_{\bar{m}_k} (-1)^{(n-k)(k-1)} = \sum_k \alpha_{m_k} \beta_{\bar{m}_k} \begin{cases} 1 \text{ odd } \\ (-1)^{n(n-1)/2} \text{ even } \end{cases}$$

$$\begin{cases} \text{if } n=2l+1 & (-1)^{(n-k)(k-1)} = 1 \\ \text{if } n=2l & (-1)^{(n-k)(k-1)} = (-1)^{k-1} \end{cases}$$

$m_k \bar{m}_k =$ cyclic permutation of M

$$m_k \bar{m}_k = \{m_k, m_{k+1}, \dots, m_n, m_1, \dots, m_{k-1}\}$$

$$[\alpha \wedge \beta^{(n-1)}]_M = \sum_{k=1}^n \alpha_{m_k} \beta_{\bar{m}_k} (-1)^{(k-1)(n+1)}$$

on $\mathbb{R}^n$

$$[\alpha \wedge \beta^2]_{ijk} = \alpha_i \beta_{jk} + \alpha_j \beta_{ki} + \alpha_k \beta_{ij}$$

on $\mathbb{R}^3$

$$\alpha^1 = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$$

$$\gamma^2 = c_1 dx^2 \wedge dx^3 + c_2 dx^3 \wedge dx^1 + c_3 dx^1 \wedge dx^2$$

$$\begin{aligned} \alpha^1 \wedge \gamma^2 &= (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (c_1 dx^2 \wedge dx^3 + c_2 dx^3 \wedge dx^1 + c_3 dx^1 \wedge dx^2) \\ &= a_1 c_1 dx^1 \wedge (dx^2 \wedge dx^3) + a_2 c_2 dx^2 \wedge (dx^3 \wedge dx^1) \\ &\quad + a_3 c_3 dx^3 \wedge (dx^1 \wedge dx^2) \\ &= (a_1 c_1 + a_2 c_2 + a_3 c_3) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \vec{a} \cdot \vec{c} \leftrightarrow \alpha^1 \wedge \gamma^2 \quad \text{vector scalar product}$$

$$\beta^1 = b_1 dx^1 + b_2 dx^2 + b_3 dx^3$$

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\eta^1 = n_1 dx^1 + n_2 dx^2 + n_3 dx^3$$

from p 68-69 of notes

$$\begin{aligned} \alpha^1 \wedge \beta^1 &= \underbrace{(a_2 b_3 - a_3 b_2)}_{c_1} dx^2 \wedge dx^3 \\ &\quad + \underbrace{(a_3 b_1 - a_1 b_3)}_{c_2} dx^3 \wedge dx^1 \\ &\quad + \underbrace{(a_1 b_2 - a_2 b_1)}_{c_3} dx^1 \wedge dx^2 \end{aligned}$$

$$\alpha^1 \wedge \beta^1 \wedge \eta^1 = \eta^1 \wedge (\alpha^1 \wedge \beta^1) = \left[n_1 (a_2 b_3 - a_3 b_2) + n_2 (a_3 b_1 - a_1 b_3) + n_3 (a_1 b_2 - a_2 b_1) \right] dx^1 \wedge dx^2 \wedge dx^3$$

$$\alpha^1 \wedge \beta^1 \wedge \eta^1 \leftrightarrow \vec{n} \cdot (\vec{a} \times \vec{b}) = [\vec{n}, \vec{a}, \vec{b}] = \text{vector triple product}$$

$$\begin{aligned}
[F^*(\alpha \wedge \beta^*)](v_I) &= F^* \left[\sum_{\substack{K \leq L}} \delta_I^{KL} \alpha(v_K) \beta(v_L) \right] \\
&= \sum_{\substack{K \leq L}} \delta_I^{KL} F^* [\alpha(v_K) \beta(v_L)] = \sum_{\substack{K \leq L}} \delta_I^{KL} \alpha(F_* v_K) \beta(F_* v_L) \\
&= \alpha(F_* v_K) \wedge \beta(F_* v_L) = [F^* \alpha(v_K)] \wedge [F^* \beta(v_L)] \\
&= [(F^* \alpha) \wedge (F^* \beta)](v_I)
\end{aligned}$$

It makes no sense
to take F^* of a number

GP #5

M E Glinisky

30
40

(20)

$T^*M^n (x^m, a_\nu) \quad m, \nu = 1, \dots, n$

$$\bar{x}^i = \bar{x}^i(x^m)$$

$$\bar{a}_i = \frac{\partial \bar{x}^j}{\partial x^i} a_j$$

$$\frac{\partial \bar{a}_i}{\partial a_j} = \frac{\partial^2 \bar{x}^j}{\partial x^i \partial x^j}$$

$$\frac{\partial(\bar{x}, \bar{a})}{\partial(x, a)} = \begin{vmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial a} \\ \frac{\partial \bar{a}}{\partial x} & \frac{\partial \bar{a}}{\partial a} \end{vmatrix} = \begin{vmatrix} \frac{\partial \bar{x}}{\partial x} & 0 \\ * & \frac{\partial x}{\partial \bar{x}} \end{vmatrix}$$

$$= \frac{\partial(\bar{x}^1, \dots, \bar{x}^n)}{\partial(x^1, \dots, x^n)} \frac{\partial(x^1, \dots, x^n)}{\partial(\bar{x}^1, \dots, \bar{x}^n)}$$

$$= 1 > 0 \quad \therefore \text{orientable}$$

(21)

$P^n(\mathbb{R})$

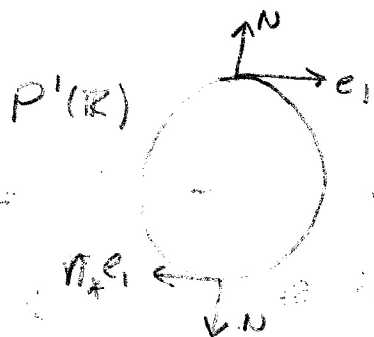
$$r: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

$$r: S^n \rightarrow S^n \quad r(x) = -x \quad \text{look at } \bar{x} = -x$$

Jacobian

$$\frac{\partial(\bar{x}^1, \dots, \bar{x}^n)}{\partial(x^1, \dots, x^n)} = \begin{vmatrix} -1 & & 0 \\ & -1 & 0 \\ 0 & & \ddots & -1 \end{vmatrix} = (-1)^{n+1} \begin{matrix} ?? \\ > 0 \text{ for } n \text{ odd} \\ < 0 \text{ for } n \text{ even} \end{matrix}$$

$P^n(\mathbb{R})$ is orientable for n odd
non-orientable for n even



note: $P^1(\mathbb{R})$ orientable

(23)

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\text{vol}^n = O(z) \sqrt{g(y)} \frac{\partial(x^1, \dots, x^n)}{\partial(y^1, \dots, y^n)} dy^1 \wedge \dots \wedge dy^n$$

$$\text{vol}^n = O(x) dx \wedge dy \wedge dz$$

$$\sqrt{g(y)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= \cos \theta [(r \cos \theta \cos \varphi)(r \sin \theta \cos \varphi) + (r \sin \theta \sin \varphi)(r \cos \theta \sin \varphi)]$$

$$r^2 \cos \theta [\cos^2 \varphi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi)]$$

$$= r^2 \cos^3 \theta (\cos^2 \varphi + \sin^2 \varphi) + r^2 \sin^3 \theta (\cos^2 \varphi + \sin^2 \varphi)$$

$$= r^2 \cos \theta (\cos^2 \theta + \sin^2 \theta) = r^2 \cos \theta$$

$$\text{vol}^n = O(r) \quad r^2 |\sin \theta| \, d\theta \, d\varphi \, d\psi$$

(24)

$$\alpha = \sum_{\mathbf{I}} a_{\mathbf{I}} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p} \quad X = \sum e_i X^i$$

$$i_X \alpha^p(X_2, \dots, X_p) = \alpha(X, X_2, \dots, X_p) = \sum_j \sum_{\mathbf{I}} X^j a_{\mathbf{I}} \sigma^{\mathbf{I}}(e_j, X_2, \dots, X_p)$$

$$= \sum_j \sum_{\mathbf{I}} \sum_{\substack{\mathbf{K} \\ \text{why?}}} X^j a_{\mathbf{I}} \delta_{1,2,\dots,p}^{K,K} \sigma^{i_1}(\tilde{X}_K) \sigma^{\tilde{\mathbf{I}}}(\tilde{X}_K) \quad \text{where } \tilde{X}_1 = e_j$$

$$= \sum_{k=1}^p \sum_j \sum_{\mathbf{I}} a_{\mathbf{I}} X^j (-1)^{k+1} \delta_{j,k}^{i_1} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_{k-1}} \wedge \sigma^{i_{k+1}} \wedge \dots \wedge \sigma^{i_p}$$

$$= \sum_k \sum_{\mathbf{I}} a_{\mathbf{I}} X^{i_k} (-1)^{k+1} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p}$$

$$= \sum_k \sum_{\mathbf{I}} a_{i_k i_1 \dots i_{k-1} i_{k+1} \dots i_p} X^{i_k} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_{k-1}} \wedge \sigma^{i_{k+1}} \wedge \dots \wedge \sigma^{i_p}$$

$$= \sum_j \sum_{\mathbf{J}} a_{j\mathbf{J}} X^j \sigma^{\mathbf{J}} = \sum_{\mathbf{J}} \underbrace{\left(\sum_j X^j a_{j\mathbf{J}} \right)}_{b_{\mathbf{J}}} \sigma^{\mathbf{J}}$$

(25)

$$(i) \quad \nabla(fg) \Leftrightarrow d(\alpha^0 \wedge \beta^0) = d\alpha^0 \wedge \beta^0 + \alpha^0 \wedge d\beta^0$$

$$\Leftrightarrow g \nabla f + f \nabla g$$

$$(ii) \quad \text{div}(f\vec{A}) \Leftrightarrow d(f^0 \wedge \alpha^2) = df^0 \wedge \alpha^2 + f^0 \wedge d\alpha^2$$

$$\Leftrightarrow \nabla f \cdot \vec{A} + f \text{div} \vec{A}$$

$$(iii) \quad \text{curl}(f\vec{A}) \Leftrightarrow d(f^0 \wedge \alpha^1) = df^0 \wedge \alpha^1 + f^0 \wedge d\alpha^1$$

$$\Leftrightarrow (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$$

$$(iv) \quad (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) \Leftrightarrow (i\vec{B} \wedge \alpha^2) \wedge (\gamma^1 \wedge \delta^1)$$

$$= -\alpha^2 \wedge i\vec{B}(\gamma^1 \wedge \delta^1) + i\vec{B}(\underbrace{\alpha^2 \wedge \gamma^1 \wedge \delta^1}_{\text{four form in } \mathbb{R}^3 = 0})$$

$$= -\alpha^2 \wedge i\vec{B}\gamma^1 \wedge \delta^1 + \alpha^2 \wedge \delta^1 \wedge i\vec{B}\gamma^1$$

$$= -\delta^1 \wedge \alpha^2 \wedge i\vec{B}\gamma^1 + \gamma^1 \wedge \alpha^2 \wedge i\vec{B}\delta^1$$

$$= (i\vec{B}\delta^1)(\gamma^1 \wedge \alpha^2) - (i\vec{B}\gamma^1)(\delta^1 \wedge \alpha^2)$$

$$= (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) \text{vol}_0 = [(\vec{B} \cdot \vec{D})(\vec{A} \cdot \vec{C}) - (\vec{B} \cdot \vec{C})(\vec{D} \cdot \vec{A})] \text{vol}_0$$

(26)

$$x^1 = u^1$$

$$x^2 = u^2$$

$$x^3 = f(u^1, u^2)$$

$$dS = \text{vol}^3 \left(\vec{N}, \frac{\partial \vec{x}}{\partial u^1}, \frac{\partial \vec{x}}{\partial u^2} \right) du^1 du^2$$

$$\frac{\vec{n}}{|\vec{n}|} = \vec{N}$$

$$\vec{n} = \frac{\partial \vec{x}}{\partial u^1} \times \frac{\partial \vec{x}}{\partial u^2}$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \partial_1 f \\ 0 & 1 & \partial_2 f \end{vmatrix} = \begin{pmatrix} -\partial_1 f \\ -\partial_2 f \\ 1 \end{pmatrix}$$

$$\text{vol}^3 \left(\vec{N}, \frac{\partial \vec{x}}{\partial u^1}, \frac{\partial \vec{x}}{\partial u^2} \right) = \begin{vmatrix} -\partial_1 f & -\partial_2 f & 1 \\ 1 & 0 & \partial_1 f \\ 0 & 1 & \partial_2 f \end{vmatrix} \frac{1}{\sqrt{(\partial_1 f)^2 + (\partial_2 f)^2 + 1}}$$

$$= \frac{(\partial_1 f)^2 + (\partial_2 f)^2 + 1}{\sqrt{(\partial_1 f)^2 + (\partial_2 f)^2 + 1}}$$

$$dS = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + 1} dx_1 dx_2$$

(2)
5

$$\phi^* \alpha^p(x_1, \dots, x_p) = \alpha^p(\phi_* x_1, \dots, \phi_* x_p)$$

$$\phi: \mathbb{R}^p \rightarrow M^n$$

$$\phi_*: \mathbb{R}_{u_0}^p \rightarrow M_{\phi(u_0)}^n$$

Since ϕ_* is of rank $< p$ $\phi_* x_p = \sum_{i=1}^{p-1} a^i \phi_* x_i$ why is x_p dependent on others?

$$= \alpha^p(\phi_* x_1, \dots, \phi_* x_{p-1}, \sum_{i=1}^{p-1} a^i \phi_* x_i)$$

$$= \sum_{i=1}^{p-1} a^i \alpha^p(\phi_* x_1, \dots, \phi_* x_{p-1}, \phi_* x_i) = 0$$

by skew symmetry of α^p

GP #6

ME Glinka

(28)

$$\mathcal{P} = \mathcal{P}^i \frac{\partial}{\partial x^i} \quad \mathcal{X} = \mathcal{X}^i \frac{\partial}{\partial x^i}$$

$$\mathcal{X}(\mathcal{P}(f)) = \mathcal{X}^j \frac{\partial}{\partial x^j} \left(\mathcal{P}^i \frac{\partial f}{\partial x^i} \right)$$

$$\mathcal{P}(\mathcal{X}(f)) = \mathcal{P}^j \frac{\partial}{\partial x^j} \left(\mathcal{X}^i \frac{\partial f}{\partial x^i} \right)$$

$$\mathcal{L}_{\mathcal{X}} \mathcal{P}(f) = [\mathcal{X}, \mathcal{P}](f) = \mathcal{X}(\mathcal{P}(f)) - \mathcal{P}(\mathcal{X}(f))$$

$$= \mathcal{X}^j \frac{\partial}{\partial x^j} \left(\mathcal{P}^i \frac{\partial f}{\partial x^i} \right) - \mathcal{P}^j \frac{\partial}{\partial x^j} \left(\mathcal{X}^i \frac{\partial f}{\partial x^i} \right)$$

$$= \mathcal{X}^j \mathcal{P}^i \frac{\partial^2 f}{\partial x^j \partial x^i} + \mathcal{X}^j \frac{\partial \mathcal{P}^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

$$- \mathcal{P}^j \mathcal{X}^i \frac{\partial^2 f}{\partial x^j \partial x^i} - \mathcal{P}^j \frac{\partial \mathcal{X}^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

$$= \mathcal{X}^j \frac{\partial \mathcal{P}^i}{\partial x^j} \frac{\partial f}{\partial x^i} - \mathcal{P}^j \frac{\partial \mathcal{X}^i}{\partial x^j} \frac{\partial f}{\partial x^i} = \left(\mathcal{X}^j \frac{\partial \mathcal{P}^i}{\partial x^j} - \mathcal{P}^j \frac{\partial \mathcal{X}^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}$$

$$= [\mathcal{X}, \mathcal{P}]^i \frac{\partial f}{\partial x^i}$$

$$\boxed{[\mathcal{X}, \mathcal{P}]^i = \mathcal{X}^j \frac{\partial \mathcal{P}^i}{\partial x^j} - \mathcal{P}^j \frac{\partial \mathcal{X}^i}{\partial x^j}}$$

GP #7 M E Glinsky

29 $(\theta \circ A - A \circ \theta)(\alpha^p \wedge \beta^q) = \theta[(\alpha^p \wedge \beta^q) + (-)^p \alpha^p \wedge (\theta \beta^q)] - A[(\theta \alpha^p) \wedge \beta^q + (-)^p \alpha^p \wedge (\theta \beta^q)]$

$\theta: \Lambda^p \rightarrow \Lambda^{p+1}$ (n even) derivation
 $A: \Lambda^p \rightarrow \Lambda^{p+1}$ (s odd) anti-derivation

$$= (\theta \circ A \alpha^p) \wedge \beta^q + (-)^{p+1} (A \alpha^p) \wedge (\theta \beta^q) + (\theta \alpha^p) \wedge (A \beta^q) + (-)^p \alpha^p \wedge (\theta \circ A \beta^q) - [(A \circ \theta \alpha^p) \wedge \beta^q + (\theta \alpha^p) \wedge (A \beta^q) + (-)^p (A \alpha^p) \wedge (\theta \beta^q) + (-)^p \alpha^p \wedge (A \circ \theta \beta^q)]$$

$$= [(\theta \circ A - A \circ \theta) \alpha^p] \wedge \beta^q + (-)^p \alpha^p \wedge [(\theta \circ A - A \circ \theta) \beta^q] + (-)^p [(-)^1 - 1] (A \alpha^p) \wedge (\theta \beta^q) + [1 - 1] (\theta \alpha^p) \wedge (A \beta^q)$$

$$= [(\theta \circ A - A \circ \theta) \alpha^p] \wedge \beta^q + (-)^p \alpha^p \wedge [(\theta \circ A - A \circ \theta) \beta^q]$$

$(\theta \circ A - A \circ \theta): \Lambda^{p+q} \rightarrow \Lambda^{p+q+1}$ odd $\therefore$ anti derivation

$B: \Lambda^q \rightarrow \Lambda^{q+1}$ (t odd) anti-derivation

$$(A \circ B + B \circ A)(\alpha^p \wedge \beta^q) = A[(B \alpha^p) \wedge \beta^q + (-)^p \alpha^p \wedge (B \beta^q)] + B[(A \alpha^p) \wedge \beta^q + (-)^p \alpha^p \wedge (A \beta^q)]$$

$$= (A \circ B \alpha^p) \wedge \beta^q + (-)^{p+t} (B \alpha^p) \wedge (A \beta^q) + (-)^p (A \alpha^p) \wedge (B \beta^q) + (-)^{2p} [\alpha^p \wedge (A \circ B) \beta^q] + (B \circ A \alpha^p) \wedge \beta^q + (-)^{p+s} (A \alpha^p) \wedge (B \beta^q) + (-)^p (B \alpha^p) \wedge (A \beta^q) + (-)^{2p} [\alpha^p \wedge (B \circ A) \beta^q]$$

$$= [(A \circ B + B \circ A) \alpha^p] \wedge \beta^q + (-)^{2p} \alpha^p \wedge [(A \circ B + B \circ A) \beta^q] + (-)^p [(-)^t + 1] [(B \alpha^p) \wedge (A \beta^q)] + (-)^p [1 + (-)^s] [(A \alpha^p) \wedge (B \beta^q)]$$

$$= [(A \circ B + B \circ A) \alpha^p] \wedge \beta^q + \alpha^p \wedge [(A \circ B + B \circ A) \beta^q]$$

$(A \circ B + B \circ A): \Lambda^{p+q} \rightarrow \Lambda^{p+q+1}$ even $\therefore$ derivation

(30)

On functions $(L_X \circ i_Y - i_Y \circ L_X)f$

$$= L_X(i_Y(f)) - i_Y(L_X(f)) = L_X(0) - i_Y(X(f))$$

$$= 0 - 0 = 0$$

On differentials of functions

$$(L_X \circ i_Y - i_Y \circ L_X)(df) = L_X(i_Y(df)) - i_Y(\overbrace{L_X(df)}^{dL_X f})$$

$$= [L_X(L_Y f) - L_X(\underbrace{d i_Y f}_{=0})] - [L_Y(L_X f) - d(\underbrace{i_Y L_X f}_{=0})]$$

$$= L_X L_Y f - L_Y L_X f = (XY - YX)f$$

$$= [X, Y]f = i_{[X, Y]} df$$

Thus both sides agree on functions and $d(\text{functions})$
 $\therefore$ agree on p -forms.

(31)

$$-d([u, v]) + u(d(v)) - v(d(u))$$

$$= -d\left[\left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j}\right) \frac{d}{dx^i}\right] + u(a_j Y^j) - v(a_j X^j)$$

$$= -a_i \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \right) + X^i \frac{d}{dx^i} (a_j Y^j) - Y^j \frac{d}{dx^j} (a_i X^i)$$

$$= X^i Y^j \frac{\partial a_j}{\partial x^i} - Y^j X^i \frac{\partial a_j}{\partial x^i}$$

$$= (X^i Y^j - Y^j X^i) d_i a_j$$

$$= X^i Y^j \delta_{ij}^{kl} d_k a_l = d\alpha\left(X^i \frac{d}{dx^i}, Y^j \frac{d}{dx^j}\right) = d\alpha(u, v)$$

32

$$L_u \alpha'(v) = u(\alpha(v)) - \alpha([u, v])$$

$$u = X^i \frac{\partial}{\partial x^i}$$

$$v = \bar{I}^j \frac{\partial}{\partial x^j}$$

$$\begin{aligned} &= u(a_i \bar{I}^i) - \alpha\left(\left(X^j \frac{\partial \bar{I}^i}{\partial x^j} - \bar{I}^j \frac{\partial X^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}\right) \\ &= X^j \frac{\partial}{\partial x^j} (a_i \bar{I}^i) - a_i \left(X^j \frac{\partial \bar{I}^i}{\partial x^j} - \bar{I}^j \frac{\partial X^i}{\partial x^j}\right) \\ &= X^j \bar{I}^i \frac{\partial a_i}{\partial x^j} + a_i \bar{I}^j \frac{\partial X^i}{\partial x^j} \\ &= \bar{I}^j \left(X^i \frac{\partial a_i}{\partial x^j} + a_i \frac{\partial X^i}{\partial x^j}\right) \\ &= \underbrace{\left[\left(X^i \frac{\partial a_i}{\partial x^j} + a_i \frac{\partial X^i}{\partial x^j}\right) dx^j\right]}_{L_u \alpha'} \underbrace{\left(\bar{I}^k \frac{\partial}{\partial x^k}\right)}_v \end{aligned}$$

33

$$A' = \vec{A} \cdot d\vec{x}$$

$$\frac{d}{dt} \int_{C_1} A' = \int_{C_1} \underbrace{\frac{\partial A'}{\partial t}}_{\frac{\partial \vec{A}}{\partial t} \cdot d\vec{x}} + \underbrace{i \vec{v} \underbrace{dA'}_{i \nabla \times A \text{ vol}_0}}_{\text{two form components } \nabla \times \vec{A}} + \underbrace{d \underbrace{i \vec{v} A'}_{0 \text{ form} = \vec{v} \cdot \vec{A}}}_{d(\vec{v} \cdot \vec{A}) = (\vec{\nabla} \cdot \vec{A}) \cdot d\vec{x} \text{ term} = \vec{\nabla}(\vec{v} \cdot \vec{A}) \cdot d\vec{x}}$$

$- i \vec{v} \beta^2$ covariant version of $\vec{v} \times \vec{B}$
 i.e. term = $[\vec{v} \times (\vec{\nabla} \times \vec{A})] \cdot d\vec{x}$

$$\begin{aligned} &= \int_{C_1} \frac{\partial \vec{A}}{\partial t} \cdot d\vec{x} + [\vec{v} \times (\vec{\nabla} \times \vec{A})] \cdot d\vec{x} + \vec{\nabla}(\vec{v} \cdot \vec{A}) \cdot d\vec{x} \\ &= \int_{C_1} \left[\frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A}) + \vec{\nabla}(\vec{v} \cdot \vec{A}) \right] \cdot d\vec{x} \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \int_{C_2} \vec{B} \cdot d\vec{S} &= \frac{d}{dt} \int_{C_2} \beta^2 = \int_{C_2} \underbrace{\frac{\partial \beta^2}{\partial t}}_{i \frac{\partial \vec{B}}{\partial t} \text{ vol}_0} + \underbrace{i \vec{v} d\beta^2}_{\nabla \cdot \vec{B} \text{ vol}_0} + \underbrace{d i \vec{v} \beta^2}_{- \vec{v} \times \vec{B} \text{ 1-form}} \\ &= \int_{C_2} \left[i \frac{\partial \vec{B}}{\partial t} + (\vec{\nabla} \cdot \vec{B}) \vec{v} - \vec{v} \times \vec{B} \right] \cdot d\vec{S} \\ &= \int \left[\frac{\partial \vec{B}}{\partial t} + (\vec{\nabla} \cdot \vec{B}) \vec{v} - \vec{v} \times \vec{B} \right] \cdot d\vec{S} \end{aligned}$$

$\beta^2 = i \vec{B} \text{ vol}_0$
 $dd' = i - \vec{v} \times \vec{B} \text{ vol}_0$

$$\begin{aligned}
 \frac{d}{dt} \int_{C_3} \rho dV &= \frac{d}{dt} \int_{C_3} \rho \text{vol}_0 = \frac{d}{dt} \int_{C_3} \rho^3 = \int_{C_3} \underbrace{\frac{d\rho^3}{dt}}_{\frac{d\rho}{dt} \text{vol}_0} + i\vec{v} \underbrace{\vec{d}\rho^3}_{=0} + \underbrace{\vec{d} i\vec{v} \rho^3}_{i\rho\vec{v} \text{vol}_0} \\
 &= \int_{C_3} \left[\frac{d\rho}{dt} + \vec{\nabla} \cdot (\rho\vec{v}) \right] \text{vol}_0 \\
 &= \int_{C_3} \left[\frac{d\rho}{dt} + \vec{\nabla} \cdot (\rho\vec{v}) \right] dV
 \end{aligned}$$

two form
 $\vec{\nabla} \cdot (\rho\vec{v}) \text{vol}_0$

(34)

$$p = p(\rho) \Rightarrow dp \wedge d\rho = 0$$

$$\frac{\partial v_\alpha}{\partial t} + v^\beta \frac{\partial v_\alpha}{\partial x^\beta} = \frac{\partial \phi}{\partial x^\alpha} - \frac{1}{\rho} \frac{\partial \rho}{\partial x^\alpha}$$

multiply
each side by
 dx^α
and sum

$$\frac{\partial v}{\partial t} + v^\beta \frac{\partial v_\alpha}{\partial x^\beta} dx^\alpha = \underbrace{\frac{\partial \phi}{\partial x^\alpha} dx^\alpha}_{d\phi} - \frac{1}{\rho} \underbrace{\frac{\partial \rho}{\partial x^\alpha} dx^\alpha}_{d\rho}$$

add Lie
derivative to both
sides

$$\frac{\partial v}{\partial t} + \mathcal{L}_{\vec{v}}(v) = \text{div} v + i\vec{v} dv - v^\beta \frac{\partial v_\alpha}{\partial x^\beta} dx^\alpha + d\phi - \frac{1}{\rho} d\rho$$

$$= \text{div} v + \underbrace{i\vec{v} dv}_{\text{div} \left(\frac{\partial v_\alpha}{\partial x^\beta} dx^\beta \wedge dx^\alpha \right)} - v^\beta \left(\frac{\partial v_\alpha}{\partial x^\beta} - \frac{\partial v_\beta}{\partial x^\alpha} \right) dx^\alpha + d\phi - \frac{1}{\rho} d\rho$$

use Cartan's formula

$$= \text{div} v - v^\beta \frac{\partial v_\beta}{\partial x^\alpha} dx^\alpha + d\phi - \frac{1}{\rho} d\rho = d \left(\frac{v^\beta v_\beta}{2} + \phi - f \right)$$

$$v^\beta v_\beta = \|v\|^2$$

where $df = \frac{1}{\rho} d\rho$
can be found since
 $\rho = \rho(p)$

$$\begin{aligned}
 (\text{true if } \frac{\partial g_{\alpha\beta}}{\partial x^\alpha} = 0) &\rightarrow \frac{1}{2} \left(v^\beta \frac{\partial v_\beta}{\partial x^\alpha} + v_\beta \frac{\partial v^\beta}{\partial x^\alpha} \right) dx^\alpha \\
 (\forall \alpha, \beta, \gamma) &\quad \frac{1}{2} d(v^\beta v_\beta) = \frac{1}{2} d(\|v\|^2)
 \end{aligned}$$

$$\frac{d}{dt} \oint_{\varphi(t)} v = \oint_{\varphi(t)} \frac{\partial v}{\partial t} + \mathcal{L}_{\vec{v}} v = \oint_{\varphi(t)} d \left(\frac{\|v\|^2}{2} + \phi - f \right) = \int \frac{\|v\|^2}{2} + \phi - f = 0$$

Stokes thm

Since φ has
no boundary

38
40

GP #8

M E Glinsky

35

(a) $d\beta^p = 0$ $d\alpha^q = 0$ (ie α^q and β^p are closed)

$$d(\alpha^q \wedge \beta^p) = \underbrace{d\alpha^q}_{=0} \wedge \beta^p + (-1)^q \alpha^q \wedge \underbrace{d\beta^p}_{=0} = 0 \quad \therefore d\alpha^q \wedge \beta^p \text{ is closed}$$

5

(b) $d\beta^p = 0$ and $d\omega^{q-1} = d\alpha^q$

$$\text{consider } d\alpha^q \wedge \beta^p = d\omega^{q-1} \wedge \beta^p = d\omega^{q-1} \wedge \beta^p + (-1)^{q-1} \omega^{q-1} \wedge \underbrace{d\beta^p}_{=0} \\ = d(\omega^{q-1} \wedge \beta^p) \quad \therefore d\alpha^q \wedge \beta^p \text{ is exact}$$

36

$$\beta^p = b_I(z) dz^I$$

$$J^p: \begin{matrix} I \times \mathbb{R}^n \\ t \quad x \quad y \end{matrix} \rightarrow \mathbb{R}^n \quad y = (1-t)x$$

K : $p+1$ forms on $I \times \mathbb{R}^n \rightarrow p$ forms on $\mathbb{R}^n$

$$K(b_J(t, x) dx^J) = 0$$

$$K(a_I(t, x) dt \wedge dx^I) = \left[\int_0^1 a_I(t, x) dt \right] dx^I$$

$$\alpha^{p-1} = K J^* \beta^p$$

$$= -K[J^*(b_I(y) dy^I)]$$

$$= -K[b_I((1-t)x) (1-t)^p dx^I - b_{jJ}((1-t)x) x^j (1-t)^{p-1} dt \wedge dx^J] ?$$

$$= \int_0^1 dt b_{jJ}((1-t)x) x^j (1-t)^{p-1} dx^J$$

$$= \int_0^1 d\bar{t} \bar{t}^{p-1} (b_{jJ}(\bar{t}x) x^j) dx^J$$

$$\bar{t} = 1-t \\ d\bar{t} = -dt$$

a_J

Write out in more detail. Paper is cheap!

Definition of pull back. Separated out dt terms.

Definition of K . $K(dt \wedge \text{term}) = 0$

change of variable $\bar{t} = 1-t$

(note: unlabel $\sum_i \sum_j$)

for $n=3$

$p=1$

$$\beta^1 = b_1 dx^1 + b_2 dx^2 + b_3 dx^3$$

$$\alpha^0 = \int_0^1 [x^1 b_1(tx) + x^2 b_2(tx) + x^3 b_3(tx)] dt$$

$$\beta^1 = d\alpha^0$$

2

$$\textcircled{p=2} \quad \beta^2 = b_{12} dx^1 \wedge dx^2 + b_{13} dx^1 \wedge dx^3 + b_{23} dx^2 \wedge dx^3$$

$$\begin{aligned} \alpha^1 &= \int_0^1 t dt \left[(x^1 b_{11} + x^2 b_{21} + x^3 b_{31}) dx^1 \right. \\ &\quad + (x^1 b_{12} + x^2 b_{22} + x^3 b_{32}) dx^2 \\ &\quad \left. + (x^1 b_{13} + x^2 b_{23} + x^3 b_{33}) dx^3 \right] \\ &= \int_0^1 t \left\{ -[x^2 b_{12}(tx) + x^3 b_{13}(tx)] dx^1 \right. \\ &\quad + [x^1 b_{12}(tx) - x^3 b_{23}(tx)] dx^2 \\ &\quad \left. + [x^1 b_{13}(tx) + x^2 b_{23}(tx)] dx^3 \right\} t dt \end{aligned}$$

$$\beta^2 = d\alpha^1$$

$$\textcircled{p=3} \quad \beta^3 = b dx^1 \wedge dx^2 \wedge dx^3$$

$$\begin{aligned} \alpha^2 &= \int_0^1 t dt \left\{ x^1 b_{123} dx^2 \wedge dx^3 + x^2 b_{213} dx^1 \wedge dx^3 \right. \\ &\quad \left. + x^3 b_{312} dx^1 \wedge dx^2 \right\} \\ &= \int_0^1 t^2 b(tx) \{ x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 \\ &\quad + x^3 dx^1 \wedge dx^2 \} dt \\ \beta^3 &= d\alpha^2 \end{aligned}$$

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$$\begin{aligned} d\phi^1 &= 2j d\phi \\ &= H_i dx^i = d(2j\phi) \end{aligned} \quad ds^2 = dr^2 + dz^2 + r^2 d\phi^2$$

$$\vec{B} = B^i \frac{\partial}{\partial x^i} \quad B^i = g^{ik} H_k \quad \text{for Riemannian Manifold}$$

K

$$ds^2 = \sum_{i,j} g_{ij} dx^i \otimes dx^j$$

$$\text{vol}^3 = o(\gamma) \sqrt{g(\gamma)} dy^1 \wedge dy^2 \wedge dy^3 \quad g(\gamma) = \det(g_{ij})$$

$$\therefore (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r^2 \end{pmatrix}$$

$$\text{vol}^3 = r dr \wedge dz \wedge d\phi$$

$$\begin{aligned} B^i &= g^{ik} H_k \Rightarrow \vec{B} = \frac{2j}{r^2} \frac{\partial}{\partial \phi} \quad \beta^2 = i_{\vec{B}} \text{vol}^3 = i_{\frac{2j}{r^2} \frac{\partial}{\partial \phi}} (r dr \wedge dz \wedge d\phi) \\ &= \frac{2j}{r} dr \wedge dz \\ d(A^1) &= d(j \ln r^2 dz) = \beta^2 \quad \boxed{A^1 = j \ln r^2} \end{aligned}$$

38

let $V^2(t)$ be any two cycle of $\mathbb{R}^3 - C$ moving with C

Consider $\int_{V^2(t)} \mathcal{B}^2$ where $\mathcal{B}^2 = i_{\vec{E}} \text{vol}^3$ $d\mathcal{B}^2 = 0$

look at $\frac{d}{dt} \int_{V^2(t)} \mathcal{B}^2 = \int_{V^2(t)} \left(\frac{d\mathcal{B}^2}{dt} + i_{\vec{v}} d\mathcal{B}^2 + d(i_{\vec{v}} \mathcal{B}^2) \right)$

$\frac{d\mathcal{B}^2}{dt} = d\mathcal{E}^1$ (Faraday's law) $\Rightarrow \int_{V^2(t)} d\mathcal{E}^1 + 0 + d(i_{\vec{v}} \mathcal{B}^2) = \int_{V^2(t)} d(\mathcal{E}^1 + i_{\vec{v}} \mathcal{B}^2)$

$\leftarrow$ no magnetic monopoles $V^2(t)$

$= \int_{\partial V^2(t)} \mathcal{E}^1 + i_{\vec{v}} \mathcal{B}^2 = 0$

$\partial V^2(t) = \emptyset$

$\therefore \int_{V^2(t)} \mathcal{B}^2 = \text{constant in time}$

let the current in the wire C go to zero
the field $\vec{B}$ will vanish hence the constant must be zero

39

$ds^2 = d\alpha^2 + \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\varphi^2)$

$\text{vol}^3 = \sin \theta \sin^2 \alpha d\alpha d\theta d\varphi$

$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \alpha & 0 \\ 0 & 0 & \sin^2 \alpha \sin^2 \theta \end{pmatrix}$

$\vec{B} = D(\alpha) \frac{d}{d\alpha} \mathcal{B}^2 = i_{D(\alpha) \frac{d}{d\alpha}} (\sin \theta \sin^2 \alpha d\alpha d\theta d\varphi)$

$= D(\alpha) \sin \theta \sin^2 \alpha d\theta d\varphi$

$d\mathcal{B}^2 = \left[\frac{dD(\alpha)}{d\alpha} \sin^2 \alpha + 2 D(\alpha) \sin \alpha \cos \alpha \right] \sin \theta d\alpha d\theta d\varphi = 0$

$\Rightarrow \frac{dD(\alpha)}{D(\alpha)} = - \frac{d(\sin^2 \alpha)}{\sin^2 \alpha} \Rightarrow D(\alpha) = \frac{A}{\sin^2 \alpha}$

$\mathcal{B}^2 = A \sin \theta d\theta d\varphi$

consider $\int_{S^2(\alpha)} \mathcal{B}^2 = \int_{S^2(\alpha)} A \sin \theta d\theta d\varphi = A \int \sin \theta d\theta d\varphi = 4\pi A$

$S^2(\alpha) = \partial V^3$

$\int_{V^3} d\mathcal{B}^2 = \int_{V^3} \rho \text{vol}^3 = 4\pi q \Rightarrow A = q \Rightarrow \mathcal{B}^2 = q \sin \theta d\theta d\varphi$

slides thm $\rightarrow$

$V^3 = \text{volume containing } q, \vec{n}$ is outward pointing normal

$$\vec{D} = \frac{q}{\sin^2 \alpha} \frac{d}{d\alpha} = D^k \frac{\partial}{\partial x^k} \quad \mathcal{E}^1 = \frac{q}{\sin^2 \alpha} d\alpha = E_\alpha dx^\alpha \quad E_\alpha = q_{,\alpha} D^\alpha$$

$$= d(-q \cot \alpha) = d\phi \quad \boxed{\phi = -q \cot \alpha}$$

$$d\mathcal{E}^1 = d\left(\frac{q}{\sin^2 \alpha} d\alpha\right) = d\left(\frac{q}{\sin^2 \alpha}\right) \wedge d\alpha = \frac{-2q}{\sin^3 \alpha} \cos \alpha d\alpha \wedge d\alpha = 0$$

= 0 by skew symmetry

40

$$\mathcal{H}^1 = 2j d\phi \quad ds^2 = d\alpha^2 + d\theta^2 \sin^2 \theta d\phi^2$$

$$(H_\alpha, H_\theta, H_\phi) = (0, 0, 2j)$$

$$(B^\alpha, B^\theta, B^\phi) = (0, 0, \frac{2j}{\sin^2 \theta})$$

S

$$\vec{B} = \vec{H} = \frac{2j}{\sin^2 \theta} \frac{\partial}{\partial \phi}$$

$$((g_{ij})) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2 \theta \end{pmatrix}$$

$$((g^{ij})) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sin^2 \theta \end{pmatrix}$$

$$\beta^2 = i_{\vec{B}} \omega$$

$$= i_{\frac{2j}{\sin^2 \theta} \frac{\partial}{\partial \phi}} (\sin \theta d\alpha \wedge d\theta \wedge d\phi)$$

$$= \frac{2j}{\sin \theta} d\alpha \wedge d\theta$$

$$\omega = \sin \theta d\alpha \wedge d\theta \wedge d\phi$$

$$((i_{\vec{B}} \beta)_I) = X^i b_{iI} \quad \vec{X} = X^i \frac{\partial}{\partial x^i} \quad \beta = b_J dx^J$$

$$d\beta^2 = d\left(\frac{2j}{\sin \theta} d\alpha \wedge d\theta\right)$$

$$= d\left(\frac{2j}{\sin \theta}\right) \wedge d\alpha \wedge d\theta = 2j \left(\frac{-\cos \theta}{\sin^2 \theta}\right) d\theta \wedge d\alpha \wedge d\theta = 0$$

= 0

41

consider $r: S^n \rightarrow S^n$ $S^n \subset \mathbb{R}^{n+1} \ni \vec{x} \mapsto -\vec{x}$
 $x \mapsto y = -x$

$$\deg(r) = \sum_{x \in r^{-1}(y)} \text{sgn } r_*(x) = \sum_{r \text{ is 1 to 1}} \text{sgn } r_*(x) = \det \left(\frac{d(y)}{d(x)} \right)$$

where $\text{sgn } r_*(x) = \begin{cases} +1 & \text{if } r_*: S^n_x \rightarrow S^n_y \text{ is orientation preserving} \\ -1 & \text{if } r_* \text{ reverses orientation} \end{cases}$

$$= \det \begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{bmatrix} = (-1)^{n+1}$$

n+1 coords
because embedded in $\mathbb{R}^{n+1}$

But S^n has n coords.

42

M^n closed (compact w/o boundary)

$M^n = \partial U$ $U \subset \mathbb{R}^{n+1}$ U is compact oriented manifold w/ boundary M^n

$\vec{v}$ unit vector field on M^n defines a map.

5 $v: M^n \rightarrow S^n \subset \mathbb{R}^{n+1} \ni x \in M^n \mapsto \vec{v}(x) \in S^n$

$$\text{Index of } \vec{v} \text{ on } M^n \equiv \deg(v) = \frac{1}{A_n} \int_{M^n} v^* \text{vol}_S^n$$

$$A_n = \int_{S^n} \text{vol}_S^n$$

$$= \frac{1}{A_n} \int_U v^* \text{vol}_S^n = \frac{1}{A_n} \int_U d[v^* \text{vol}_S^n]$$

can apply Stokes Thm since v is well defined on U

$$= \frac{1}{A_n} \int_U v^* [d \text{vol}_S^n] = 0$$

because $d \text{vol}_n$ on n -dimensional manifold equals 0

Consider compact region of $\mathbb{R}^{n+1}$ defined by $U - \{B_\alpha\}$ (Set U with small ball B_α excised about P_α)

the boundary of this region will be $M + \{\partial B_\alpha\}$ (Union of M and boundaries of B_α). $\partial(U - \{B_\alpha\})$ is oriented by the outward pointing normal to $U - \{B_\alpha\}$. But this is opposite to the normal defined for B_α . This reverses the the defined orientation for ∂B_α $\therefore$ one has the following minus sign

$$0 = \text{Index}[\vec{v} \text{ on } U - \{B_\alpha\}] = \frac{1}{A_n} \int_{\partial(U - \{B_\alpha\})} v^* \text{vol}_S^n$$

$$= \frac{1}{A_n} \int_M v^* \text{vol}_S^n - \frac{1}{A_n} \int_{\{\partial B_\alpha\}} v^* \text{vol}_S^n$$

∂B_α are disjoint

$$= \frac{1}{A_n} \int_M v^* \text{vol}_S^n - \sum_\alpha \frac{1}{A_n} \int_{\partial B_\alpha} v^* \text{vol}_S^n$$

$$= \text{Index}(\vec{v} \text{ on } M) - \sum_\alpha \text{Index}(\vec{v} \text{ on } \partial B_\alpha)$$

$$\Rightarrow \text{Index}(\vec{v} \text{ on } M) = \sum_{\alpha=1}^k \text{Index}(\vec{v} \text{ on } \partial B_\alpha)$$

44 (i)

$$\begin{aligned} \phi: \mathbb{R}^n &\rightarrow M^n \\ \nu: M^n &\rightarrow S^n \\ i: S^n &\rightarrow \mathbb{R}^{n+1} \end{aligned}$$

$$\begin{aligned} v &= i \circ \nu \circ \phi: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \\ (u^1, \dots, u^n) &\text{ coords for } \mathbb{R}^n \\ (v^1, \dots, v^{n+1}) &\text{ coords for } \mathbb{R}^{n+1} \end{aligned}$$

$$v^j = v^j(u^1, \dots, u^n)$$

$$\vec{v} = v^j \frac{\partial}{\partial u^j}$$

$$\text{now } v_* \left(\frac{\partial}{\partial u^i} \right) = \frac{\partial v^j}{\partial u^i} \frac{\partial}{\partial v^j} = \frac{\partial \vec{v}}{\partial u^i} \quad \text{Eq 6.4}$$

where have you used S^n ?

$$\text{need } \nu_* \left(\frac{\partial}{\partial u^i} \right), \nu: M^n \rightarrow S^n$$

$$(ii) \text{ Index}(\vec{v} \text{ on } M^n) = \frac{1}{A_n} \int_{M^n} v^* \text{vol}_S^n$$

$$= \frac{1}{A_n} \int v^* \text{vol}_S^n = \frac{1}{A_n} \sum_{\{U, \phi\}} \int_{U, \phi} v^* \text{vol}_S^n$$

$$\therefore \text{contribution for patch } U \text{ is } = \frac{1}{A_n} \int_U \phi^* v^* \text{vol}_S^n$$

$$= \frac{1}{A_n} \int_U \phi^* \nu^* \text{vol}_S^n \left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

$$= \frac{1}{A_n} \int_U \phi^* \nu^* [i^* i_N \text{vol}_{\mathbb{R}^{n+1}}^{n+1}] \left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

$$= \frac{1}{A_n} \int_U \underbrace{\phi^* \nu^* i^*}_{(i \circ \nu \circ \phi)^*} (i_N \text{vol}_{\mathbb{R}^{n+1}}^{n+1}) \left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

$$= \frac{1}{A_n} \int_U i_N \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\nu_* \frac{\partial}{\partial u^1}, \dots, \nu_* \frac{\partial}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

$$= \frac{1}{A_n} \int_U \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\vec{N}, \frac{\partial \vec{v}}{\partial u^1}, \dots, \frac{\partial \vec{v}}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

now $\vec{N}$ to S^n embedded in $\mathbb{R}^{n+1}$ is $\vec{v} = \nu^* \frac{\partial}{\partial u^i}$

3

$$\therefore \text{Index}(\vec{v} \text{ on } M^n) = \sum_{\alpha} \frac{1}{A_n} \int_{U_\alpha} \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\vec{v}, \frac{\partial \vec{v}}{\partial u^1}, \dots, \frac{\partial \vec{v}}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

if $\vec{v}$ is not of unit magnitude but is non vanishing on M^n then $\frac{\vec{v}}{\|\vec{v}\|}$ is of unit magnitude and should be used in place of $\vec{v}$

$$\text{Since } \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\frac{\vec{v}}{\|\vec{v}\|}, \frac{\partial(\vec{v}/\|\vec{v}\|)}{\partial u^1}, \dots, \frac{\partial(\vec{v}/\|\vec{v}\|)}{\partial u^n} \right)$$

$$= \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\frac{\vec{v}}{\|\vec{v}\|}, \frac{1}{\|\vec{v}\|} \frac{\partial \vec{v}}{\partial u^1} + \vec{v} \frac{\partial \|\vec{v}\|}{\partial u^1}, \dots, \frac{1}{\|\vec{v}\|} \frac{\partial \vec{v}}{\partial u^n} + \vec{v} \frac{\partial \|\vec{v}\|}{\partial u^n} \right)$$

skew
symmetry
of forms

$$= \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\frac{\vec{v}}{\|\vec{v}\|}, \frac{1}{\|\vec{v}\|} \frac{\partial \vec{v}}{\partial u^1}, \dots, \frac{1}{\|\vec{v}\|} \frac{\partial \vec{v}}{\partial u^n} \right)$$

$$= \frac{1}{\|\vec{v}\|^{n+1}} \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\vec{v}, \frac{\partial \vec{v}}{\partial u^1}, \dots, \frac{\partial \vec{v}}{\partial u^n} \right)$$

$$\Rightarrow \text{Index}(\vec{v} \text{ on } M^n) = \sum_{\alpha} \frac{1}{A_n} \int_{U_\alpha} \frac{1}{\|\vec{v}\|^{n+1}} \text{vol}_{\mathbb{R}^{n+1}}^{n+1} \left(\vec{v}, \frac{\partial \vec{v}}{\partial u^1}, \dots, \frac{\partial \vec{v}}{\partial u^n} \right) du^1 \wedge \dots \wedge du^n$$

(45) consider $\vec{v}_x = (1-x)\vec{N} + x\vec{V}$ $\vec{v}_0 = \vec{N}$ $\vec{v}_1 = \vec{V}$
 where $\|\vec{N}\| = \|\vec{V}\| = 1$

assume $\vec{v}_{x_0} = 0 = (1-x_0)\vec{N} + x_0\vec{V}$

either ① $\vec{N}$ and $\vec{V}$ are L.I. $v_{x_0} \neq 0$ $\otimes$

or ② $\vec{N}$ and $\vec{V}$ are L.D

then either

① $\vec{N} = \vec{V} \Rightarrow$

what about $\vec{N} = \lambda \vec{V}$ $\lambda \neq 0$

$0 = (1-x_0)\vec{N} + x_0\vec{N} \Rightarrow 1=0$ $\otimes$

or ② $\vec{N} = -\vec{V}$ $\otimes$ by assumption

$\therefore v_x \neq 0$ $0 \leq x \leq 1$ $\checkmark$

$\Rightarrow \begin{cases} \text{index } \vec{v}_0 = \text{index } \vec{v}_1 \Rightarrow \boxed{\text{index } \vec{N} = \text{index } \vec{V} = +1} \\ \text{for } \vec{v} = \vec{N} \quad v: S^1 \rightarrow S^1 \text{ is identity } \therefore \text{index of } \vec{N} \text{ on } S^1 = \frac{1}{A_1} \int_{S^1} v^* \text{vol}_{S^1}^1 = \frac{A_1}{A_1} = 1 \end{cases}$

(46)

$C_1, \vec{r} = \vec{r}_1(\theta) \quad 0 \leq \theta \leq 2\pi$

$C_2: S^1 \rightarrow \mathbb{R}^3$

$i: S^2 \rightarrow \mathbb{R}^3$

$\phi: \mathbb{R}^2 \rightarrow T^2$

$\alpha: T^2 \rightarrow S^2$

$C_2, \vec{r} = \vec{r}_2(\varphi) \quad 0 \leq \varphi \leq 2\pi$

$$\mathcal{L}(\theta, \varphi) = \frac{\vec{r}_2(\varphi) - \vec{r}_1(\theta)}{\|\vec{r}_2(\varphi) - \vec{r}_1(\theta)\|} \equiv \frac{\vec{r}_{12}}{r_{12}} \equiv \hat{r}_{12}$$

$$f = i \circ \alpha \circ \phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\mathcal{L}_K(C_1, C_2) \equiv \deg(\mathcal{L}) = \frac{1}{A_2} \int_{T^2} \mathcal{L}^* \text{vol}_{S^2}^2 \quad \text{where } A_2 = \int_{S^2} \text{vol}_{S^2}^2 = 4\pi$$

$$= \frac{1}{4\pi} \int_{T^2} \mathcal{L}^* \text{vol}_{S^2}^2 = \frac{1}{4\pi} \int_{T^2} \mathcal{L}^* i^* i_{\hat{r}_{12}} \text{vol}_{\mathbb{R}^3}^3 = \frac{1}{4\pi} \int_U \phi^* \alpha^* i^* i_{\hat{r}_{12}} \text{vol}^3\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right) d\theta \wedge d\varphi$$

$$= \frac{1}{4\pi} \int_U \text{vol}^3(\hat{r}_{12}, f_* \frac{\partial}{\partial \theta}, f_* \frac{\partial}{\partial \varphi}) = \frac{1}{4\pi} \int_U \text{vol}(\hat{r}_{12}, \frac{\partial \hat{r}_{12}}{\partial \theta}, \frac{\partial \hat{r}_{12}}{\partial \varphi}) d\theta \wedge d\varphi$$

$$= \frac{1}{4\pi} \int_U \text{vol}(\hat{r}_{12}, \frac{1}{r_{12}} \left[\frac{d\vec{r}_{12}}{d\theta} - \hat{r}_{12} \left(\frac{d\vec{r}_{12}}{d\theta} \cdot \hat{r}_{12} \right) \right], \frac{1}{r_{12}} \left[\frac{d\vec{r}_{12}}{d\varphi} - \hat{r}_{12} \left(\frac{d\vec{r}_{12}}{d\varphi} \cdot \hat{r}_{12} \right) \right]) d\theta \wedge d\varphi$$

$$= \frac{1}{4\pi} \int_U \text{vol}(\hat{r}_{12}, \frac{1}{r_{12}} \frac{d\vec{r}_{12}}{d\theta}, \frac{1}{r_{12}} \frac{d\vec{r}_{12}}{d\varphi}) d\theta \wedge d\varphi \quad (\text{from p. 44 directly})$$

$$= \frac{1}{4\pi} \int_U \frac{1}{(r_{12})^3} \left[\left(\vec{r}_{12} \times \frac{d\vec{r}_{12}}{d\theta} \right) \cdot \frac{d\vec{r}_{12}}{d\varphi} \right] d\theta \wedge d\varphi = \frac{1}{4\pi} \int_U \frac{1}{r_{12}^3} \left(\vec{r}_{12} \times \frac{d\vec{r}_{12}}{d\varphi} \right) \cdot \frac{d\vec{r}_{12}}{d\theta} d\theta \wedge d\varphi$$

by
chain
rule

$$= \frac{1}{4\pi} \int_{S'} \left[\int_{S'} \frac{1}{r_{12}^3} (\vec{r}_{12} \times \frac{d\vec{r}_2}{d\varphi} d\varphi) \cdot \frac{d\vec{r}_1}{d\theta} d\theta \right]$$

$$\stackrel{5}{=} \frac{1}{4\pi} \int_{C_1} \left[\int_{C_2} \frac{1}{r_{12}^3} (\vec{r}_{12} \times d\vec{r}_2) \right] \cdot d\vec{r}_1$$

(note: could also have directly applied problem #44 to get this result much more quickly)

(47) assume that there is a current $I=1$ in C_2

(Hint: $\vec{H}(\vec{r}_1) = \int_{C_2} \frac{\vec{r}_{12} \times d\vec{r}_2}{r_{12}^3}$)

Ampere's Law $d\vec{H} = 4\pi \vec{j} + \frac{d\vec{D}}{dt}$

$$\int_{C_1} d\vec{H} = \int_{C_1} \left[\int_{C_2} \frac{1}{r_{12}^3} (\vec{r}_{12} \times d\vec{r}_2) \right] \cdot d\vec{r}_1 = \int_{C_2} d\vec{H} = \int_{C_2} 4\pi \vec{j}$$

Hint $\nearrow$ $\nwarrow$ Ampere's Law

$$= 4\pi \int j^z = 4\pi (\omega^2 \cdot C_2)$$

~~scribbles~~

$$\Rightarrow 4\pi \left\{ \frac{1}{4\pi} \int_{C_1} \left[\int_{C_2} \frac{1}{r_{12}^3} (\vec{r}_{12} \times d\vec{r}_2) \right] \cdot d\vec{r}_1 \right\} = 4\pi L_u(C_1, C_2)$$

5

$$= 4\pi (\omega^2 \cdot C_2)$$

$$\Rightarrow \boxed{\omega^2 \cdot C_2 = L_u(C_1, C_2)}$$

(49)

$$\theta_p \equiv dz^p - \sum f_i^p(x, y) dx^i$$

$$\theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_r = dz^1 \wedge dz^2 \wedge \dots \wedge dz^r + (\text{mixed } dy \wedge dx \text{ terms}) \\ + (dx \text{ term})$$

5

it is obvious that $dz^1 \wedge dz^2 \wedge \dots \wedge dz^r \neq 0$
because of the form of the remaining terms there
will be no term with all dz 's $\therefore \theta_1, \dots, \theta_r \neq 0$
 $\therefore \theta_p$ are independent

60/60

GP Midterm Problems

ME Glinsky

Good work.

M1

$$w = dp_i \wedge dq^i \quad dw = \underbrace{d(dp_i)}_0 \wedge dq^i + (-1)^n dp_i \wedge \underbrace{d(dq^i)}_0 = 0$$

5

$$(i_X w)_j = \sum_k X^k w_{kj} \quad \text{where} \quad (w_{kj}) = \downarrow \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{matrix} q \\ p \end{matrix}$$

$$w_I = w(e_I) = \sum_i dp_i(e_j) dq^i(e_k) - dq^i(e_j) dp_i(e_k)$$

$$\det(w_{kj}) = 1 \neq 0 \quad \therefore \text{transformation is non-singular}$$

M2

$$w^n = \underbrace{w \wedge \dots \wedge w}_{n\text{-times}} = (dp_1 \wedge dq^1) \wedge \dots \wedge (dp_n \wedge dq^n) \quad (*)$$

can not repeat dp_i or dq^i in grassman product. because term will equal zero. Since w^n is a form on $2n$ dimensional manifold w^n will be of form

$A (dp_1 \wedge dq^1) \wedge \dots \wedge (dp_n \wedge dq^n)$. The order of the $(dp_i \wedge dq^i)$ can be changed without altering the sign since it will take an even # of exchanges.

5

But what is the value of A . This is equivalent to how many times permutations of distinct double wedge products can be formed. Imagine the process as follows pick a $dp_i \wedge dq^i$ from the first term in $(*)$ (n possibilities). Then another $dp_i \wedge dq^i$ from the second term in $(*)$ (will be $n-1$ possibilities since i_2 cannot equal i_1). Continue this process until the last term is reached. $\therefore$ there will be $n(n-1)\dots(3)(2)(1) = n!$ permutations of double wedge products. Now need to get it into

$$dq^1 \wedge \dots \wedge dq^n \wedge dp_1 \wedge \dots \wedge dp_n \quad \text{order this will add a } (-1)^{\frac{n(n+1)}{2}} \text{ sign} \quad \therefore w^n = (-1)^{\frac{n(n+1)}{2}} n! dq^1 \wedge \dots \wedge dq^n \wedge dp_1 \wedge \dots \wedge dp_n$$

first calculate $i_X \omega$

M3

$$X = \sum_i \frac{dq^i}{dt} \frac{\partial}{\partial q^i} + \frac{dp_i}{dt} \frac{\partial}{\partial p_i}$$

$$i_X \omega = i \left(\frac{dq^i}{dt} \frac{\partial}{\partial q^i} + \frac{dp_i}{dt} \frac{\partial}{\partial p_i} \right) (dp_i \wedge dq^i)$$

$$(i_X \omega)_j = X^i \omega_{ij} \Rightarrow i_X \omega = \begin{pmatrix} \frac{dq^i}{dt} & \frac{dp_i}{dt} \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} dq^i \\ dp_i \end{pmatrix}$$

now calculate $-dH$

$$-dH = -\left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \right)$$

$$= -\frac{dq^i}{dt} dp_i + \frac{dp_i}{dt} dq^i$$

finally $i_X \omega = -dH \Leftrightarrow \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i = -\frac{dp_i}{dt} dq^i + \frac{dq^i}{dt} dp_i$

5

equating components

$$\left. \begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i} \\ \frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i} \end{aligned} \right\} (\forall i)$$

M4

assume X defines a hamiltonian flow

$$L_X \omega = L_X (dp_i \wedge dq^i) = L_X (dp_i) \wedge dq^i + dp_i \wedge (L_X dq^i)$$

$$= d(L_X p_i) \wedge dq^i + dp_i \wedge d(L_X q^i)$$

$$= d(X(p_i)) \wedge dq^i + dp_i \wedge d(X(q^i)) \quad \left\{ X = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right.$$

$$= d\left(-\frac{\partial H}{\partial q^i}\right) \wedge dq^i + dp_i \wedge d\left(\frac{\partial H}{\partial p_i}\right)$$

$$= -d\left(\frac{\partial H}{\partial q^i} dq^i\right) + (-)^i d\left(\frac{\partial H}{\partial p_i} dp_i\right)$$

$$= -d\left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i\right)$$

$$= -d(dH) = 0$$

$$\begin{aligned} i_X d\omega + di_X \omega \\ = d(-dH) = 0 \end{aligned}$$

5

$$\mathcal{L}_X \omega^n = \mathcal{L}_X (\omega_1 \wedge \dots \wedge \omega_n)$$

$$= \sum_{i=1}^n \omega_1 \wedge \dots \wedge \underbrace{\mathcal{L}_X \omega_i}_{\text{ith position}} \wedge \dots \wedge \omega_n$$

$$= 0$$

$\mathcal{L}_X \omega = 0$ from part a

(M5)

look at $H: T^*M^n \rightarrow \mathbb{R}$ assume H is C^{2n} or greater
 i.e. by AP Morse Sard's thm of p22, the set of $E \in H(T^*M^n)$
 that are not regular values has measure zero.

E is not a regular value $\Leftrightarrow H^{-1}(E)$ has a critical point

$x \in H^{-1}(E)$ is a critical point $\Leftrightarrow F_*: T^*M^n_x \rightarrow \mathbb{R}_{H(x)}$
 is not onto

F_* is not onto $\Leftrightarrow dH = 0$

$\therefore$ the set of E when $dH = 0$ for some $x \in H^{-1}(E)$
 is a set of measure 0.

(M6)

$$i: V_E^{2n-1} \rightarrow T^*M^n$$

$$\tau^{2n-1} \equiv i^* \tau^{2n-1}$$

let σ and σ' be two $(2n-1)$ forms that satisfy $\omega^n = dH \wedge \sigma^{2n-1}$

Consider $0 = dH \wedge (\sigma - \sigma')$

$$0 = dH \wedge (\sigma - \sigma') [N, T_2, \dots, T_{2n}] \rightarrow \begin{cases} dH(T_i) = 0 \\ dH(N) \neq 0 \end{cases}$$

$$= dH(N) (\sigma - \sigma') [T_2, \dots, T_{2n}]$$

$$0 = (\sigma - \sigma') [T_2, \dots, T_{2n}]$$

$$= (\sigma - \sigma') [i_* T_2, \dots, i_* T_{2n}]$$

$$= (i^* \sigma - i^* \sigma') [T_2, \dots, T_{2n}]$$

$\left\{ \begin{array}{l} dH(T_i) = 0 \\ dH(N) \neq 0 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} i_* T = T \text{ def of inclusion map} \end{array} \right.$

$\therefore i^* \sigma = i^* \sigma' \Rightarrow \tau$ is a well defined form

also vector in V_E^{2n-1}

(M7)

$dH(T_i) = 0$
and $i_* \omega(T_2, \dots, T_{2n}) = \omega(T_2, \dots, T_{2n})$

$$dH \wedge i \frac{\nabla H}{\|\nabla H\|^2} \text{vol}^{2n} \left(\frac{\nabla H}{\|\nabla H\|}, T_2, \dots, T_{2n} \right)$$

$$= dH \left(\frac{\nabla H}{\|\nabla H\|} \right) \text{vol}^{2n} \left(\frac{\nabla H}{\|\nabla H\|}, T_2, \dots, T_{2n} \right)$$

$$= \frac{dH(\nabla H)}{\|\nabla H\|} \frac{\text{vol}^{2n}}{\|\nabla H\|} \left(\frac{\nabla H}{\|\nabla H\|}, T_2, \dots, T_{2n} \right)$$

$$= \frac{\|\nabla H\|^2}{\|\nabla H\|} \frac{\text{vol}^{2n}}{\|\nabla H\|} \left(\frac{\nabla H}{\|\nabla H\|}, T_2, \dots, T_{2n} \right)$$

$$= \text{vol}^{2n} \left(\frac{\nabla H}{\|\nabla H\|}, T_2, \dots, T_{2n} \right)$$

$$= (\pm 1) (1)$$

$\uparrow$
depends on orientation

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}$$

$$df = \frac{\partial f}{\partial x^i} dx^i$$

$$(df(\nabla f)) = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial f}{\partial x^i} = \|\nabla f\|^2$$

since vectors are orthonormal

M8

$$\left. \begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i} \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ \frac{dH}{dt} &= \frac{\partial H}{\partial t} \end{aligned} \right\} \text{Hamilton's equations} + \left\{ \frac{dH}{dt} = \frac{\partial H}{\partial t} \right\}$$

$$X = \frac{dq_i}{dt} \frac{\partial}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}$$

assume hamilton's equations
then $X = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}$
and $\frac{dH}{dt} = X(H) = \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$
($dp_i \wedge dq_i - dH \wedge dt$)

Consider $i_X \Omega = i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \right)$

$$(i_X \Omega)_j = X^i \Omega_{ij}$$

$$(\Omega_{ij}) = \begin{pmatrix} 0 & -I & \frac{\partial H}{\partial q_i} \\ I & 0 & -\frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_i} & \frac{\partial H}{\partial p_i} & 0 \end{pmatrix}$$

$$i_X \Omega(V) = X^i \Omega_{ij} V^j$$

$$i_X \Omega = \left(\frac{\partial H}{\partial p_i} \mid -\frac{\partial H}{\partial q_i} \mid 1 \right) \begin{pmatrix} 0 & -I & \frac{\partial H}{\partial q_i} \\ I & 0 & -\frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_i} & \frac{\partial H}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} dq_i \\ dp_i \\ dt \end{pmatrix}$$

$$= \left(-\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \right) dq_i + \left(-\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i} \right) dp_i$$

$$\left(-\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} \right) dt = 0$$

Now assume $0 = i_X \Omega = i \left(\frac{dq_i}{dt} \frac{\partial}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \right) (dp_i \wedge dq_i - dH \wedge dt)$

$$= \left(\frac{dq_i}{dt} \mid \frac{dp_i}{dt} \mid 1 \right) \begin{pmatrix} 0 & -I & \frac{\partial H}{\partial q_i} \\ I & 0 & -\frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_i} & \frac{\partial H}{\partial p_i} & 0 \end{pmatrix} \begin{pmatrix} \frac{dq_i}{dt} \\ \frac{dp_i}{dt} \\ dt \end{pmatrix}$$

$$= \left(\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} \right) dq_i + \left(-\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i} \right) dp_i$$

$$\left(-\frac{dq_i}{dt} \frac{\partial H}{\partial q_i} - \frac{dp_i}{dt} \frac{\partial H}{\partial p_i} \right) dt$$

$$= \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dq^i}{dt} \frac{\partial H}{\partial q^i} + \frac{dp_i}{dt} \frac{\partial H}{\partial p_i} = \frac{dH}{dt} - \frac{dH}{dt} = 0$$

in summary $\{H, E\} \Rightarrow \left\{ \frac{dH}{dt} = \frac{\partial H}{\partial t} \right\}$ and $\{i_X \Omega = 0\}$

$$\{i_X \Omega = 0\} \Rightarrow \{H, E\} \text{ and } \left\{ \frac{dH}{dt} = \frac{\partial H}{\partial t} \right\}$$

or $\{H, E\} + \left\{ \frac{dH}{dt} = \frac{\partial H}{\partial t} \right\} \Leftrightarrow \{i_X \Omega = 0\}$

(M9)

$$\mathcal{L}_X \Omega = \mathcal{L}_X (dp_i \wedge dq^i - dH \wedge dt)$$

$$= d(\mathcal{L}_X dp_i) \wedge dq^i + dp_i \wedge d(\mathcal{L}_X q^i) - d(\mathcal{L}_X H) \wedge dt - dH \wedge d(\mathcal{L}_X t)$$

$$i_X d\Omega + d(i_X \Omega)$$

$$= d(\mathcal{L}_X (p_i)) \wedge dq^i + dp_i \wedge d(\mathcal{L}_X (q^i)) - d(\mathcal{L}_X (H)) \wedge dt - dH \wedge d(\mathcal{L}_X t)$$

5

$$= d\left(-\frac{\partial H}{\partial q^i}\right) \wedge dq^i + dp_i \wedge d\left(\frac{\partial H}{\partial p_i}\right) - d\left(\frac{\partial H}{\partial t}\right) \wedge dt - dH \wedge \underbrace{d(1)}_{=0}$$

$$= -d\left(\frac{\partial H}{\partial q^i} dq^i\right) - d\left(\frac{\partial H}{\partial p_i} dp_i\right) - d\left(\frac{\partial H}{\partial t} dt\right)$$

$$= -d\left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt\right) = -d(dH) = 0$$

stokes thm

(M10)

$$\int_A \Lambda = \int_A d\Lambda = \int_A \Omega = 0 \Rightarrow \oint_{\partial A} \Lambda = \oint_{\partial A} \Phi = 0$$

Ω will operate on two tangent vectors to A but $\Omega(T_1, T_2) = 0$ as shown below

let $T_1 = a_1 X + b_1 Y$
 $T_2 = a_2 X + b_2 Y$

$$X = \frac{dp_i}{dt} \frac{\partial}{\partial p_i} + \frac{dq^i}{dt} \frac{\partial}{\partial q^i} + \frac{\partial}{\partial t} \quad (Y, X \text{ are independent and tangent to } A)$$

X generates a hamiltonian flow, $i_X \Omega = 0$

$$\Omega(T_1, T_2) = a_1 a_2 \Omega(X, X) + b_1 a_2 \Omega(Y, X) + b_2 a_1 \Omega(X, Y) + b_1 b_2 \Omega(Y, Y)$$

$$= 0 - b_1 a_2 i_X \Omega(Y) + b_2 a_1 i_X \Omega(Y) + 0 = 0$$

5

M11

5

Cartan's formula

now apply Stokes then to second term

$$\delta S \equiv \left[\frac{d}{d\epsilon} \int_C \Lambda \right]_{\epsilon=0} = \int_C \mathcal{L}_J \Lambda = \int_C i_J d\Lambda + d i_J \Lambda$$

$$d\Lambda \equiv \Omega \quad \rightarrow \quad = \int_C i_J d\Lambda + \int_{\partial C} i_J \Lambda \xrightarrow{0}$$

$$= \int_C i_J \Omega$$

$J = a_i \frac{d}{dp_i}$ on ∂C

$$i_J \Lambda = i_{a_i \frac{d}{dp_i}} p_i dq^i - H dt$$

$$= (p_i dq^i - H dt) (a_i \frac{d}{dp_i})$$

$$= 0$$

M12

$$\delta S = \int_C i_J \Omega$$

$$= \int_a^b i_J \Omega(\tau(s)) ds = \int_a^b \Omega(J(s), \tau(s)) ds$$

$$= - \int_a^b \Omega(\tau(s), J(s)) ds = - \int_a^b i_{\tau} \Omega(J(s)) ds$$

$$= - \int_a^b (i_{\tau} \Omega)_i J^i(s) ds = 0 \quad \forall J^i(s) \Rightarrow J^i(a) = J^i(b) = 0$$

for $i=0, \dots, n$
and $J^i(a), J^i(b)$ finite
 $i=n+1, \dots, 2n$
(p coordinates)

then $(i_{\tau} \Omega)_i \equiv 0 (\forall i)$

$\therefore i_{\tau} \Omega = 0$

fundamental lemma of calculus of variations

5

Two complications to this proof can arise:

- (1) The curve extends over more than one coordinate patch. In this case use partition of unity to integrate. The form on each coordinate patch can be found to vanish by restricting the variation to only be non-zero on that patch.
- (2) The final integral for which the fundamental lemma is applied is actually a sum of integrals - one for each component. Again as in complication #1 restrict $J^i(s)$ to be non-zero for only one value of i . This leads to the conclusion that each component must be identically zero separately.

GP #1

M E Glinisky

48

$$\begin{aligned}x &= \cos \omega t \\y &= \sin \omega t \\z &= kt\end{aligned}$$

$$\vec{r}(t) = \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ kt \end{pmatrix} \quad \vec{v}(t) = \begin{pmatrix} -\omega \sin \omega t \\ \omega \cos \omega t \\ k \end{pmatrix}$$

$$\vec{a}(t) = \begin{pmatrix} -\omega^2 \cos \omega t \\ -\omega^2 \sin \omega t \\ 0 \end{pmatrix}$$

$$v^2 = \omega^2 + k^2$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega \sin \omega t & \omega \cos \omega t & k \\ -\omega^2 \cos \omega t & -\omega^2 \sin \omega t & 0 \end{vmatrix} = \begin{pmatrix} k\omega^2 \sin \omega t \\ -k\omega^2 \cos \omega t \\ \omega^3 \end{pmatrix}$$

$$\|\vec{v} \times \vec{a}\|^2 = k^2 \omega^4 + \omega^6$$

$$K = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\omega^2(k^2 + \omega^2)^{1/2}}{(k^2 + \omega^2)^{3/2}} = \frac{\omega^2}{k^2 + \omega^2}$$

60
60

GP #2

M E G l i n s k y

(49). $-e i u \mathcal{F} = -e i u [\mathcal{E} \wedge dt + \frac{1}{c} \mathcal{B}]$

$$= -e i \gamma \frac{d}{dt} + \gamma \vec{v} [\mathcal{E} \wedge dt + \frac{1}{c} \mathcal{B}]$$

$$= -e \gamma (i \frac{d}{dt} + i \vec{v}) (\mathcal{E} \wedge dt + \frac{1}{c} \mathcal{B})$$

$$= -e \gamma \left[i \frac{d}{dt} (\mathcal{E} \wedge dt) + i \frac{d}{dt} \frac{1}{c} \mathcal{B} + i \vec{v} (\mathcal{E} \wedge dt) + i \vec{v} \frac{1}{c} \mathcal{B} \right]$$

= 0 since
B does not
contain $\frac{d}{dt}$

$$= -e \gamma \left[\underbrace{\mathcal{E} \left(\frac{1}{dt} \right) dt}_{=0 \text{ since } \mathcal{E} \text{ does not contain } dt} - \mathcal{E} dt \left(\frac{1}{dt} \right) + (i \vec{v} \mathcal{E}) dt - \underbrace{\mathcal{E} dt (\vec{v})}_{=0 \text{ since } \vec{v} \text{ does not contain } \frac{d}{dt}} + \frac{1}{c} i \vec{v} \mathcal{B} \right]$$

$$= -\gamma e (i \vec{v} \mathcal{E}) dt + \gamma e [\mathcal{E} - \frac{1}{c} i \vec{v} \mathcal{B}] = f \quad \text{QED}$$

(50)

$$x = \gamma(x' + v t')$$

$$y = y'$$

$$t = \gamma(t' + v x')$$

$$z = z'$$

$$\mathcal{F} = \mathcal{E} \wedge dt + \mathcal{B} = \mathcal{E}' \wedge dt' + \mathcal{B}'$$

$$d\vec{x} = \frac{\partial \vec{x}}{\partial x^i} dx^i$$

$$\begin{aligned} dz &= dz' & dx &= \gamma(dt' + v dx') \\ dz &= dz' & dx &= \gamma(dx' + v dt') \end{aligned}$$

$$\mathcal{E} \wedge dt + \mathcal{B} = (E_1 dx + E_2 dy + E_3 dz) \wedge dt + B_1 dz \wedge dx + B_2 dx \wedge dy + B_3 dy \wedge dz$$

$$= [E_1 \gamma(dx' + v dt') + E_2 dy' + E_3 dz'] \wedge \gamma(dt' + v dx') + B_1 dz' \wedge \gamma(dt' + v dx') + B_2 \gamma(dx' + v dt') \wedge dy' + B_3 \gamma(dx' + v dt') \wedge dz'$$

$$= \gamma^2 (1-v^2) E_1 dx' \wedge dt' + \gamma E_2 dy' \wedge dt' + \gamma E_3 dz' \wedge dt' + \gamma v E_2 dz' \wedge dx' - \gamma v E_3 dx' \wedge dy' - \gamma v E_1 dx' \wedge dy' + \gamma v E_3 dz' \wedge dx' + B_1 dv' \wedge dz' + \gamma B_2 dz' \wedge dx' + \gamma B_3 dx' \wedge dy'$$

$$= [E_1 dx' + \gamma(E_2 - v B_3) dy' + \gamma(E_3 + v B_2) dz'] \wedge dt' + B_1 dy' \wedge dz' + \gamma(B_2 + v E_3) dz' \wedge dx' + \gamma(B_3 - v E_2) dx' \wedge dy'$$

$$= (E'_1 dx' + E'_2 dy' + E'_3 dz') \wedge dt' + B'_1 dy' \wedge dz' + B'_2 dz' \wedge dx' + B'_3 dx' \wedge dy'$$

$$= \mathcal{E}' \wedge dt' + \mathcal{B}' \Rightarrow$$

$$E'_1 = E_1$$

$$E'_2 = \gamma(E_2 - v B_3)$$

$$E'_3 = \gamma(E_3 + v B_2)$$

$$B'_1 = B_1$$

$$B'_2 = \gamma(B_2 + v E_3)$$

$$B'_3 = \gamma(B_3 - v E_2)$$

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$$\begin{aligned}x^1 &= a \sin \theta \cos \varphi \\x^2 &= a \sin \theta \sin \varphi \\x^3 &= a \cos \theta\end{aligned}$$

$$\cos \varphi = \frac{\sum_{i=1}^3 \frac{\partial x^i}{\partial u^1} \frac{\partial x^i}{\partial u^2}}{\|\vec{x}_u\| \|\vec{x}_v\|} = \vec{x}_u \cdot \vec{x}_v$$

$$\vec{x}_\theta = a \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix} \quad \vec{x}_\varphi = a \begin{pmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{pmatrix}$$

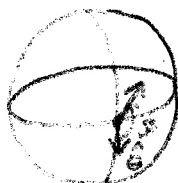
$$\vec{x}_\theta \cdot \vec{x}_\varphi = \vec{x}_\varphi \cdot \vec{x}_\theta = a^2 [-\sin \theta \cos \theta \cos \varphi \sin \varphi + \sin \theta \cos \theta \sin \varphi \cos \varphi + 0] = 0$$

$$\vec{x}_\theta \cdot \vec{x}_\theta = a^2 [\cos^2 \theta \cos^2 \varphi + \cos^2 \theta \sin^2 \varphi + \sin^2 \theta] = a^2 [\cos^2 \theta + \sin^2 \theta] = a^2$$

$$\vec{x}_\varphi \cdot \vec{x}_\varphi = a^2 [\sin^2 \theta \sin^2 \varphi + \sin^2 \theta \cos^2 \varphi] = a^2 \sin^2 \theta$$

$$((g_{ij})) = \begin{pmatrix} \vec{x}_\theta \cdot \vec{x}_\theta & \vec{x}_\theta \cdot \vec{x}_\varphi \\ \vec{x}_\varphi \cdot \vec{x}_\theta & \vec{x}_\varphi \cdot \vec{x}_\varphi \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 \theta \end{pmatrix}$$

52



$$\cos \varphi = \frac{\langle \hat{\theta}, \frac{d\vec{x}}{dt} \rangle}{\|\hat{\theta}\| \|\frac{d\vec{x}}{dt}\|} = \cos t = A$$

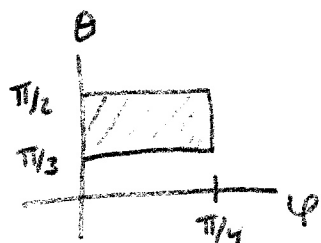
$$\langle \hat{\theta}, \frac{d\vec{x}}{dt} \rangle \frac{1}{A} = \|\frac{d\vec{x}}{dt}\|$$

$$\begin{aligned}\frac{d\vec{x}}{dt} &= \frac{\partial \vec{x}}{\partial u^1} \frac{du^1}{dt} \\&= \frac{\partial \vec{x}}{\partial \theta} + \frac{\partial \vec{x}}{\partial \varphi} \frac{d\varphi}{d\theta} \\&= a \left(\hat{\theta} + \frac{d\varphi}{d\theta} \hat{\varphi} \right)\end{aligned}$$

$$\begin{aligned}\text{let } t &= \theta \\u^1 &= \theta \\u^2 &= \varphi\end{aligned}$$

$$\begin{aligned}\langle \hat{\theta}, \frac{d\vec{x}}{dt} \rangle &= \langle \hat{\theta}, a \left(\hat{\theta} + \frac{d\varphi}{d\theta} \hat{\varphi} \right) \rangle = a \langle \hat{\theta}, \hat{\theta} \rangle + a \frac{d\varphi}{d\theta} \langle \hat{\theta}, \hat{\varphi} \rangle \\&= a \cdot 1 + a \cdot 0 = a\end{aligned}$$

$$L = \int_{\frac{\pi}{2}}^0 \|\frac{d\vec{x}}{d\theta}\| d\theta = \int_{\frac{\pi}{2}}^0 \frac{a}{\cos \varphi} d\theta = -\frac{\pi}{2} \frac{a}{\cos \varphi} = \frac{\pi a/2}{1/\cos \varphi}$$



$$|A| = (g_{uu} g_{vv} - g_{uv}^2)^{1/2} du dv$$

$$= (g_{\theta\theta} g_{\varphi\varphi} - g_{\theta\varphi}^2)^{1/2} d\theta d\varphi = a^2 \sin \theta d\theta d\varphi$$

$$A = \int_{\theta=\pi/3}^{\pi/2} \int_{\varphi=0}^{\pi/4} a^2 \sin \theta d\theta d\varphi = \frac{\pi}{4} a^2 \int_{\cos \theta=1/2}^{\cos \theta=0} 1(\cos \theta) d\cos \theta = \frac{\pi}{8} a^2$$

$$a = 6.37 \times 10^6 \text{ m} \quad A = 1.6 \times 10^{13} \text{ m}^2$$

53

(54) $z = x^2 - 2y^2$ $u = x$ $\vec{x} = \begin{pmatrix} u \\ v \\ u^2 - 2v^2 \end{pmatrix}$
 $u^2 = y$

$$g_{\alpha\beta} = \frac{\partial \vec{x}}{\partial u^\alpha} \cdot \frac{\partial \vec{x}}{\partial u^\beta} = \vec{x}_\alpha \cdot \vec{x}_\beta \quad \vec{x}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix} \quad \vec{x}_v = \begin{pmatrix} 0 \\ 1 \\ -4v \end{pmatrix}$$

$$g_{uu} = 1 + 4u^2 \quad g_{uv} = g_{vu} = -8uv$$

$$g_{vv} = 1 + 16v^2$$

$$((g_{\alpha\beta})) = \begin{pmatrix} 1+4u^2 & -8uv \\ -8uv & 1+16v^2 \end{pmatrix} \quad ((g_{\alpha\beta}))_{(u,v)=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{N} = \vec{x}_u \times \vec{x}_v \quad \hat{N} = \frac{\vec{N}}{\|\vec{N}\|} \quad \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & -4v \end{vmatrix} = \begin{pmatrix} -2u \\ 4v \\ 1 \end{pmatrix}$$

$$\hat{N} = \frac{1}{\sqrt{1+4u^2+16v^2}} \begin{pmatrix} -2u \\ 4v \\ 1 \end{pmatrix}$$

$$\vec{x}_{uu} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad \vec{x}_{uv} = \vec{x}_{vu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}_{vv} = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$$

$$b_{vu} = b_{uv} = 0$$

$$b_{uu} = \frac{2}{\sqrt{1+4u^2+16v^2}} \quad b_{vv} = \frac{-4}{\sqrt{1+4u^2+16v^2}}$$

$$((b_{\alpha\beta})) = \frac{1}{\sqrt{1+4u^2+16v^2}} \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix} \quad ((b_{\alpha\beta}))_0 = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

$$((g^{\alpha\beta}))_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ((b^{\alpha}_{\beta}))_0 = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

(55) $x^3 = z(x^1, x^2) = z(0,0) + \sum_{\alpha=1}^2 \frac{\partial z}{\partial x^\alpha} \Big|_0 x^\alpha + \frac{1}{2} \sum_{\alpha,\beta=1}^2 \frac{\partial^2 z}{\partial x^\alpha \partial x^\beta} \Big|_0 x^\alpha x^\beta + \mathcal{O}(3)$

Taylor series

$0 = \left\langle \frac{\partial \vec{x}}{\partial u^\alpha}, \hat{N}(0,0) \right\rangle$ by definition of coord system $z(0,0) = \frac{\partial z(0,0)}{\partial x^\alpha} = 0 \quad (\forall \alpha)$

$= \frac{\partial z}{\partial u^\alpha} \Big|_{(0,0)} \quad \hat{N}(0,0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \therefore \frac{\partial^2 z(0,0)}{\partial x^\alpha \partial x^\beta} = \begin{pmatrix} \frac{\partial^2 z}{\partial x^\alpha \partial x^\beta} \\ \frac{\partial^2 z}{\partial x^\beta \partial x^\alpha} \\ \frac{\partial^2 z}{\partial x^\alpha \partial x^\beta} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b_{\alpha\beta}$

$\therefore x^3 = \frac{1}{2} \sum_{\alpha\beta} b_{\alpha\beta} x^\alpha x^\beta + \mathcal{O}(3)$

(56)

$$K = \det b = -8 \quad H = \text{tr } b = -2$$

$$\hat{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$K = B(\hat{T}, \hat{T}) = b_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 2\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) = -1$$

$$\hat{T} = \frac{d\vec{x}}{ds} = \frac{d\vec{x}}{du^2} \frac{du^2}{ds} \quad \left. \begin{array}{l} \text{at } (0,0) \\ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{du}{ds} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{dv}{ds} \end{array} \right\} \frac{du}{ds} = \frac{1}{\sqrt{2}} \quad \frac{dv}{ds} = \frac{1}{\sqrt{2}}$$

(57)

$$K = B(\hat{T}, \hat{T}) = B(\cos\theta \hat{T}_1 + \sin\theta \hat{T}_2, \cos\theta \hat{T}_1 + \sin\theta \hat{T}_2)$$

$$\hat{T} = \cos\theta \hat{T}_1 + \sin\theta \hat{T}_2$$

5

$$= \cos^2\theta B(\hat{T}_1, \hat{T}_1) + \sin^2\theta B(\hat{T}_2, \hat{T}_2)$$

$$+ \sin\theta \cos\theta B(\hat{T}_1, \hat{T}_2) = K_1 \cos^2\theta + K_2 \sin^2\theta$$

$$B(\hat{T}_1, \hat{T}_2) = 0 \quad \left. \begin{array}{l} \text{since } \hat{T}_1 \text{ and } \hat{T}_2 \text{ are } \perp \\ \text{by definition} \end{array} \right\} B(\hat{T}_\alpha, \hat{T}_\beta)$$

$$B(\hat{T}_1, \hat{T}_1) = K_1$$

$$B(\hat{T}_2, \hat{T}_2) = K_2$$

$$= \langle \hat{T}_\alpha, b(\hat{T}_\beta) \rangle \quad \left. \begin{array}{l} \text{def} \\ \text{principal} \\ \text{direction} \end{array} \right\} = K_\beta \langle \hat{T}_\alpha, \hat{T}_\beta \rangle$$

$$= K_\beta \delta_{\alpha\beta}$$

(58)

$$\vec{x} = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$

$$\vec{x}_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}$$

$$\vec{x}_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$$

$$\vec{N} = \vec{x}_u \times \vec{x}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix} \quad \hat{N} = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\vec{x}_{uu} = \begin{pmatrix} 0 \\ 0 \\ f_{uu} \end{pmatrix}$$

$$\vec{x}_{vv} = \begin{pmatrix} 0 \\ 0 \\ f_{vv} \end{pmatrix}$$

$$\vec{x}_{uv} = \vec{x}_{vu} = \begin{pmatrix} 0 \\ 0 \\ f_{uv} \end{pmatrix}$$

$$((b_{\alpha\beta})) = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \frac{1}{\sqrt{1+f_u^2+f_v^2}}$$

$$b_{\alpha\beta} = \frac{f_{\alpha\beta}}{W^{1/2}}$$

$$((g_{\alpha\beta})) = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix}$$

$$\det(g_{\alpha\beta}) = 1 + f_u^2 + f_v^2 + f_u^2 f_v^2 - f_u^2 f_v^2 = W$$

$$\det(b_{\alpha\beta}) = \frac{\det(f_{\alpha\beta})}{W}$$

$$K = \frac{\det(b_{\alpha\beta})}{\det(g_{\alpha\beta})} = \frac{\det(f_{\alpha\beta})}{W^2}$$

$$(g^{dR}) = \frac{1}{W} \begin{pmatrix} 1+f_u^2 & -f_u f_v \\ -f_u f_v & 1+f_v^2 \end{pmatrix}$$

$$b_{\alpha\beta}^d = \frac{1}{W^{3/2}} \begin{pmatrix} f_{uu}(1+f_v^2) - f_{uv}f_{uv} & -f_{uv}f_{uv} + f_{vv}(1+f_u^2) \\ f_{uv}(1+f_v^2) - f_{vv}f_u & -f_{uv}f_u + f_{vv}(1+f_u^2) \end{pmatrix}$$

$$H = \text{tr}(b) = \sum_{\alpha} b_{\alpha}^{\alpha} = \frac{1}{W^{3/2}} [f_{uu}(1+f_v^2) - 2f_{uv}f_{uv} + f_{vv}(1+f_u^2)]$$

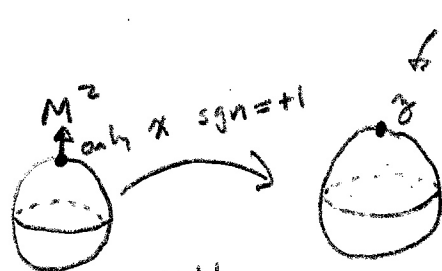
(59)

$$\nu: M^2 \rightarrow S^2$$

$$\begin{matrix} y \in S^2 \\ x \in M^2 \end{matrix}$$

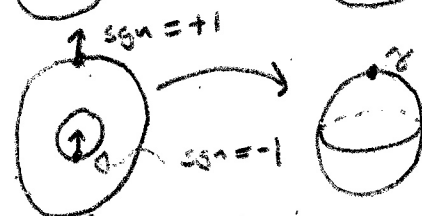
let y be a regular value of ν
and $x \in \nu^{-1}(y)$

$$\text{deg}(\nu) = \sum_{x \in \nu^{-1}(y)} \text{sgn} \nu_*(x)$$

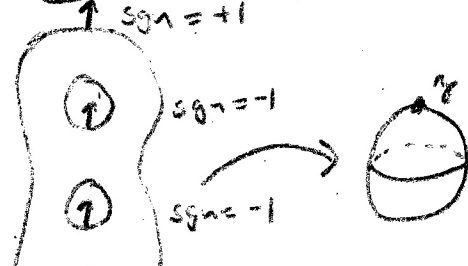


$$\therefore \text{deg}(\nu) = +1$$

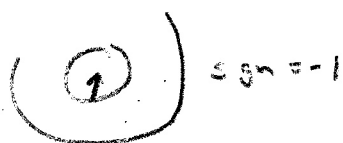
$$\text{sgn} \nu_*(x) = \begin{cases} +1 & \text{if } \nu_x: M_x^2 \rightarrow S_y^2 \\ & \text{is orientation preserving} \\ -1 & \text{if not} \end{cases}$$



$$\therefore \text{deg}(\nu) = 0$$



$$\therefore \text{deg}(\nu) = 1 - g$$



GP #3

ME Glinsky

10
10

(60)

$$\vec{X}_{\alpha\beta} = \vec{X}_\gamma \Gamma_{\beta\alpha}^\gamma + b_{\alpha\beta} \hat{n} \quad \frac{d\hat{n}}{du^\beta} = -b_{\beta\alpha}^\alpha \vec{X}_\alpha$$

$$\text{define } R_{\alpha\beta\gamma}^\tau = \frac{\partial \Gamma_{\alpha\beta}^\tau}{\partial u^\gamma} - \frac{\partial \Gamma_{\alpha\gamma}^\tau}{\partial u^\beta} + \Gamma_{\alpha\beta}^\sigma \Gamma_{\sigma\gamma}^\tau - \Gamma_{\alpha\gamma}^\sigma \Gamma_{\sigma\beta}^\tau$$

$$U_{\alpha\beta\gamma}^\tau = b_{\gamma\beta}^\tau b_{\alpha\beta} - b_{\beta\gamma}^\tau b_{\alpha\beta}$$

$$\nabla_{\alpha\beta\gamma} = \Gamma_{\alpha\beta}^\sigma b_{\sigma\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \Gamma_{\alpha\gamma}^\sigma b_{\sigma\beta} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta}$$

$$\vec{X}_{\alpha\beta\gamma} = \frac{\partial^3 \vec{X}}{\partial u^\alpha \partial u^\beta \partial u^\gamma} = \frac{\partial \vec{X}_{\alpha\beta}}{\partial u^\gamma} = \frac{\partial}{\partial u^\gamma} (\vec{X}_\tau \Gamma_{\beta\alpha}^\tau + b_{\alpha\beta} \hat{n})$$

$$= \underbrace{\frac{\partial \vec{X}_\tau}{\partial u^\gamma} \Gamma_{\beta\alpha}^\tau}_{\vec{X}_{\tau\gamma}} + \vec{X}_\tau \frac{\partial \Gamma_{\beta\alpha}^\tau}{\partial u^\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} \hat{n} + b_{\alpha\beta} \underbrace{\frac{\partial \hat{n}}{\partial u^\gamma}}_{-b_{\beta\gamma}^\sigma \vec{X}_\sigma}$$

$$= \vec{X}_\sigma \Gamma_{\gamma\tau}^\sigma \Gamma_{\beta\alpha}^\tau + b_{\tau\gamma} \Gamma_{\beta\alpha}^\tau \hat{n} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} \hat{n} - b_{\alpha\beta} b_{\gamma\sigma}^\sigma \vec{X}_\sigma + \vec{X}_\tau \frac{\partial \Gamma_{\beta\alpha}^\tau}{\partial u^\gamma}$$

$$= \left(\frac{\partial \Gamma_{\beta\alpha}^\sigma}{\partial u^\gamma} + \Gamma_{\gamma\tau}^\sigma \Gamma_{\beta\alpha}^\tau - b_{\alpha\beta} b_{\gamma\sigma}^\sigma \right) \vec{X}_\sigma + \left(b_{\tau\gamma} \Gamma_{\beta\alpha}^\tau - \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} \right) \hat{n}$$

$$\vec{X}_{\alpha\beta\gamma} - \vec{X}_{\alpha\gamma\beta} = \left(\Gamma_{\gamma\tau}^\sigma \Gamma_{\beta\alpha}^\tau - \Gamma_{\beta\tau}^\sigma \Gamma_{\gamma\alpha}^\tau + \frac{\partial \Gamma_{\beta\alpha}^\sigma}{\partial u^\gamma} - \frac{\partial \Gamma_{\gamma\alpha}^\sigma}{\partial u^\beta} - b_{\alpha\beta} b_{\gamma\sigma}^\sigma + b_{\alpha\gamma} b_{\beta\sigma}^\sigma \right) \vec{X}_\sigma$$

$$+ \left(b_{\tau\gamma} \Gamma_{\beta\alpha}^\tau - b_{\tau\beta} \Gamma_{\gamma\alpha}^\tau - \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} \right) \hat{n}$$

$$= \left[\left(\frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\gamma} - \frac{\partial \Gamma_{\alpha\gamma}^\sigma}{\partial u^\beta} + \Gamma_{\alpha\beta}^\tau \Gamma_{\tau\gamma}^\sigma - \Gamma_{\gamma\alpha}^\tau \Gamma_{\tau\beta}^\sigma \right) - (b_{\gamma\beta}^\sigma b_{\alpha\beta} - b_{\beta\gamma}^\sigma b_{\alpha\gamma}) \right] \vec{X}_\sigma$$

$$+ \left(\Gamma_{\gamma\beta}^\tau b_{\tau\gamma} + \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} - \Gamma_{\alpha\gamma}^\tau b_{\tau\beta} - \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} \right) \hat{n}$$

$$= (R_{\alpha\beta\gamma}^\sigma - U_{\alpha\beta\gamma}^\sigma) \vec{X}_\sigma + \nabla_{\alpha\beta\gamma} \hat{n}$$

(61)

$$R^{\tau}_{\alpha\beta\gamma} = \frac{\partial \Gamma^{\tau}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\tau}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\tau}_{\alpha\beta} \Gamma^{\tau}_{\sigma\gamma} - \Gamma^{\tau}_{\alpha\gamma} \Gamma^{\tau}_{\sigma\beta}$$

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2) = g_{\alpha\beta} du^{\alpha} du^{\beta}$$

$$\Gamma^{\tau}_{\alpha\beta} = \frac{1}{2} g^{\alpha\tau} \left(\frac{\partial g_{\alpha\beta}}{\partial u^{\tau}} + \frac{\partial g_{\alpha\tau}}{\partial u^{\beta}} - \frac{\partial g_{\beta\tau}}{\partial u^{\alpha}} \right)$$

$$((g_{\alpha\beta})) = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix} \quad ((g^{\alpha\beta})) = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/a^2 \sin^2\theta \end{pmatrix}$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\alpha\theta} \left(\frac{\partial g_{\alpha\theta}}{\partial u^{\theta}} + \frac{\partial g_{\alpha\theta}}{\partial u^{\theta}} - \frac{\partial g_{\theta\theta}}{\partial u^{\alpha}} \right) = 0$$

$$\Gamma^{\theta}_{\varphi\theta} = \Gamma^{\theta}_{\theta\varphi} = \frac{1}{2} g^{\alpha\theta} \left(\frac{\partial g_{\alpha\theta}}{\partial u^{\varphi}} + \frac{\partial g_{\alpha\varphi}}{\partial u^{\theta}} - \frac{\partial g_{\theta\varphi}}{\partial u^{\alpha}} \right) = \frac{1}{2} g^{\varphi\theta} \frac{\partial g_{\varphi\theta}}{\partial u^{\theta}} = 0$$

$$\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2} g^{\alpha\theta} \left(\frac{\partial g_{\alpha\varphi}}{\partial u^{\varphi}} + \frac{\partial g_{\alpha\varphi}}{\partial u^{\varphi}} - \frac{\partial g_{\varphi\varphi}}{\partial u^{\alpha}} \right) = \frac{1}{2} g^{\theta\theta} \left(- \frac{\partial g_{\varphi\varphi}}{\partial u^{\theta}} \right)$$

$$= \frac{1}{2} \cdot \frac{-1}{a^2} a^2 2 \sin\theta \cos\theta$$

$$= -\sin\theta \cos\theta$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2} g^{\alpha\varphi} \left(\frac{\partial g_{\alpha\varphi}}{\partial u^{\varphi}} + \frac{\partial g_{\alpha\varphi}}{\partial u^{\varphi}} - \frac{\partial g_{\varphi\varphi}}{\partial u^{\alpha}} \right)$$

$$= \frac{1}{2} g^{\theta\varphi} \left(\frac{\partial g_{\varphi\varphi}}{\partial u^{\theta}} \right) = 0$$

$$\Gamma^{\varphi}_{\theta\theta} = \Gamma^{\varphi}_{\theta\theta} = \frac{1}{2} g^{\alpha\varphi} \left(\frac{\partial g_{\alpha\theta}}{\partial u^{\theta}} + \frac{\partial g_{\alpha\theta}}{\partial u^{\theta}} - \frac{\partial g_{\theta\theta}}{\partial u^{\alpha}} \right) = \frac{1}{2} g^{\varphi\varphi} \frac{\partial g_{\theta\theta}}{\partial u^{\varphi}} = \frac{1}{2} \frac{1}{a^2 \sin^2\theta} (2a^2 \sin\theta \cos\theta)$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{1}{2} g^{\alpha\varphi} \left(\frac{\partial g_{\alpha\theta}}{\partial u^{\varphi}} + \frac{\partial g_{\alpha\theta}}{\partial u^{\varphi}} - \frac{\partial g_{\theta\varphi}}{\partial u^{\alpha}} \right) = 0$$

5

$$R^{\theta\varphi}_{\theta\varphi} = g^{\theta\alpha} R^{\varphi}_{\alpha\theta\varphi} = g^{\theta\theta} R^{\varphi}_{\theta\theta\varphi} + g^{\theta\varphi} R^{\varphi}_{\varphi\theta\varphi} = \left(\frac{1}{a^2} \right) (1) = \boxed{\frac{1}{a^2} = K = R^{\theta\varphi}_{\theta\varphi}}$$

$$R^{\varphi}_{\theta\theta\varphi} = \frac{\partial \Gamma^{\varphi}_{\theta\varphi}}{\partial u^{\theta}} - \frac{\partial \Gamma^{\varphi}_{\theta\theta}}{\partial u^{\varphi}} + \Gamma^{\varphi}_{\theta\varphi} \Gamma^{\varphi}_{\sigma\theta} - \Gamma^{\varphi}_{\theta\theta} \Gamma^{\varphi}_{\sigma\varphi}$$

$$= \Gamma^{\varphi}_{\theta\varphi} \Gamma^{\varphi}_{\varphi\theta} + \frac{\partial \Gamma^{\varphi}_{\theta\varphi}}{\partial u^{\theta}} = \frac{\cos^2\theta}{\sin^2\theta} - \frac{1}{\sin^2\theta} = 1$$

$$R^{\varphi}_{\varphi\theta\varphi} = \frac{\partial \Gamma^{\varphi}_{\varphi\varphi}}{\partial u^{\theta}} - \frac{\partial \Gamma^{\varphi}_{\varphi\theta}}{\partial u^{\varphi}} + \Gamma^{\varphi}_{\varphi\varphi} \Gamma^{\varphi}_{\sigma\theta} - \Gamma^{\varphi}_{\varphi\theta} \Gamma^{\varphi}_{\sigma\varphi} = 0$$

GP #4

M E Glinsky

(40/50)

62

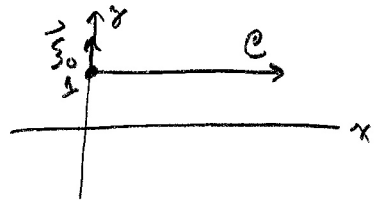
$$ds^2 = \frac{1}{z^2} (dx^2 + dz^2)$$

$$\xi_0 = \frac{\partial}{\partial z} = \xi(x=0, z=1)$$

$$C: (x, z)_t = (t, 1)$$

$$\text{find } \xi(t) \rightarrow \xi(0) = \xi_0 \text{ and}$$

$$\frac{d\xi}{dt} = 0$$



$$\frac{d\xi}{dt} = \xi^x \frac{dx}{dt} + \xi^z \frac{dz}{dt}$$

$$= \xi^x(x, z) \frac{dx}{dt} + \xi^z(x, z) \frac{dz}{dt}$$

$$= \xi^x(t, 1) \frac{dx}{dt} + \xi^z(t, 1) \frac{dz}{dt}$$

$$\frac{d\xi^x}{dt} = 0 = \frac{d\xi^x}{dt} + \left(\frac{dx}{dt} \frac{dz}{dt} \right) \begin{pmatrix} \Gamma_{xx}^x & \Gamma_{xz}^x \\ \Gamma_{zx}^x & \Gamma_{zz}^x \end{pmatrix} \begin{pmatrix} \xi^x \\ \xi^z \end{pmatrix}$$

$$\frac{d\xi^z}{dt} = 0 = \frac{d\xi^z}{dt} + \left(\frac{dx}{dt} \frac{dz}{dt} \right) \begin{pmatrix} \Gamma_{xx}^z & \Gamma_{xz}^z \\ \Gamma_{zx}^z & \Gamma_{zz}^z \end{pmatrix} \begin{pmatrix} \xi^x \\ \xi^z \end{pmatrix}$$

$$\frac{dx}{dt} = 1$$

$$\frac{dz}{dt} = 0$$

$$\Gamma_{\mu\nu}^\tau = \frac{1}{2} g^{\tau\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$$

$$(\Gamma_{\alpha\beta}^\gamma) = \begin{pmatrix} 1/z^2 & 0 \\ 0 & 1/z^2 \end{pmatrix}$$

$$(\Gamma_{\alpha\beta}^{\bar{\gamma}}) = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix}$$

$$\Gamma_{xx}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{xx}}{\partial x} + \frac{\partial g_{xx}}{\partial x} - \frac{\partial g_{xx}}{\partial x} \right) = 0$$

$$\begin{aligned} \Gamma_{xz}^x &= \Gamma_{zx}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{xx}}{\partial z} + \frac{\partial g_{xz}}{\partial x} - \frac{\partial g_{xz}}{\partial x} \right) \\ &= \frac{1}{2} z^2 \cdot \frac{-2}{z^3} = -\frac{1}{z} \end{aligned}$$

$$\Gamma_{zz}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{zx}}{\partial z} + \frac{\partial g_{xz}}{\partial z} - \frac{\partial g_{zz}}{\partial x} \right) = 0$$

$$\Gamma_{xx}^z = \frac{1}{2} g^{zz} \left(\frac{\partial g_{xx}}{\partial z} + \frac{\partial g_{zx}}{\partial x} - \frac{\partial g_{xz}}{\partial z} \right) = \frac{1}{2} z^2 \cdot \frac{2}{z^3} = \frac{1}{z}$$

$$\Gamma_{xz}^z = \Gamma_{zx}^z = \frac{1}{2} g_{zz} \left(\frac{\partial g_{xz}}{\partial z} + \frac{\partial g_{zx}}{\partial x} - \frac{\partial g_{xx}}{\partial z} \right) = 0$$

$$\Gamma_{zz}^z = \frac{1}{2} g_{zz} \left(\frac{\partial g_{zz}}{\partial z} + \frac{\partial g_{zz}}{\partial z} - \frac{\partial g_{zz}}{\partial z} \right) = \frac{1}{2} z^2 \left(\frac{-2}{z^3} \right) = -\frac{1}{z}$$

$$-\Gamma_{xx}^z = \Gamma_{xz}^x = \Gamma_{zx}^x = \Gamma_{zz}^x = -\frac{1}{z} \quad \text{all other } \Gamma^s = 0$$

$$0 = \frac{d\xi^x}{dt} + (1 \ 0) \begin{pmatrix} 0 & -\frac{1}{z} \\ -\frac{1}{z} & 0 \end{pmatrix} \begin{pmatrix} \xi^x \\ \xi^z \end{pmatrix} = \dot{\xi}^x - \frac{\xi^z}{z} = \dot{\xi}^x - \xi^z$$

$$0 = \frac{d\xi^z}{dt} + (1 \ 0) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & -\frac{1}{z} \end{pmatrix} \begin{pmatrix} \xi^x \\ \xi^z \end{pmatrix} = \dot{\xi}^z + \frac{\xi^x}{z} = \dot{\xi}^z + \xi^x$$

$$\xi^z = -\frac{d\xi^x}{dt} = -\frac{d^2 \xi^x}{dt^2}$$

$$\xi^x = -\frac{d\xi^z}{dt} = -\frac{d^2 \xi^z}{dt^2}$$

$$\xi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad a=0 \quad c=1$$

$$\xi^x = a \cos t + b \sin t$$

$$\xi^z = c \cos t + d \sin t$$

$$\xi^x = b \sin t$$

$$\xi^z = \cos t + d \sin t$$

$$\cos t + d \sin t = b \cos t \Rightarrow b=1 \quad d=0$$

$$\xi(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

(63)

$$\begin{aligned} X^j Y_{;j}^i - Y^j X_{;j}^i &= X^j \left(\frac{\partial Y^i}{\partial x^j} + \Gamma_{j\tau}^i Y^\tau \right) - Y^j \left(\frac{\partial X^i}{\partial x^j} + \Gamma_{j\tau}^i X^\tau \right) \\ &= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) + \Gamma_{j\tau}^i Y^\tau X^j - \Gamma_{j\tau}^i X^\tau Y^j \end{aligned}$$

$$= [X, Y]^i + \Gamma_{j\tau}^i X^j Y^\tau - \Gamma_{j\tau}^i X^\tau Y^j$$

$$= [X, Y]^i + \underbrace{(\Gamma_{j\tau}^i - \Gamma_{\tau j}^i)}_{=0} X^j Y^\tau = [X, Y]^i$$

$$\Rightarrow [X, Y] = \nabla_X Y - \nabla_Y X$$

(64)

$$\nabla_{\partial_\beta} d\alpha = d\sigma \Gamma_{\beta\alpha}^\sigma$$

Leibnitz rule

$$\begin{aligned} \nabla_{\partial_\gamma} \nabla_{\partial_\beta} d\alpha - \nabla_{\partial_\beta} \nabla_{\partial_\gamma} d\alpha &= \nabla_{\partial_\gamma} (d\sigma \Gamma_{\beta\alpha}^\sigma) - \nabla_{\partial_\beta} (d\sigma \Gamma_{\gamma\alpha}^\sigma) \\ &= \Gamma_{\beta\alpha}^\sigma (\nabla_{\partial_\gamma} d\sigma) + d\sigma (d\gamma \Gamma_{\beta\alpha}^\sigma) - \Gamma_{\gamma\alpha}^\sigma (\nabla_{\partial_\beta} d\sigma) - d\sigma (d\beta \Gamma_{\gamma\alpha}^\sigma) \\ &= \Gamma_{\beta\alpha}^\sigma \Gamma_{\gamma\sigma}^\tau d\tau + d\sigma \frac{d\Gamma_{\beta\alpha}^\sigma}{d\alpha^\sigma} - \Gamma_{\gamma\alpha}^\sigma \Gamma_{\beta\sigma}^\tau d\tau - d\sigma \frac{d\Gamma_{\gamma\alpha}^\sigma}{d\alpha^\sigma} \end{aligned}$$

 Γ torsion free

$$\begin{aligned} &= \left(\Gamma_{\beta\alpha}^\sigma \Gamma_{\gamma\sigma}^\tau - \Gamma_{\gamma\alpha}^\sigma \Gamma_{\beta\sigma}^\tau + \frac{d\Gamma_{\beta\alpha}^\tau}{d\alpha^\tau} - \frac{d\Gamma_{\gamma\alpha}^\tau}{d\alpha^\tau} \right) d\tau \\ &= \left(\Gamma_{\alpha\beta}^\sigma \Gamma_{\sigma\gamma}^\tau - \Gamma_{\alpha\gamma}^\sigma \Gamma_{\sigma\beta}^\tau + \frac{d\Gamma_{\alpha\beta}^\tau}{d\alpha^\tau} - \frac{d\Gamma_{\alpha\gamma}^\tau}{d\alpha^\tau} \right) d\tau \\ &\stackrel{5}{=} R_{\alpha\gamma\beta}^\tau d\tau \end{aligned}$$

$$\nabla_X (d\alpha Z^\alpha) = \nabla_X (d\alpha) Z^\alpha + d\alpha (\nabla_X Z^\alpha)$$

(65)

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z = \nabla_{[X, Y]} Z$$

$$(\nabla_{\partial_i} Z) X^j = \nabla_{\partial_i} Z^j X^j$$

$$= [\nabla_{X^i \partial_i} \nabla_{Y^j \partial_j} - \nabla_{Y^j \partial_j} \nabla_{X^i \partial_i} - \nabla_{(X^i \partial_i Y^j - Y^j \partial_i X^i) \partial_j}] Z$$

$$= \nabla_{X^i \partial_i} [(\nabla_{\partial_j} Z) X^j] - \nabla_{Y^j \partial_j} [(\nabla_{\partial_i} Z) X^i] - (\nabla_{\partial_j} Z) (X^i \partial_i Y^j - Y^j \partial_i X^i)$$

$$= X^i \nabla_{\partial_i} [Y^j \nabla_{\partial_j} Z] - Y^j \nabla_{\partial_j} [X^i \nabla_{\partial_i} Z] - (X^i \partial_i Y^j - Y^j \partial_i X^i) (\nabla_{\partial_j} Z)$$

$$= X^i [Y^j \nabla_{\partial_i} \nabla_{\partial_j} Z + (\partial_i Y^j) \nabla_{\partial_j} Z] - Y^j [X^i \nabla_{\partial_j} \nabla_{\partial_i} Z + \partial_j X^i \nabla_{\partial_i} Z - X^i \partial_i Y^j (\nabla_{\partial_j} Z) + Y^j \partial_i X^i \nabla_{\partial_j} Z]$$

$$= X^i Y^j (\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}) (Z^k d\alpha)$$

$$= X^i Y^j [\nabla_{\partial_i} (Z^k \nabla_{\partial_j} d\alpha + \partial_j Z^k d\alpha) - \nabla_{\partial_j} (Z^k \nabla_{\partial_i} d\alpha + \partial_i Z^k d\alpha)]$$

$$= X^i Y^j [Z^k \nabla_{\partial_i} \nabla_{\partial_j} d\alpha + (\partial_i Z^k) \nabla_{\partial_j} d\alpha + (\partial_j Z^k) \nabla_{\partial_i} d\alpha + (\partial_i \partial_j Z^k) d\alpha - Z^k \nabla_{\partial_j} \nabla_{\partial_i} d\alpha - (\partial_j Z^k) \nabla_{\partial_i} d\alpha - (\partial_i Z^k) \nabla_{\partial_j} d\alpha - (\partial_j \partial_i Z^k) d\alpha]$$

$$= X^i Y^j Z^k [\nabla_{\partial_i} \nabla_{\partial_j} d\alpha - \nabla_{\partial_j} \nabla_{\partial_i} d\alpha] + X^i Y^j (\partial_i \partial_j - \partial_j \partial_i) Z^k d\alpha$$

$$= X^i Y^j Z^k (\nabla_{d_i} \nabla_{d_j} - \nabla_{d_j} \nabla_{d_i}) d_k \leftarrow \{d_i d_j Z^k = d_j d_i Z^k\}$$

$$= X^i Y^j Z^k R_{kij}^{\tau} d_{\tau}$$

$$= (R_{\alpha\gamma\beta}^{\tau} X^{\alpha} Y^{\beta} Z^{\gamma}) d_{\tau} = [R(X, Y)Z]^{\tau} d_{\tau} = R(X, Y)Z$$

$$\therefore [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = R(X, Y)$$

5

GP #5

M.E. Glinisky

(66)

$e_i = \partial_i$ coordinate frame $[e_i, e_j] = 0$

$$\omega = ((\omega^k_j)) = ((\Gamma^k_{ij} \sigma^i))$$

$$\theta^k_j = d\omega^k_j + \omega^k_i \wedge \omega^i_j$$

$$= d(\Gamma^k_{ij} \sigma^i) + \Gamma^k_{li} \sigma^l \wedge \Gamma^i_{mj} \sigma^m$$

$$= (d\Gamma^k_{ij}) \wedge \sigma^i + \Gamma^k_{ij} d\sigma^i + \frac{1}{2} (\Gamma^k_{li} \Gamma^i_{mj} - \Gamma^k_{mi} \Gamma^i_{lj}) \sigma^l \wedge \sigma^m$$

$$= e_l(\Gamma^k_{ij}) \sigma^l \wedge \sigma^i + \Gamma^k_{ij} (0) + \frac{1}{2} (\Gamma^k_{li} \Gamma^i_{mj} - \Gamma^k_{mi} \Gamma^i_{lj}) \sigma^l \wedge \sigma^m$$

$$\uparrow d\sigma^i(e_j, e_k) = -\sigma^i([e_j, e_k])$$

$$= 0$$

because coord frame

$$= \frac{1}{2} \left(\frac{\partial \Gamma^k_{ij}}{\partial x^l} - \frac{\partial \Gamma^k_{li}}{\partial x^j} \right) \sigma^l \wedge \sigma^i + \frac{1}{2} (\Gamma^k_{li} \Gamma^i_{mj} - \Gamma^k_{mi} \Gamma^i_{lj}) \sigma^l \wedge \sigma^m$$

$$\uparrow \text{Eq 38.3}$$

because coord frame

$$= \frac{1}{2} \left[\frac{\partial \Gamma^k_{sj}}{\partial x^r} - \frac{\partial \Gamma^k_{rj}}{\partial x^s} + \Gamma^k_{ri} \Gamma^i_{sj} - \Gamma^k_{si} \Gamma^i_{rj} \right] \sigma^r \wedge \sigma^s$$

$$= \frac{1}{2} R^k_{jrs} \sigma^r \wedge \sigma^s$$

$\uparrow$ not exactly Eq 34.3 will be the same if Γ 's symmetric

(67)

$$\bar{\theta} = d\bar{\omega} + \bar{\omega} \wedge \bar{\omega} \quad \text{and} \quad \theta = d\omega + \omega \wedge \omega$$

already know $\bar{\omega} = P^{-1}\omega P + P^{-1}dP$
 $P^{-1}P = PP^{-1} = I$

$$\begin{pmatrix} \bar{e} = eP \\ \bar{\omega} = P^{-1}\omega \\ \bar{\sigma} = P^{-1}\sigma \end{pmatrix} \text{ transformation}$$

$$\begin{aligned} \bar{\theta} &= d(P^{-1}\omega P + P^{-1}dP) + (P^{-1}\omega P + P^{-1}dP) \wedge (P^{-1}\omega P + P^{-1}dP) \\ &= P^{-1}(dP\omega + d\omega P) + dP^{-1} \wedge \omega P + P^{-1}\omega P \wedge P^{-1}\omega P \\ &\quad + P^{-1}\omega P \wedge P^{-1}dP + P^{-1}dP \wedge P^{-1}\omega P + P^{-1}dP \wedge P^{-1}dP \\ &= -P^{-1}\omega \wedge dP + P^{-1}d\omega P - P^{-1}dP \wedge P^{-1}\omega P + P^{-1}\omega P \wedge P^{-1}\omega P \\ &\quad + P^{-1}\omega \wedge dP + P^{-1}dP \wedge P^{-1}\omega P + \underbrace{dP \wedge dP}_{=0} \end{aligned}$$

one forms

$$\begin{aligned} &= P^{-1}d\omega P + P^{-1}\omega \wedge \omega P \\ &= P^{-1}(d\omega + \omega \wedge \omega)P = P^{-1}\theta P \end{aligned}$$

(68)

$$\bar{e} = (dr, d\theta) = (\partial_r x dx + \partial_r y dy, \partial_\theta x dx + \partial_\theta y dy)$$

$$= (dx, dy) \begin{pmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{pmatrix} = eP$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$P = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$P^{-1} = \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$dP = \begin{pmatrix} -\sin \theta d\theta & -\sin \theta dr - r \cos \theta d\theta \\ \cos \theta d\theta & \cos \theta dr - r \sin \theta d\theta \end{pmatrix}$$

$$\bar{\omega} = P^{-1}dP = \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -\sin \theta d\theta & -\sin \theta dr - r \cos \theta d\theta \\ \cos \theta d\theta & \cos \theta dr - r \sin \theta d\theta \end{pmatrix}$$

$$= \frac{1}{r} \begin{pmatrix} -r \cos \theta \sin \theta d\theta + r \sin \theta \cos \theta d\theta & -r \sin \theta \cos \theta dr - r^2 \cos^2 \theta d\theta \\ \sin^2 \theta d\theta + r \cos^2 \theta d\theta & + r \sin \theta \cos \theta dr - r^2 \sin^2 \theta d\theta \end{pmatrix}$$

$$= \frac{1}{r} \begin{pmatrix} 0 & -r^2 d\theta \\ d\theta & dr \end{pmatrix} = \begin{pmatrix} 0 & -r d\theta \\ \frac{d\theta}{r} & \frac{dr}{r} \end{pmatrix}$$

$$d\bar{\omega} = \begin{pmatrix} 0 & -dr \wedge d\theta \\ -\frac{dr \wedge d\theta}{r^2} & 0 \end{pmatrix}$$

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$$\bar{\omega} \wedge \bar{\omega} = \begin{pmatrix} 0 & -r d\theta \\ \frac{d\theta}{r} & \frac{dr}{r} \end{pmatrix} \wedge \begin{pmatrix} 0 & -r d\theta \\ \frac{d\theta}{r} & \frac{dr}{r} \end{pmatrix} = \begin{pmatrix} 0 & -dr \wedge d\theta \\ \frac{dr \wedge d\theta}{r^2} & 0 \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & -dr \wedge d\theta \\ -\frac{dr \wedge d\theta}{r^2} & 0 \end{pmatrix} = -d\bar{\omega} \Rightarrow d\bar{\omega} + \bar{\omega} \wedge \bar{\omega} = \bar{\theta} = 0$$

(69)

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad (y > 0)$$

let

$$u = \ln y$$

$$v = x$$

$$dv = dx$$

$$du = dy/y$$

$$y = e^u$$

$$ds^2 = \frac{(dv)^2 + (y du)^2}{y^2} = du^2 + \frac{1}{2} dv^2$$

$$= du^2 + e^{-2u} dv^2$$

$$= du^2 + G(u, v) dv^2$$

$$G(u, v) = e^{-u}$$

$$R = -\frac{1}{G} \frac{\partial^2 G}{\partial u^2} = -e^u (e^{-u}) = -1$$

$$(g_{ij}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad (g^{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Gamma_{\mu\beta}^{\tau} = \frac{1}{2} g^{\tau\gamma} \left(\frac{\partial g_{\mu\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\mu}} - \frac{\partial g_{\mu\beta}}{\partial u^{\alpha}} \right)$$

This is not
the method
of frames.

$$\Gamma_{xx}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{xx}}{\partial u^x} + \frac{\partial g_{xx}}{\partial u^x} - \frac{\partial g_{xx}}{\partial u^x} \right) = 0$$

$$\Gamma_{zx}^x = \Gamma_{xz}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{xx}}{\partial u^z} + \frac{\partial g_{xx}}{\partial u^x} - \frac{\partial g_{xz}}{\partial u^x} \right) = \frac{1}{2} z^2 \frac{-2}{z^3} = -\frac{1}{z}$$

$$\Gamma_{zz}^x = \frac{1}{2} g^{xx} \left(\frac{\partial g_{xx}}{\partial u^z} + \frac{\partial g_{xx}}{\partial u^z} - \frac{\partial g_{zz}}{\partial u^x} \right) = 0$$

$$\Gamma_{zz}^z = \frac{1}{2} g^{zz} \left(\frac{\partial g_{zz}}{\partial u^z} + \frac{\partial g_{zz}}{\partial u^z} - \frac{\partial g_{zz}}{\partial u^z} \right) = \frac{1}{2} z^2 \frac{-2}{z^3} = -\frac{1}{z}$$

$$\Gamma_{zx}^z = \Gamma_{xz}^z = \frac{1}{2} g^{zz} \left(\frac{\partial g_{zz}}{\partial u^z} + \frac{\partial g_{zz}}{\partial u^x} - \frac{\partial g_{xz}}{\partial u^x} \right) = 0$$

$$\Gamma_{xx}^z = \frac{1}{2} g^{zz} \left(\frac{\partial g_{xx}}{\partial u^x} + \frac{\partial g_{xx}}{\partial u^x} - \frac{\partial g_{xx}}{\partial u^x} \right) = -\frac{1}{2} z^2 \left(\frac{-2}{z^3} \right) = \frac{1}{z}$$

$$\boxed{\Gamma_{zx}^x = \Gamma_{xz}^x = \Gamma_{zz}^z = -\Gamma_{xx}^z = -\frac{1}{z} \quad \text{all others} = 0}$$

$$\omega^k_j = \Gamma^k_{ij} \sigma^i$$

$$\begin{aligned} \omega^x_x &= \Gamma^x_{zx} \sigma^z + \Gamma^x_{xx} \sigma^x \\ &= -\frac{1}{z} dz \end{aligned}$$

$$\omega^z_z = \Gamma^z_{zz} \sigma^z + \Gamma^z_{xz} \sigma^x = -\frac{1}{z} dz$$

$$\omega^x_z = \Gamma^x_{zz} \sigma^z + \Gamma^x_{xz} \sigma^x = -\frac{1}{z} dx$$

$$\omega^z_x = \Gamma^z_{zx} \sigma^z + \Gamma^z_{xx} \sigma^x = \frac{1}{z} dx$$

$$\Theta = d\omega + \omega \wedge \omega$$

$$(\omega^k_j) = \frac{1}{z} \begin{pmatrix} -dz & -dx \\ dx & -dz \end{pmatrix}$$

$$\begin{aligned} (\Theta^k_j) &= \begin{pmatrix} 0 & \frac{1}{z^2} dz \wedge dx \\ -\frac{1}{z^2} dz \wedge dx & 0 \end{pmatrix} + \frac{1}{z^2} \underbrace{\begin{pmatrix} -dz & -dx \\ dx & -dz \end{pmatrix} \wedge \begin{pmatrix} -dz & -dx \\ dx & -dz \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \\ &= \frac{1}{z^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dx \wedge dz \\ &= \frac{1}{2z^2} \begin{pmatrix} 0 & -dx \wedge dz + dz \wedge dx \\ dx \wedge dz - dz \wedge dx & 0 \end{pmatrix} = \frac{1}{2} ((R^k_{jrs} \sigma^r \wedge \sigma^s)) \end{aligned}$$

$$\frac{1}{2} \frac{1}{z^2} (-dx \wedge dz + dz \wedge dx) = \frac{1}{2} [R^x_{zxx} dx \wedge dx + R^x_{zxz} dx \wedge dz + R^x_{zzx} dz \wedge dx + R^x_{zzz} dz \wedge dz]$$

$$\Rightarrow R^x_{zxz} = -\frac{1}{z^2}$$

$$R^{xz}_{xz} = g^{zx} R^x_{zxz} = z^2 \left(-\frac{1}{z^2}\right) = -1$$

same as before

GP #6 M. E. Glinsky

72 $l(\epsilon) = \int_0^L \left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle^{1/2} ds$

connection is riemannian $\frac{\partial}{\partial \epsilon} \langle X, Y \rangle = \left\langle \frac{\nabla X}{\partial \epsilon}, Y \right\rangle + \left\langle X, \frac{\nabla Y}{\partial \epsilon} \right\rangle$

$\forall$ vector field X, Y defined on surface of variation

connection is synthetic $\frac{\nabla}{\partial u} \left(\frac{\partial \vec{x}}{\partial v} \right) - \frac{\nabla}{\partial v} \left(\frac{\partial \vec{x}}{\partial u} \right) = 0$

$$\begin{aligned} l'(\epsilon) &= \frac{\partial l(\epsilon)}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_0^L \left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle^{1/2} ds = \int_0^L \frac{\partial}{\partial \epsilon} \left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle^{1/2} ds \\ &= \int_0^L \frac{1}{2} \left(\left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle \right)^{-1/2} \frac{\partial}{\partial \epsilon} \left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle ds \\ &= \int_0^L \frac{1}{2} \left\| \frac{\partial \vec{x}}{\partial s} \right\|^{-1} \left[\left\langle \frac{\nabla}{\partial \epsilon} \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle + \left\langle \frac{\partial \vec{x}}{\partial s}, \frac{\nabla}{\partial \epsilon} \frac{\partial \vec{x}}{\partial s} \right\rangle \right] ds \\ &= \int_0^L \left\| \frac{\partial \vec{x}}{\partial s} \right\|^{-1} \left\langle \frac{\nabla}{\partial \epsilon} \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle ds = \int_0^L \left\langle \frac{\nabla}{\partial \epsilon} \frac{\partial \vec{x}}{\partial s}, \frac{\partial \vec{x}}{\partial s} \right\rangle ds \\ &= \int_0^L \left\langle \frac{\nabla}{\partial s} \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\partial \vec{x}}{\partial s} \right\rangle ds = \int_0^L \left[\frac{\partial}{\partial s} \left\langle \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\partial \vec{x}}{\partial s} \right\rangle - \left\langle \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\nabla}{\partial s} \frac{\partial \vec{x}}{\partial s} \right\rangle \right] ds \\ &= \left\langle \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\partial \vec{x}}{\partial s} \right\rangle \Big|_0^L - \int_0^L \left\langle \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\nabla}{\partial s} \frac{\partial \vec{x}}{\partial s} \right\rangle ds \end{aligned}$$

$\left\| \frac{\partial \vec{x}(s, \epsilon=0)}{\partial s} \right\| = 1$ since s is arclength

C is a geodesic if $l'(0) = 0$ for all variations that vanish at the endpoints

(i.e., $\frac{\partial \vec{x}(0, \epsilon)}{\partial \epsilon} = 0 = \frac{\partial \vec{x}(L, \epsilon)}{\partial \epsilon}$)

$\Rightarrow 0 = \int_0^L \left\langle \frac{\partial \vec{x}}{\partial \epsilon}, \frac{\nabla}{\partial s} \frac{\partial \vec{x}}{\partial s} \right\rangle ds$ ($\forall \frac{\partial \vec{x}}{\partial \epsilon}$ not satisfying the above condition)

i.e. by fundamental lemma of calculus of variations

$\frac{\nabla}{\partial s} \left(\frac{\partial \vec{x}}{\partial s} \right) = 0$

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$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad (e_1, e_2) = \left(y \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = e$$

$$ds^2 = (\sigma^1)^2 + (\sigma^2)^2 \quad \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{y} dx \\ \frac{1}{y} dy \end{pmatrix} = \sigma$$

$$\boxed{\begin{aligned} de &= e \otimes \omega \\ d\sigma &= -\omega \wedge \sigma \quad \nabla \text{ symmetric} \end{aligned}}$$

$$\sigma^i(e_j) = \delta^i_j \quad \text{with } g_{ij} = \delta_{ij}$$

$$\Rightarrow \omega^i_j = -\omega^j_i \quad \text{since orthonormal}$$

find ω

$$d\sigma = d \begin{pmatrix} \frac{1}{y} dx \\ \frac{1}{y} dy \end{pmatrix} = \begin{pmatrix} -\frac{1}{y^2} dy \wedge dx \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma^1 \wedge \sigma^2 \\ 0 \end{pmatrix} \Rightarrow \omega^1_1 = \omega^2_2 = 0$$

$$\omega^1_2 = -\omega^2_1$$

$$= -\omega \wedge \sigma = - \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix} \wedge \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\omega^1_2 \\ \omega^1_2 & 0 \end{pmatrix} \wedge \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} -\omega^1_2 \wedge \sigma^2 \\ \omega^1_2 \wedge \sigma^1 \end{pmatrix} = \begin{pmatrix} -a \sigma^1 \wedge \sigma^2 \\ -b \sigma^1 \wedge \sigma^2 \end{pmatrix} \Rightarrow \begin{aligned} a &= -1 \\ b &= 0 \end{aligned}$$

$$\omega^1_2 = -\sigma^1$$

$$\text{let } \omega^1_2 = a\sigma^1 + b\sigma^2$$

$$(\omega^i_j) = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$

$$d(ev) = e \otimes (dv + \omega v) = \nabla X$$

$$\frac{\nabla X}{ds} = \nabla_T X = \nabla X(T)$$

$$\nabla(v^i e_i) = (e_1, e_2) \otimes \left[\begin{pmatrix} dv^1 \\ dv^2 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right]$$

$$= (e_1, e_2) \otimes \begin{pmatrix} dv^1 - v^2 \sigma^1 \\ dv^2 + v^1 \sigma^1 \end{pmatrix}$$

$$\nabla_T X = (e_1, e_2) \otimes \begin{pmatrix} dv^1(T) - v^2 \sigma^1(T) \\ dv^2(T) + v^1 \sigma^1(T) \end{pmatrix}$$

$$T = e_2$$

$$J = \frac{1}{y} e_1$$

$$\nabla_T(e_2) = (e_1, e_2) \otimes \begin{pmatrix} 0 - (1) \sigma^1(e_2) \\ 0 - (0) \sigma^1(e_2) \end{pmatrix} = -e_1 \sigma^1(e_2) = 0$$

$$X = e_2 \begin{cases} v^1 = 0 & dv^1 = 0 \\ v^2 = 1 & dv^2 = 0 \end{cases}$$

$$\Rightarrow \frac{\nabla(e_2)}{ds} = 0$$

QED

$$\text{now } X = J \begin{cases} v^1 = 1/2 & dv^1 = -\frac{1}{2} dy \\ v^2 = 0 & dv^2 = 0 \end{cases}$$

$$\begin{aligned} \frac{\nabla J}{ds} = \nabla_T J &= (e_1 \ e_2) \otimes \begin{pmatrix} -\frac{1}{2} dy(e_2) + 0 \\ 0 & +\frac{1}{2} \sigma'(e_2) \end{pmatrix} \\ &= (e_1 \ e_2) \otimes \begin{pmatrix} -\frac{1}{2} \sigma^2(e_2) \\ \frac{1}{2} \sigma'(e_2) \end{pmatrix} = -\frac{1}{2} e_1 = -J \end{aligned}$$

$$\nabla_T \nabla_T J = \nabla_T (-J) = -\nabla_T J = -(-J) = J$$

$$\begin{aligned} \theta = d\omega + \omega \wedge \omega &\Rightarrow ((\theta^i_j)) = \begin{pmatrix} 0 & -d\sigma^1 \\ d\sigma^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -d\sigma^1 \\ d\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^1 \wedge \sigma^2 \\ \sigma^1 \wedge \sigma^2 & 0 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \theta(J, T) = (e_1 \ e_2) \begin{pmatrix} 0 & -\sigma^1 \wedge \sigma^2(J, T) \\ \sigma^1 \wedge \sigma^2(J, T) & 0 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$$

$$= (e_1 \ e_2) \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$$

$$\begin{aligned} &\sigma^1 \wedge \sigma^2(J, T) \\ &= \sigma^1(J) \sigma^2(T) - \sigma^2(J) \sigma^1(T) \\ &= \sigma^1(J) = \sigma^1\left(\frac{1}{2} e_1\right) = \frac{1}{2} \end{aligned}$$

$$\theta(J, T)T = (e_1 \ e_2) \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \sigma^1(e_2) \\ \sigma^2(e_1) \end{pmatrix}$$

$$= (e_1 \ e_2) \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} e_1 = -J = R(J, T)T$$

$$\therefore \nabla_T^2 J + R(J, T)T = J - J = 0 \quad \underline{QED}$$

5

GP #7 M E Glinisky

74

$$\delta^i_{j;k} = \nabla_k(\delta^i_j) = \frac{d(\delta^i_j)}{dx^k} + \Gamma_{kl}^i \delta^l_j - \Gamma_{kj}^l \delta^i_l$$

$$= 0 + \Gamma_{kl}^i - \Gamma_{kj}^l = 0$$

75

$$b_{\alpha\beta|\gamma} = \nabla_\gamma b_{\alpha\beta} = \frac{db_{\alpha\beta}}{dx^\gamma} - \Gamma_{\gamma\alpha}^\lambda b_{\lambda\beta} - \Gamma_{\gamma\beta}^\lambda b_{\alpha\lambda}$$

$$b_{\alpha\gamma|\beta} = \nabla_\beta b_{\alpha\gamma} = \frac{db_{\alpha\gamma}}{dx^\beta} - \Gamma_{\beta\alpha}^\lambda b_{\lambda\gamma} - \Gamma_{\beta\gamma}^\lambda b_{\alpha\lambda}$$

$$b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \Rightarrow \frac{db_{\alpha\beta}}{dx^\gamma} - \Gamma_{\gamma\alpha}^\lambda b_{\lambda\beta} - \Gamma_{\gamma\beta}^\lambda b_{\alpha\lambda} = \frac{db_{\alpha\gamma}}{dx^\beta} - \Gamma_{\beta\alpha}^\lambda b_{\lambda\gamma} - \Gamma_{\beta\gamma}^\lambda b_{\alpha\lambda}$$

symmetric

$$\frac{db_{\alpha\beta}}{dx^\gamma} - \Gamma_{\gamma\alpha}^\sigma b_{\sigma\beta} = \frac{db_{\alpha\gamma}}{dx^\beta} - \Gamma_{\beta\alpha}^\sigma b_{\sigma\gamma} + (\Gamma_{\gamma\beta}^\sigma - \Gamma_{\beta\gamma}^\sigma) b_{\alpha\sigma}$$

QED

76

$$0 = \nabla_\gamma(\delta^i_k) = \nabla_\gamma(g^{ij}g_{jk}) \stackrel{\text{product rule}}{=} g_{jk} \nabla_\gamma g^{ij} + g^{ij} \nabla_\gamma g_{jk}$$

Eq. 44.7

for $0 = g_{jk} \nabla_\gamma g^{ij}$ since $((g_{jk}))$ is non-singular $\Rightarrow \boxed{\nabla_\gamma g^{ij} = 0 = g^{ij}{}_{|\gamma}}$

77

$$(\text{Div } b)^a = k^{ia}{}_{|i} = (g^{i\sigma} g^{\alpha\tau} b_{\sigma\tau})_{|i}$$

$$= g^{i\sigma} g^{\alpha\tau} b_{\sigma\tau|i} + \underbrace{g^{i\sigma} g^{\alpha\tau}}_{=0} b_{\sigma\tau} + \underbrace{g^{i\sigma}}_{=0 \text{ by prob 76}} g^{\alpha\tau} b_{\sigma\tau}$$

$$= g^{i\sigma} g^{\alpha\tau} b_{\sigma\tau|i} \stackrel{\text{prob 75}}{=} g^{\alpha\tau} g^{i\sigma} b_{\sigma i|\tau} = g^{\alpha\tau} b^i{}_{i|\tau}$$

$$= g^{\alpha\tau} \frac{db^i{}_i}{dx^\tau} = g^{\alpha\tau} \frac{dH}{dx^\tau} = (\nabla H)^a \quad \text{QED}$$

$$R^i_k \equiv R^{ij}{}_{ik}$$

$$R_{ju} \equiv R^i{}_{jik}$$

$$R \equiv R^k{}_k = g^{kj} R_{jk} = g^{kj} R^i{}_{jik} = R^{ik}{}_{ik}$$

$$R^i{}_{jk\ell m} + R^i{}_{jmk\ell} + R^i{}_{j\ell mk} = 0$$

raise 2nd index

$$R^{ij}{}_{k\ell m} + R^{ij}{}_{mk\ell} + R^{ij}{}_{\ell mk} = 0$$

contract $i \leftrightarrow k$
 $j \leftrightarrow \ell$

$$R^{ij}{}_{ijm} + R^{ij}{}_{mij} + R^{ij}{}_{jmi} = 0$$

$$R^{ij}{}_{ijm} = R^{ij}{}_{imj} + R^{ij}{}_{mji}$$

skew symmetry of $R^{ij}{}_{kl}$ in kl

$$R_{im} = R^j{}_{mij} + R^i{}_{mji}$$

def of R and $R^i{}_j$

and skew symmetry of $R^{ij}{}_{kl}$ in kl and ij

$$= 2 R^j{}_{mij}$$

$$\left(\text{ie, } R^j{}_u = R^{ij}{}_{ik} = R^{ji}{}_{ki} \right)$$

Eg 47.13

$$\Rightarrow \frac{\partial R}{\partial x^m} = 2 R^j{}_{mij}$$

$$\nabla^2 f = g^{\alpha\beta} \left(\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} - \Gamma_{\alpha\beta}^\gamma \frac{\partial f}{\partial u^\gamma} \right)$$

$$\nabla^2 \vec{x} = g^{\alpha\beta} \left(\frac{\partial^2 \vec{x}}{\partial u^\alpha \partial u^\beta} - \Gamma_{\alpha\beta}^\gamma \frac{\partial \vec{x}}{\partial u^\gamma} \right)$$

$$= g^{\alpha\beta} (\vec{x}_{\alpha\beta} - \Gamma_{\alpha\beta}^\gamma \vec{x}_\gamma)$$

$$= g^{\alpha\beta} (\vec{x}_\gamma \Gamma_{\alpha\beta}^\gamma + b_{\alpha\beta} \hat{N} - \vec{x}_\gamma \Gamma_{\alpha\beta}^\gamma)$$

$$= g^{\alpha\beta} b_{\alpha\beta} \hat{N}$$

$$= b^\beta{}_\beta \hat{N} = H \hat{N}$$

79.5

$$\mathcal{L} = F_{ik} F^{ik} \sqrt{|g|}$$

$$S = \int_M \mathcal{L} d^4x$$

$$\frac{\delta S}{\delta g^{ik}} = -T_{ik} \sqrt{|g|}$$

$$S = \int_{M^4} F_{ik} F^{ik} \sqrt{|g|} d^4x$$

$$g_{ij} = g_{ij}^0 + \epsilon n_{ij}$$

$$\dot{g}_{ij} = n_{ij}$$

$$\frac{dS}{d\epsilon} = \frac{d}{d\epsilon} \left(\int_{M^4} F_{ik} F^{ik} \sqrt{|g|} d^4x \right)_{\epsilon=0} = \int_{M^4} (F_{ik} F^{ik} \sqrt{|g|})^{\cdot} d^4x$$

$$(F_{ik} F^{ik} \sqrt{-g})^{\cdot} = (F_{ik} F_{je} g^{ij} g^{ke} \sqrt{-g})^{\cdot}$$

$$= F_{ik} F_{je} (g^{ij} g^{ke} \sqrt{-g})^{\cdot} = F_{ik} F_{je} (g^{ij} \dot{g}^{ke} \sqrt{-g}$$

$$= F_{ik} F_{je} [\dot{g}^{ij} g^{ke} \sqrt{-g} + g^{ij} \dot{g}^{ke} \sqrt{-g} + g^{ij} g^{ke} (\sqrt{-g})^{\cdot}]$$

$$- \frac{1}{2} g_{ab} \dot{g}^{ab} \sqrt{-g}$$

$$= (F_{ik} F_j{}^k \dot{g}^{ij} + F^i{}_i F_{ik} \dot{g}^{ke} - \frac{1}{2} F_{ik} F^{ik} g_{ab} \dot{g}^{ab}) \sqrt{-g}$$

$$= (F_{ik} F_j{}^k + F^k{}_j F_{ki} - \frac{1}{2} F_{ke} F^{ke} g_{ij}) \dot{g}^{ij} \sqrt{-g}$$

$$= (2 F_{ik} F_j{}^k - \frac{1}{2} F_{ke} F^{ke} g_{ij}) \dot{g}^{ij} \sqrt{|g|}$$

$\leftarrow F_{ik}$ skew symmetric

$$= \int_{M^4} 2 (F_{ik} F_j{}^k - \frac{1}{4} F_{ke} F^{ke} g_{ij}) \dot{g}^{ij} \sqrt{|g|} d^4x = \int_{M^4} \frac{\delta S}{\delta g^{ij}} \dot{g}^{ij} d^4x$$

$$T_{ij} = 2 (- F_{ik} F_j{}^k + \frac{1}{4} F_{ke} F^{ke})$$

GP # 8 M.E. Glinck

80

$$\bar{\nabla}_X Y = \nabla_X Y - \langle \nabla_X Y, N \rangle N \quad \langle X, N \rangle = \langle Y, N \rangle = \langle Z, N \rangle = 0 \Big|_{\text{on } V^{n-1}}$$

$$\nabla_X (Y) = \nabla Y(X)$$

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X \stackrel{\text{def of } \bar{\nabla}}{=} [\nabla_X Y - \langle \nabla_X Y, N \rangle N] - [\nabla_Y X - \langle \nabla_Y X, N \rangle N]$$

$$= (\nabla_X Y - \nabla_Y X) - \langle \nabla_X Y - \nabla_Y X, N \rangle N$$

Since ∇ is
Riemannian
 ∇ is symmetric

$$= [X, Y] - \langle [X, Y], N \rangle N = [X, Y] - (0)N$$

$$\Rightarrow \nabla_X Y - \nabla_Y X = [X, Y]$$

expand $X = X^i e_i \quad Y = Y^j e_j$

sum over $i = 1, \dots, n-1$

$$(e_i, N) = 0 \quad \forall i = 1, \dots, n-1$$

where $e_i = \partial_i$

$$[X, Y](f) = X^i e_i (Y^j e_j(f)) - Y^j e_j (X^i e_i(f))$$

$$= X^i e_i (Y^j) e_j(f) + X^i Y^j e_i(e_j(f))$$

$$- Y^j e_j (X^i) e_i(f) - Y^j X^i e_j(e_i(f)) \quad \left. \begin{array}{l} \text{because} \\ \text{partial} \\ \text{derivatives} \\ \text{commute} \end{array} \right\}$$

$$= (X^i e_i (Y^j) e_j - Y^j e_j (X^i) e_i)(f)$$

$$= \{X^i e_j (Y^j) - Y^j e_j (X^i)\} e_i(f) = [X, Y]^i e_i(f)$$

↑ no component
⊥ to V^{n-1}

$$\langle [X, Y], N \rangle = [X, Y]^i \langle e_i, N \rangle = 0$$

QED

$$X(Y, Z) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

∇ is Riemannian

$$= \langle \bar{\nabla}_X Y + \langle \nabla_X Y, N \rangle N, Z \rangle + \langle Y, \bar{\nabla}_X Z + \langle \nabla_X Z, N \rangle N \rangle$$

$$= \langle \bar{\nabla}_X Y, Z \rangle + \langle Y, \bar{\nabla}_X Z \rangle +$$

$$+ [\langle \nabla_X Y, N \rangle \underbrace{\langle N, Z \rangle}_{=0} + \langle \nabla_X Z, N \rangle \underbrace{\langle Y, N \rangle}_{=0}] N$$

$$= \langle \bar{\nabla}_X Y, Z \rangle + \langle Y, \bar{\nabla}_X Z \rangle$$

81

$$ds^2 = \varphi^2 dt^2 + \sum_{\alpha\beta=1}^{n-1} h_{\alpha\beta} dx^\alpha dx^\beta$$

$$\frac{d}{dt} \sqrt{\det h_{\alpha\beta}} = \underbrace{\frac{\partial (\det h_{\alpha\beta})}{\partial h^{\mu\nu}}}_{-\frac{1}{2} h^{\mu\nu} \sqrt{\det h_{\alpha\beta}}} \underbrace{\frac{\partial h_{\mu\nu}}{\partial t}}_{-2\varphi b_{\mu\nu}}$$

from eq 45.19 from eq 47.11

$$= \varphi \sqrt{\det h_{\alpha\beta}} h^{\mu\nu} b_{\mu\nu}$$

$$= \varphi \sqrt{\det h_{\alpha\beta}} b_{\mu}{}^{\mu}$$

$$= \varphi \sqrt{\det h_{\alpha\beta}} \text{tr}(b)$$

$$= \varphi H \sqrt{\det h_{\alpha\beta}}$$

$$\Rightarrow \boxed{\frac{d}{dt} (\ln \varphi) = H}$$

82

$$R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klij} = 0$$

1st Bianchi Identi. 4
Eq 44.8

$$R_{ijkl} + R_{iljk} + R_{klij} = 0 \quad (1)$$

$$(i \leftrightarrow j) \quad R_{jikl} + R_{jlki} + R_{kijl} = 0 \quad (2)$$

$$(i \leftrightarrow k) \quad R_{kijl} + R_{keji} + R_{kilj} = 0 \quad (3)$$

$$(i \leftrightarrow l) \quad R_{lijki} + R_{lijk} + R_{kijl} = 0 \quad (4)$$

$$(2) \Rightarrow -R_{ijkl} - R_{ljk i} - R_{jkil} = 0 \quad (5)$$

$$(3) \Rightarrow -R_{jkil} - R_{kijl} - R_{iljk} = 0 \quad (6)$$

$$(4) \Rightarrow -R_{ljk i} - R_{iljk} - R_{kijl} = 0 \quad (7)$$

$$(1) + (5) \Rightarrow R_{iljk} + R_{kijl} = R_{jkil} + R_{ljk i} \quad (8)$$

$$(1) + (6) \Rightarrow R_{iljk} + R_{ijkl} = R_{jkil} + R_{kijl} \quad (9)$$

$$(1) + (7) \Rightarrow R_{ijkl} + R_{kijl} = R_{kijl} + R_{ljk i} \quad (10)$$

use $R_{hijk} = -R_{ihjk}$
and $R_{hijk} = -R_{ihkj}$

$$(8) - (9) \Rightarrow R_{iklj} - R_{ijlk} = R_{ljik} - R_{lji k} \quad (11)$$

$$(11) + (10) \Rightarrow 2 R_{iklj} = 2 R_{ljik}$$

$$\Rightarrow \boxed{R_{iklj} = R_{ljik}} \quad \underline{Q.E.D.}$$

$$(83) \quad (i) \quad \langle R_M(X, Y)Z, N \rangle = (\bar{\nabla}_X B)(Y, Z) - (\bar{\nabla}_Y B)(X, Z)$$

$$R_{ijkl} X^k Y^l Z^j N^i = \langle R(X, Y)Z, N \rangle$$

$$N = \frac{\partial}{\partial x} / \left\| \frac{\partial}{\partial x} \right\| = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial x^0} \quad = R_{0jke} X^k Y^l Z^j \frac{1}{\sqrt{g_{00}}}$$

$$(\bar{\nabla}_X B)(Y, Z) - (\bar{\nabla}_Y B)(X, Z)$$

$$= (\bar{\nabla}_{X^k d_k} B)(Y^l d_l, Z^j d_j) - (\bar{\nabla}_{Y^l d_l} B)(X^k d_k, Z^j d_j)$$

$$= [(\bar{\nabla}_{d_k} B)(d_l, d_j) - (\bar{\nabla}_{d_l} B)(d_k, d_j)] X^k Y^l Z^j$$

$$= (B_{ljik} - B_{klij}) X^k Y^l Z^j$$

using the Codazzi eqn for $U^{n-1} \subset M^n$

$$\sqrt{g_{00}} (b_{ljik} - b_{klij}) X^k Y^l Z^j = R_{0jke} X^k Y^l Z^j$$

since this is true for all $X, Y, Z \perp$ to $\frac{\partial}{\partial x}$

$$\Rightarrow R_{0jke} = (b_{klij} - b_{ljik}) \sqrt{g_{00}}$$

$$\frac{1}{\sqrt{g_{00}}} R_{0jke} = b_{klij} - b_{ljik} \quad \text{since } b \text{ is self adjoint}$$

$$(ii) \quad g_{ij} = \begin{pmatrix} g_{00} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \quad g^{ij} = \begin{pmatrix} g_{00}^{-1} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}$$

$$\begin{aligned}
 -h^{\alpha\sigma} R_{\sigma\alpha\beta\gamma} &= h^{\alpha\sigma} R_{\sigma\alpha\beta\gamma} = h^{\alpha\sigma} (b_{\sigma\alpha\beta\gamma} - b_{\sigma\gamma\alpha\beta}) \\
 &= b^{\alpha}{}_{\alpha\beta\gamma} - b^{\alpha}{}_{\gamma\alpha\beta} \\
 &= -(g^{\alpha\sigma} R_{\sigma\alpha\beta\gamma} + g^{\alpha\sigma} R_{\sigma\gamma\alpha\beta}) \\
 &= -g^{\alpha i} R_{i\alpha\beta\gamma} = -R^{\alpha}{}_{\alpha\beta\gamma} \\
 \Rightarrow -R^{\alpha}{}_{\alpha\beta\gamma} &= b^{\alpha}{}_{\alpha\beta\gamma} - b^{\alpha}{}_{\gamma\alpha\beta}
 \end{aligned}$$

$$(iii) \quad R_{\alpha\beta} = R^i{}_{\alpha i \beta} = R^0{}_{0\alpha\beta} + R^{\alpha}{}_{\alpha\beta} = 0 + R^{\alpha}{}_{\alpha\beta}$$

$$R^0{}_{0\alpha\beta} = g^{0i} R_{i0\alpha\beta} = g^{00} R_{00\alpha\beta} = g^{00} R_{\alpha\beta 00} = g^{00}(0) = 0$$

$$R_{ijkl} = \langle R(d_i, d_j) d_k, d_l \rangle$$

$$R_{0000} = \langle \underbrace{R(d_0, d_0)}_{=0} d_0, d_0 \rangle = 0$$

= 0 because R is a form (antisymmetric)

84

$$\text{let } \alpha^p = d_{\mathbb{I}} \sigma_{\mathbb{I}}$$

$$*(\alpha^p) = d_{\mathbb{I}} *(\sigma_{\mathbb{I}})$$

helpful facts:

$$d \wedge *p = \langle d, p \rangle \text{vol}^n$$

$$* \sigma_{\mathbb{I}} = \pm \sigma_{\mathbb{J}} \quad (\mathbb{I}, \mathbb{J} \text{ complementary})$$

$$\Rightarrow *(\sigma_{\mathbb{I}}) = A \sigma_{\mathbb{J}}$$

so only need consider $*(\sigma_{\mathbb{I}})$

$$\sigma_{\mathbb{I}} \wedge * \sigma_{\mathbb{I}} = \langle \sigma_{\mathbb{I}}, \sigma_{\mathbb{I}} \rangle \text{vol}^n = \pm \sigma_{\mathbb{I}} \wedge \sigma_{\mathbb{I}} = \pm \delta_{\mathbb{I}\mathbb{I}}^{\dots n} \text{vol}^n \quad \left. \vphantom{\sigma_{\mathbb{I}} \wedge * \sigma_{\mathbb{I}}} \right\} \text{(no sum)}$$

$$\sigma_{\mathbb{I}} \wedge *(\sigma_{\mathbb{I}}) = \langle \sigma_{\mathbb{I}}, \pm \sigma_{\mathbb{I}} \rangle \text{vol}^n = A \sigma_{\mathbb{I}} \wedge \sigma_{\mathbb{I}} = A \delta_{\mathbb{I}\mathbb{I}}^{\dots n} \text{vol}^n$$

$$\frac{\|\sigma_{\mathbb{I}}\|^2}{\pm \|\sigma_{\mathbb{I}}\|^2} = \frac{\pm \varepsilon_{\mathbb{I}\mathbb{I}}}{A \varepsilon_{\mathbb{I}\mathbb{I}}} \Rightarrow A = \frac{\|\sigma_{\mathbb{I}}\|^2}{\|\sigma_{\mathbb{I}}\|^2} \frac{\varepsilon_{\mathbb{I}\mathbb{I}}}{\varepsilon_{\mathbb{I}\mathbb{I}}}$$

since $\mathbb{I}$ has p element and $\mathbb{J}$ has $n-p$ elements $\varepsilon_{\mathbb{I}\mathbb{I}} = (-1)^{p(n-p)} \varepsilon_{\mathbb{J}\mathbb{J}}$

Since σ is O.N. $\frac{\|\sigma_{\mathbb{I}}\|^2}{\|\sigma_{\mathbb{I}}\|^2} = \begin{cases} +1 & \text{Riemannian} \\ -1 & \text{pseudo-Riemannian} \end{cases}$

$$\Rightarrow *(\alpha^p) = \alpha^p \begin{cases} (-1)^{p(n-p)} & \text{Riemannian } M^n \\ (-1)^{p(n-p)+1} & \text{Pseudo-Riemannian } M^n \end{cases}$$

25

$$*F = *(E \wedge dt + B^2)$$

$$= *(E \wedge dt) + *B$$

$$= *(E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt) + *(B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy)$$

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \text{vol}^n$$

$$*(\sigma^{i_1, \dots, i_p} \wedge \sigma^{j_1, \dots, j_{n-p}}) = \pm \sigma^{j_1, \dots, j_{n-p}}$$

$$dx \wedge dt \wedge *(dx \wedge dt) = \langle dx \wedge dt, dx \wedge dt \rangle dt \wedge dx \wedge dy \wedge dz = -dx \wedge dt \wedge [dx \wedge dt, dx \wedge dt] dz \wedge dy \wedge dz$$

$$\langle \alpha, \beta \rangle = \alpha_i \beta^i$$

$$d\tau^2 = dt^2 - d\vec{x} \cdot d\vec{x}$$

$$= dx \wedge dt \wedge (dy \wedge dz)$$

$$\Rightarrow * (dx \wedge dt) = dy \wedge dz$$

$$dy \wedge dt \wedge *(dy \wedge dt) = \langle dy \wedge dt, dy \wedge dt \rangle dt \wedge dx \wedge dy \wedge dz = dy \wedge dt \wedge (\langle dy \wedge dt, dy \wedge dt \rangle dx \wedge dz) = dy \wedge dt \wedge (dz \wedge dx)$$

$$\Rightarrow * (dy \wedge dt) = dz \wedge dx$$

$$dz \wedge dt \wedge *(dz \wedge dt) = \langle dz \wedge dt, dz \wedge dt \rangle dt \wedge dx \wedge dy \wedge dz = -dz \wedge dt \wedge [\langle dz \wedge dt, dz \wedge dt \rangle dx \wedge dy] = dz \wedge dt \wedge (dx \wedge dy)$$

$$\Rightarrow * (dz \wedge dt) = dx \wedge dy$$

$$dy \wedge dz \wedge *(dy \wedge dz) = \langle dy \wedge dz, dy \wedge dz \rangle dt \wedge dx \wedge dy \wedge dz = dy \wedge dz \wedge (\langle dy \wedge dz, dy \wedge dz \rangle dt \wedge dx) = dy \wedge dz \wedge (dt \wedge dx)$$

$$\Rightarrow * (dy \wedge dz) = -dx \wedge dt$$

$$\begin{aligned}
 dz \wedge dx \wedge *(dz \wedge dx) &= \langle dz \wedge dx, dz \wedge dx \rangle dt \wedge dx \wedge dz \wedge dx \\
 &= dz \wedge dx \wedge (\langle dz \wedge dx, dz \wedge dx \rangle dt \wedge dx) \\
 &= dz \wedge dx \wedge (dt \wedge dx)
 \end{aligned}$$

$$\Rightarrow \boxed{*(dz \wedge dx) = -dx \wedge dt}$$

$$\begin{aligned}
 dx \wedge dz \wedge *(dx \wedge dz) &= \langle dx \wedge dz, dx \wedge dz \rangle dt \wedge dx \wedge dz \wedge dz \\
 &= dx \wedge dz \wedge (\langle dx \wedge dz, dx \wedge dz \rangle dt \wedge dz) \\
 &= dx \wedge dz \wedge (dt \wedge dz)
 \end{aligned}$$

$$\Rightarrow \boxed{*(dx \wedge dz) = -dz \wedge dt}$$

$$\begin{aligned}
 &= E_1 *(dx \wedge dt) + E_2 *(dz \wedge dt) + E_3 *(dz \wedge dx) \\
 &\quad + B_1 *(dz \wedge dx) + B_2 *(dz \wedge dx) + B_3 *(dx \wedge dz)
 \end{aligned}$$

$$\begin{aligned}
 &= (E_1 dz \wedge dx + E_2 dz \wedge dx + E_3 dx \wedge dz) \\
 &\quad - (B_1 dx \wedge dt + B_2 dz \wedge dt + B_3 dz \wedge dt)
 \end{aligned}$$

$$= \mathcal{D}^2 - \mathcal{H}' \wedge dt$$

$$\Rightarrow \boxed{*\mathcal{F}_1 = -\mathcal{H}' \wedge dt + \mathcal{D}^2}$$

GP #9 M E Glinsky

(i) $d^* f^0 \stackrel{48.10}{=} (-)^{3(0+1)+1} * d f^0 = * d f^0 = * d(\underbrace{f \text{ vol}_0}_{n\text{-form}})$
 $= * d(\gamma^3) = 0$
 $d\gamma^3 \stackrel{\uparrow}{=} 0 \text{ in } \mathbb{R}^3$

48.11 45.6

(ii) $d^* \alpha^1 \stackrel{48.11}{=} -\nabla_j A^j = -A^j{}_{;j} = -\nabla \cdot \vec{A} = -\partial_j A^j$
 $\nearrow \text{since } \mathbb{R}^3 \quad \Gamma^k{}_l = 0$

(iii) $d^* \beta^2 = d^*(i_{\vec{B}_0} \text{vol}_0) = (-)^{3(2+1)+1} * d^*(i_{\vec{B}_0} \text{vol}_0)$
 $= * d^*(i_{\vec{B}_0} \text{vol}_0) = * d^*(\beta^1) = * d[(-)^{1(3-1)} \beta^1]$
 $= * d\beta^1 = *(i_{\nabla \times \vec{B}_0} \text{vol}_0)$

(iv) $d^* \gamma^3 = d^*(\beta^0) = (-)^{3(3+1)+1} * d^*(\beta^0)$
 $= (-) * d[(-)^{0(3-0)} \beta^0] = - * d\beta^0$
 $= -i_{\nabla f} \text{vol}_0$

(v) $\Delta f^0 = (dd^* + d^*d)f^0 = \underbrace{dd^*f^0}_{=0} + \underbrace{d^*df^0}_{\Gamma^k{}_l=0}$
 $= d^*[d_i f dx^i] = -\nabla_j [g^{ji} d_i f] = -g^{ji} \partial_j \partial_i f$
 $= -\partial^i \partial_i f = -\nabla^2 f$
 $g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } \mathbb{R}^3$

(vi) $\Delta \alpha^1 = (dd^* + d^*d)\alpha^1 = dd^*\alpha^1 + d^*d\alpha^1$
 $= -d(\nabla \cdot \vec{A}) + d^*[i_{\nabla \times \vec{A}} \text{vol}_0]$
 $= *[i_{\nabla \times \nabla \times \vec{A}} \text{vol}_0] - *[i_{\nabla(\nabla \cdot \vec{A})} \text{vol}_0]$
 $= *[i_{\nabla \times \nabla \times \vec{A} - \nabla(\nabla \cdot \vec{A})} \text{vol}_0]$
 $= g_{ji} [\nabla \times \nabla \times \vec{A} - \nabla(\nabla \cdot \vec{A})]^i dx^j = [\nabla \times \nabla \times \vec{A} - \nabla(\nabla \cdot \vec{A})]_j dx^j$

note: since $\mathbb{R}^3$ contravariant and covariant components of a vector are the same

$$\begin{aligned}
 (vii) \quad \Delta \beta^2 &= (d d^* + d^* d) \beta^2 = d d^* \beta^2 + d^* d \beta^2 \\
 &= d[* (i \nabla \times \vec{B} \text{ vol}_0)] + d^* [+ \nabla \cdot \vec{B} \text{ vol}_0] \\
 &= i \nabla \times \nabla \times \vec{B} \text{ vol}_0 - i \nabla (\nabla \cdot \vec{B}) \text{ vol}_0 \\
 &= i \nabla \times \nabla \times \vec{B} - \nabla (\nabla \cdot \vec{B}) \text{ vol}_0 \rightarrow 0
 \end{aligned}$$

$$\begin{aligned}
 (viii) \quad \Delta(*f) &= d d^* + d^* d = d d^*(*f) + d^* d(*f) \\
 &= d(-i \nabla f \text{ vol}_0) = -\nabla \cdot (\nabla f) \text{ vol}_0 \\
 &= -\nabla^2 f \text{ vol}_0 = -\nabla^2 f * 1
 \end{aligned}$$

(87) F^p is a p -form on a (pseudo-) Riemannian M^n

$$(D_\mu F^p)^K = F^{jK}{}_{ij} = \nabla_j F^{jK} =$$

$$= \frac{\partial F^{jK}}{\partial x^j} + F^{lK} \Gamma_{jl}^j + [F^{j l k_2 \dots k_{p-1}} \Gamma_{jl}^{k_1} + \dots + F^{j k_1 \dots k_{p-2} l} \Gamma_{jl}^{k_{p-1}}]$$

$$= \frac{\partial F^{jK}}{\partial x^j} + F^{lK} \frac{\partial}{\partial x^l} (\ln \sqrt{g}) + [nm]$$

$$= \frac{1}{\sqrt{g}} \left\{ \sqrt{g} \frac{\partial F^{jK}}{\partial x^j} + F^{jK} \frac{\partial (\sqrt{g})}{\partial x^j} + \sqrt{g} [nm] \right\}$$

$$= \frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial x^j} (\sqrt{g} F^{jK}) + \sqrt{g} F^{j l k_2 \dots k_{p-1}} \Gamma_{jl}^{k_1} + \dots + \sqrt{g} F^{j k_1 \dots k_{p-2} l} \Gamma_{jl}^{k_{p-1}} \right\}$$

$$F^{...j...l...} \Gamma_{jl}^k = -F^{...l...j...} \Gamma_{lj}^k = -F^{...j...l...} \Gamma_{jl}^k$$

$\uparrow$
 symmetric in bottom index
 F anti symmetric since form
 $\uparrow$
 l and j are dummy index so switch

$$\Rightarrow F^{...j...l...} \Gamma_{jl}^k = 0$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} F^{jK})$$

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Since $P^3(\mathbb{R})$ is connected $H_0(P^3(\mathbb{R}), \mathbb{Z}) = \mathbb{Z} p_0$
 since $\partial B_2^+ = 2A_1$, $2A_1 \sim 0$ and $A_1 = \partial(\frac{1}{2} B_2^+)$ but
 $\frac{1}{2} B_2^+ \notin C_2(P^3(\mathbb{R}), \mathbb{Z})$

$$H_1(P^3(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}_2 A_1$$

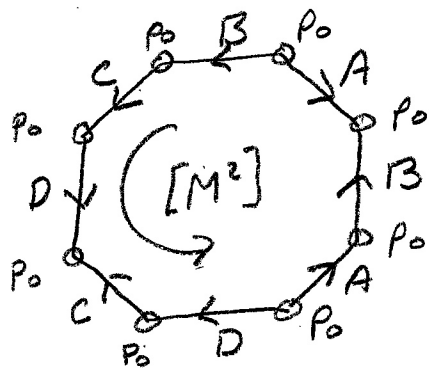
Since $\partial(B_2^+) = 2A_1$ and $\partial(B_2^-) = -2A_1$ (i.e., $B_2^\pm$ are not
 2-cycles) $H_2(P^3(\mathbb{R}), \mathbb{Z}) = 0$

$$\partial[P^3] = B_2^+ + B_2^- = 0 \quad \therefore H_3(P^3, \mathbb{Z}) = \mathbb{Z} P^3$$

In summary

$$\begin{aligned} H_0(P^3(\mathbb{R}), \mathbb{Z}) &= \mathbb{Z} p_0 \\ H_1(P^3(\mathbb{R}), \mathbb{Z}) &= \mathbb{Z}_2 A_1 \\ H_2(P^3(\mathbb{R}), \mathbb{Z}) &= 0 \\ H_3(P^3(\mathbb{R}), \mathbb{Z}) &= \mathbb{Z} P^3 \end{aligned}$$

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Again since $\partial[M^2] = 2B$

$$H_2(M^2, \mathbb{R}) = 0$$

$$H_2(M^2, \mathbb{Z}) = 0$$

$$H_2(M^2, \mathbb{Z}_2) = \mathbb{Z}_2 M^2$$

Since M^2 is connected

$$H_0(M^2, \mathbb{R}) = \mathbb{R} p_0$$

$$H_0(M^2, \mathbb{Z}) = \mathbb{Z} p_0$$

$$H_0(M^2, \mathbb{Z}_2) = \mathbb{Z}_2 p_0$$

A, B, C, D are 1-cycles but

$$\begin{aligned} \partial[M^2] &= B + C + D - C - D + A + B - A \\ &= 2B \end{aligned}$$

$$B = \partial[\frac{1}{2} M^2] \in C_1(M^2, \mathbb{R})$$

$$\notin C_1(M^2, \mathbb{Z})$$

$$\Rightarrow H_1(M^2, \mathbb{R}) = \mathbb{R} A + \mathbb{R} C + \mathbb{R} D$$

$$H_1(M^2, \mathbb{Z}) = \mathbb{Z} A + \mathbb{Z}_2 B + \mathbb{Z} C + \mathbb{Z} D$$

$$H_1(M^2, \mathbb{Z}_2) = \mathbb{Z}_2 A + \mathbb{Z}_2 B + \mathbb{Z}_2 C + \mathbb{Z}_2 D$$

In summary,

$$H_0(M^2, \mathbb{R}) = \mathbb{R} p_0$$

$$H_1(M^2, \mathbb{R}) = \mathbb{R} A + \mathbb{R} C + \mathbb{R} D$$

$$H_2(M^2, \mathbb{R}) = 0$$

$$k_0 = 1$$

$$b_1 = 3$$

$$b_2 = 0$$

$$H_0(M^2, \mathbb{Z}) = \mathbb{Z} p_0$$

$$H_1(M^2, \mathbb{Z}) = \mathbb{Z} A + \mathbb{Z}_2 B + \mathbb{Z} C + \mathbb{Z} D$$

$$H_2(M^2, \mathbb{Z}) = 0$$

Since M^2 is not a 2-cycle M^2 is not orientable

GP #1

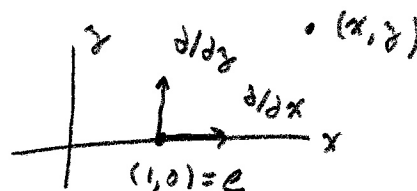
M. E. Glinck

35
35

10 $A(1) = \left\{ \begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix}, x > 0 \right\}$

$$\begin{pmatrix} \dot{x} & \dot{z} \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \dot{x} \frac{\partial}{\partial x} + \dot{z} \frac{\partial}{\partial z}$$

$$\begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix} \Leftrightarrow (x, z)$$



Consider curve through e $C(t) = \begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix}$ whose tangent at e is $\frac{d}{dx}$

$$\left[R_{\begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix}} \right]_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{d}{dt} \bigg|_{t=0} \begin{pmatrix} 1+t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dot{x} & \dot{z} \\ 0 & 0 \end{pmatrix} \Leftrightarrow x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}$$

Consider curve through e $C(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ whose tangent at e is $\frac{d}{dz}$

$$\left[R_{\begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix}} \right]_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{d}{dt} \bigg|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \frac{d}{dz}$$

$\Rightarrow$ Basis for right invariant vector fields on $A(1)$ is given by

$$\boxed{X_1 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \quad X_2 = \frac{\partial}{\partial z}}$$

$\Rightarrow$ Basis for right invariant 2-forms on $A(1)$ is given by

$$\left. \begin{aligned} 1 &= \sigma^1(X_1) = (a dx + b dz) \left(x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \right) = ax + bz \\ 0 &= \sigma^1(X_2) = (a dx + b dz) \left(\frac{\partial}{\partial z} \right) = b \\ 0 &= \sigma^2(X_1) = (c dx + d dz) \left(x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \right) = cx + dz \\ 1 &= \sigma^2(X_2) = (c dx + d dz) \left(\frac{\partial}{\partial z} \right) = d \end{aligned} \right\}$$

$$\boxed{\begin{aligned} \sigma^1 &= \frac{dx}{x} \\ \sigma^2 &= -\frac{z}{x} dx + dz \end{aligned}}$$

$$\boxed{\text{"Right Haar Measure"} = \sigma^1 \wedge \sigma^2 = \frac{dx \wedge dz}{x}}$$

5

(91)

$$J^2 = -I \quad J^3 = -J \quad J^4 = I \quad \text{etc}$$

$$J^{2n} = (-1)^n I$$

$$e^{\theta J} = I + \theta J + \frac{1}{2!}(\theta J)^2 + \dots$$

$$= I + \frac{\theta^2}{2!} J^2 + \frac{\theta^4}{4!} J^4 + \dots$$

$$+ J \left[\theta I + \frac{\theta^3}{3!} J^2 + \frac{\theta^5}{5!} J^4 \right]$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} J^{2n} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} J^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} (-1)^n I + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} J (-1)^n I$$

$$= I \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + J \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} (-1)^n$$

$$= I \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left[\frac{d}{d\theta} \cos \theta \right]_{\theta=0} + J \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left[\frac{d}{d\theta} \sin \theta \right]_{\theta=0}$$

$$= I \cos \theta + J \sin \theta$$

$$\Rightarrow e^{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

GP #2 M E Glinsky

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$$d\sigma^u(\bar{X}_R, \bar{X}_S) = -\sigma^u([X_R, X_S])$$

$$= -\sigma^u(\bar{X}_T C_{RS}^T) = -C_{RS}^T \sigma^u(X_T)$$

$$= -C_{RS}^T \delta_T^u = -C_{RS}^u = -C_{XY}^u \delta_R^X \delta_S^Y$$

$$= -C_{XY}^u \sigma^X(\bar{X}_R) \sigma^Y(\bar{X}_S)$$

$$= -\frac{1}{2} \{ C_{XY}^u \sigma^X(\bar{X}_R) \sigma^Y(\bar{X}_S) + C_{YX}^u \sigma^Y(\bar{X}_R) \sigma^X(\bar{X}_S) \}$$

$$= -\frac{1}{2} C_{XY}^u [\sigma^X(\bar{X}_R) \sigma^Y(\bar{X}_S) - \sigma^Y(\bar{X}_R) \sigma^X(\bar{X}_S)]$$

$$= -\frac{1}{2} C_{XY}^u \sigma^X \wedge \sigma^Y(\bar{X}_R, \bar{X}_S)$$

$$= -\frac{1}{2} \sum_{X,Y} C_{XY}^u \sigma^X \wedge \sigma^Y(\bar{X}_R, \bar{X}_S) \quad \left. \begin{array}{l} \text{skew symmetry} \\ \text{of } C \text{ and } \sigma \wedge \sigma \end{array} \right\}$$

$$= \left[- \sum_{X < Y} C_{XY}^u \sigma^X \wedge \sigma^Y \right] (\bar{X}_R, \bar{X}_S)$$

$$0 = d^2 \sigma^u = d \left[-\frac{1}{2} C_{RS}^u \sigma^R \wedge \sigma^S \right]$$

$$= -\frac{1}{2} C_{RS}^u [d\sigma^R \wedge \sigma^S - \sigma^R \wedge d\sigma^S]$$

$$= -\frac{1}{2} C_{RS}^u \left[-\frac{1}{2} C_{LM}^R \sigma^L \wedge \sigma^M \wedge \sigma^S + \frac{1}{2} C_{LM}^S \sigma^R \wedge \sigma^L \wedge \sigma^M \right]$$

$$= \frac{1}{4} [C_{RS}^u C_{LM}^R \sigma^L \wedge \sigma^M \wedge \sigma^S - C_{RS}^u C_{LM}^S \sigma^R \wedge \sigma^L \wedge \sigma^M]$$

$$= \frac{1}{4} [C_{RS}^u C_{LM}^R \sigma^L \wedge \sigma^M \wedge \sigma^S + C_{SR}^u C_{LM}^S \sigma^R \wedge \sigma^L \wedge \sigma^M]$$

$$= \frac{1}{4} [C_{RS}^u C_{LM}^R \sigma^L \wedge \sigma^M \wedge \sigma^S + C_{RS}^u C_{LM}^R \sigma^S \wedge \sigma^L \wedge \sigma^M]$$

$$= \frac{1}{4} [C_{RS}^u C_{LM}^R + C_{RS}^u C_{LM}^R] \sigma^L \wedge \sigma^M \wedge \sigma^S$$

$$= \frac{1}{2} C_{RS}^u C_{LM}^R \sigma^L \wedge \sigma^M \wedge \sigma^S$$

$$= \frac{1}{2} \sum_{L < M < S} C_R^u [C_{LM}^R] \sigma^L \wedge \sigma^M \wedge \sigma^S \Rightarrow C_R^u [C_{LM}^R] = 0$$

$$0 = d^2 \sigma^u(X_L, X_M, X_R) = d\sigma$$

$$\stackrel{\text{eq. 18.10}}{=} -d\sigma^u([X_L, X_M], X_R) + d\sigma^u([X_L, X_R], X_M) - d\sigma^u([X_M, X_R], X_L)$$

$$= \sigma^u([X_L, X_M], X_R) - \sigma^u([X_L, X_R], X_M) + \sigma^u([X_M, X_R], X_L)$$

$$= \sigma^u([X_L, X_M], X_R) + \sigma^u([X_R, X_L], X_M) + \sigma^u([X_M, X_R], X_L)$$

$$\Rightarrow [X_L, X_M], X_R + [X_M, X_R], X_L + [X_R, X_L], X_M = 0$$

$$0 = C_{LM}^T [X_T, X_R] + C_{MR}^T [X_T, X_L] + C_{RL}^T [X_T, X_M]$$

$$= C_{LM}^T C_{TR}^S X_S + C_{MR}^T C_{TL}^S X_S + C_{RL}^T C_{TM}^S X_S$$

$$= (C_{TR}^S C_{LM}^T + C_{TL}^S C_{MR}^T + C_{TM}^S C_{RL}^T) X_S$$

$$\Rightarrow (C_{TR}^S C_{LM}^T + C_{TL}^S C_{MR}^T + C_{TM}^S C_{RL}^T) X_S = 0$$

$$(9.3) \quad e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (A^2)^n + A \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (A^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-\Delta I)^n + A \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-\Delta I)^n$$

$$= \begin{cases} I \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\Delta^{1/2})^{2n} + \frac{A}{\Delta^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\Delta^{1/2})^{2n+1} & (\Delta > 0) \\ I \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\sqrt{|\Delta|})^{2n} + \frac{A}{\sqrt{|\Delta|}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\sqrt{|\Delta|})^{2n+1} & (\Delta < 0) \end{cases}$$

$$= \begin{cases} (\cos \sqrt{\Delta}) I + \left(\frac{1}{\sqrt{\Delta}} \sin \sqrt{\Delta}\right) A & (\Delta > 0) \\ (\cosh \sqrt{|\Delta|}) I + \left(\frac{1}{\sqrt{|\Delta|}} \sinh \sqrt{|\Delta|}\right) A & (\Delta < 0) \end{cases}$$

$$\Delta = 0?$$

$$\text{tr}(e^A) = \text{tr}(aI + bA) = a \text{tr}(I) + b \text{tr}(A)$$

$$= a(2) = 2a = \begin{cases} 2 \cos \sqrt{\Delta} \\ 2 \cosh \sqrt{|\Delta|} \end{cases} \geq \begin{cases} 2(-1) \\ 2(1) \end{cases} \geq -2$$

$$\text{Consider } g_0 = \begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\text{tr } g_0 = -2 - \frac{1}{2} < -2$$

$$\therefore g_0 \neq e^A$$

$$\text{since } \text{tr } e^A \geq -2$$

$$\text{tr } g_0 < -2$$

(94)

$$[X_1, X_2] = \lim_{t \rightarrow 0} \frac{1}{t^2} [e^{tX_1} e^{tX_2} e^{-tX_1} e^{-tX_2} - I]$$

$$= \lim_{t \rightarrow 0} \frac{1}{t^2} [(I + tX_1 + \dots)(I + tX_2 + \dots)(I - tX_1 + \dots)(I - tX_2 + \dots) - I]$$

$$= \lim_{t \rightarrow 0} \frac{1}{t^2} \left\{ [I + t(X_1 + X_2) + t^2(X_1X_2 + \frac{X_1^2}{2} + \frac{X_2^2}{2}) + O(t^3)] [I - t(X_1 + X_2) + t^2(X_1X_2 + \frac{X_1^2}{2} + \frac{X_2^2}{2}) + O(t^3)] - I \right\}$$

$$= \lim_{t \rightarrow 0} \frac{1}{t^2} \left\{ I + t(0) + t^2(X_1^2 + X_2^2 + 2X_1X_2 - (X_1 + X_2)^2) + O(t^3) \right\}$$

$$= \lim_{t \rightarrow 0} \frac{1}{t^2} \left\{ t^2(X_1X_2 - X_2X_1) + O(t^3) \right\}$$

$$= \lim_{t \rightarrow 0} \left\{ (X_1X_2 - X_2X_1) + O(t) \right\} = X_1X_2 - X_2X_1$$

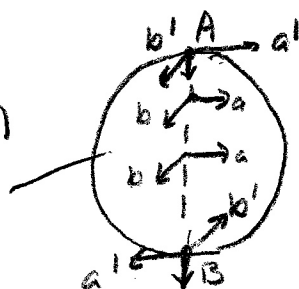
(95)

Since $\psi_v = C_{vu} \psi_u$ where C_{vu} is a non-singular linear transformation, if $\psi_u = 0$ then $\psi_v = 0$.

In other words, the zero section is a global cross-section.

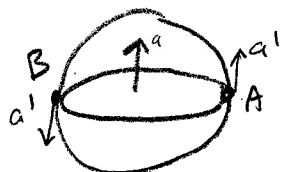
(96)

(i)



Since b and a only need to be twisted around to coincide with a' b' at points A and B . The normal bundle is trivial.

(ii)



Obviously in going from A to B a must equal zero.

The normal bundle is non-trivial.

(iii) The normal bundle to S^1 in all V^3 (orientable) is trivial. It is to twist the vectors.

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(97) $C_{uv} = e^{\frac{ie}{\hbar} f^{uv}} \quad A^v = A^u + df^{vu}$

$$A^u = g(1 - \cos \theta) d\varphi$$

$$A^v = -g(1 + \cos \theta) d\varphi$$

$$\Rightarrow df^{uv} = A^u - A^v = g(1 - \cos \theta) d\varphi + g(1 + \cos \theta) d\varphi = 2g d\varphi = d(2g\varphi)$$

$$\Rightarrow f^{uv} = 2g\varphi + \text{constant}$$

5 $\Rightarrow C_{uv} = e^{\frac{2ieg}{\hbar} \varphi} \times \text{complex constant}$

(98) $U = \mathbb{R}^3 - \text{neg } z \text{ axis}$
 $V = \mathbb{R}^3 - \text{pos } z \text{ axis}$

We don't ask $\psi_v = \psi_u$, only $\psi_v = C_{uv} \psi_u$
 and ψ_v must be well defined

0 now let $\psi_u(x) \equiv 1$ but $\psi_v(x) = C_{vu}(x) \psi_u(x) = e^{\frac{2ieg}{\hbar} \varphi(x)}$
 which does not equal 1 unless $\frac{2ieg}{\hbar} \varphi = 2\pi n$ this can not
 be true for $\varphi \neq 0$ unless $\frac{2ieg}{\hbar} = 0$ but $\frac{2ieg}{\hbar} \neq 0$.
 Therefore ψ_u cannot be extended to a global cross-section

(99) Apply thm on page 50 with $G = \mathbb{R} = \text{Lie group of real numbers}$
 $H = \mathbb{Z}$ is the closed subgroup of $\mathbb{R}$. Now $\mathbb{R}/\mathbb{Z} = S^1$, therefore
 $\mathbb{R}$ is a principle bundle with structure group $\mathbb{Z}$ and base space S^1 .

5

does it?

100

0

Let $G = O(n)$ act transitively on $M = Gr(p, n)$. Let $x_0 \in M$ be the plane oriented along the first p -axis. x_0 is fixed under orientation change of the first p -axis and the last $(n-p)$ axis, in other words under action of the subgroup

this is more than
orientation
change.

$H = O(p) \times O(n-p) \subset G$. Therefore $M \approx G/H$ or

product,
not sum.

$$Gr(p, n) \approx O(n) / [O(p) \oplus O(n-p)]$$

$$\text{Now } Gr(1, n) \approx O(n) / [O(1) \times O(n-1)]$$

5

$$= \frac{[O(n)]}{\begin{bmatrix} \pm 1 & 0 \\ 0 & O(n-1) \end{bmatrix}} = p^{n-1}(\mathbb{R})$$

$$\dim[Gr(p, n)] = \dim\{O(n) / [O(p) \oplus O(n-p)]\}$$

$$= \dim O(n) - \dim O(p) - \dim O(n-p)$$

$$= \frac{n(n-1)}{2} - \frac{p(p-1)}{2} - \frac{(n-p)(n-p-1)}{2}$$

$$= \frac{1}{2} [n^2 - n - p^2 + p - (n^2 - 2np + p^2 - n + p)]$$

5

$$= \frac{1}{2} (-2p^2 + 2np) = p(n-p) \quad \angle$$

(F1)

$$F^2 = E^1 \wedge dt + B^2 \quad \text{in } M_0^4 \quad *F_1 = -\mathcal{H}^1 \wedge dt + D^2$$

$$F \wedge *F = (E^1 \wedge dt + B^2) \wedge (-\mathcal{H}^1 \wedge dt + D^2)$$

$$= -E^1 \wedge dt \wedge \mathcal{H}^1 \wedge dt - B^2 \wedge \mathcal{H}^1 \wedge dt + E^1 \wedge dt \wedge D^2 + B^2 \wedge D^2$$

↑ equals 0 since Λ is skew symmetric and dt occurs twice

↑ equals zero because $B^2 = i\vec{B} \text{ vol}_0^{\mathbb{R}^3}$ and $D^2 = i\vec{E} \text{ vol}_0^{\mathbb{R}^3}$ have only dx, dy, dz terms but $B^2 \wedge D^2$ is a 4-form i.e. two terms must be repeated

$$= -B^2 \wedge \mathcal{H}^1 \wedge dt + E^1 \wedge dt \wedge D^2$$

$$= dt \wedge (B^2 \wedge \mathcal{H}^1 - D^2 \wedge E^1)$$

$$B^2 \wedge \mathcal{H}^1 = (B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy) \wedge (B_1 dx + B_2 dy + B_3 dz)$$

$$= (B_1^2 + B_2^2 + B_3^2) dx \wedge dy \wedge dz = |B|^2 dx \wedge dy \wedge dz$$

in a similar way $D^2 \wedge E^1 = |E|^2 dx \wedge dy \wedge dz$

$$B^2 \wedge \mathcal{H}^1 = \vec{B} \cdot \vec{E} dx \wedge dy \wedge dz$$

$$D^2 \wedge E^1 = \vec{E} \cdot \vec{E} dx \wedge dy \wedge dz$$

$$= dt \wedge (|B|^2 - |E|^2) dx \wedge dy \wedge dz$$

$$= (|B|^2 - |E|^2) dt \wedge dx \wedge dy \wedge dz$$

10

$$F \wedge F = (E^1 \wedge dt + B^2) \wedge (E^1 \wedge dt + B^2)$$

$$= E^1 \wedge dt \wedge E^1 \wedge dt + E^1 \wedge dt \wedge B^2 + B^2 \wedge E^1 \wedge dt + B^2 \wedge B^2$$

$$= E^1 \wedge dt \wedge B^2 + B^2 \wedge E^1 \wedge dt$$

$$= dt \wedge (-E^1 \wedge B^2 - B^2 \wedge E^1)$$

$$= dt \wedge (-2 B^2 \wedge E^1)$$

$$= dt \wedge (-2 \vec{B} \cdot \vec{E} dx \wedge dy \wedge dz) = -2 \vec{B} \cdot \vec{E} dt \wedge dx \wedge dy \wedge dz$$

(F2)

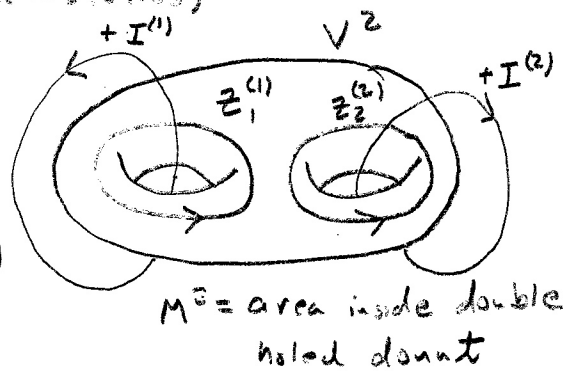
$\vec{B}$ tangent to $V^2 = \partial M^3$ (magnetostatic)

$\beta^2 = i_{\vec{B}} \text{vol}_3$ is normal to ∂M

$\eta^1 = * \beta^2$ is tangent to ∂M

$$d * \eta^1 = d * \beta^2 = \pm d \beta^2 = 0 \quad (\text{Maxwell Eq})$$

$$d \eta^1 = 4\pi j^2 = 0 \quad (\text{no currents in } M^3 \text{ with Maxwell Eq})$$



i. η^1 is a tangent harmonic 1-form on a compact Riemannian Manifold with boundary

A basis for $H_1(M^3; \mathbb{R})$ is given by $z_1^{(1)}$ and $z_2^{(2)}$ (since $b_2 = 2$)

By thm on page 156

$\exists$ 1 tangent harmonic 1-form η^1

with prescribed periods $\int_{z_p^{(i)}} \eta^1 = 4\pi I^{(i)}$

Since this manifold is embedded in $\mathbb{R}^3$

$$4\pi I^{(i)} = \int_{z_p^{(i)}} \eta^1 = \int_{A^{(i)}} d\eta^1 = \int_{A^{(i)}} 4\pi j^2 \Rightarrow I^{(i)} = \int_{A^{(i)}} j^2$$

$$z_p^{(i)} = \partial A^{(i)}$$

where $A^{(i)}$ is an area in $\mathbb{R}^3$

(10)

(F3)

(i) $F: S^2 \rightarrow T^2$ (F^* exists)

$$\dim[deR^p(M)] = \dim[H^p(M; \mathbb{R})] = b_p(M) \quad (\text{p 148})$$

Since $b_1(T^2) = 2$ (p. 139) let β_1, β_2 be a basis for $deR^1(T^2)$ and z_1, z_2 its dual basis such that $\int \beta_i(z_j) = \delta_{ij}^m$.

$\dim(deR^1(S^2)) = b_1(S^2) = 0$ (p 139) which implies that all closed 1-forms on S^2 are exact

Obviously $d(F^*\beta_i) = F^*(d\beta_i) = 0$ since $\beta_i \in deR^1(T^2)$. In other words, $F^*\beta_i$ are closed. Since $b_1(S^2) = 0$ they must be exact.

(ii) $\deg(F) \equiv \int_{S^2} F^*\gamma^2$ where $\int_{T^2} \gamma^2 = 1$

Let $\gamma^2 = \beta_1 \wedge \beta_2$

Then $\deg(F) = \int_{S^2} F^*(\beta_1 \wedge \beta_2) = \int_{S^2} F^*\beta_1 \wedge F^*\beta_2$

$\left\{ \begin{array}{l} \beta_i = df_i \\ \text{ie, } \beta_i \text{ is exact} \end{array} \right\} \Rightarrow \int_{S^2} df_1 \wedge df_2 = \int_{S^2} d(f_1 df_2) \xrightarrow{\text{Stokes thm}} \int_{\partial S^2} f_1 df_2 = 0$
 $\uparrow$ def of d and $d^2 = 0$

(F4)

X generates a flow φ_t such that $X\langle Y, Z \rangle = 0$
(i.e., φ_t is a one parameter group of isometries)

(ii) Let Y, Z be invariant under flow defined by φ_t
(i.e., $\mathcal{L}_X Y = \mathcal{L}_X Z = 0$)

By problem 63 $\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X$

$$\Rightarrow \nabla_X Y = \nabla_Y X \text{ and } \nabla_X Z = \nabla_Z X$$

$$\begin{aligned} \text{Since } 0 = X\langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ &= \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \end{aligned}$$

one finds that $0 = \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle$

(5)

(iii) Now note that $\nabla_Y X$ and $\nabla_Z X$ depends only on the vectors Y and Z but on the derivatives of the vector field X . Therefore Y, Z only need be known locally (i.e., they need not be vector fields).

$$\begin{aligned} \text{(iii)} \quad 0 &= \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = Y^i X^j{}_{;i} Z^k g_{jk} + Y^j Z^i X^k{}_{;i} g_{jk} \\ &= Y^i Z^k (X^j{}_{;i} g_{jk} + X^k{}_{;i} g_{ji}) \\ &= Y^i Z^k (X_{k;i} + X_{i;k}) \end{aligned}$$

Since Y and Z are arbitrary $X_{k;i} + X_{i;k} = 0$ ($\forall i, k$)

$$\begin{aligned} \text{(iv)} \quad 0 &= g^{ik} (X_{k;i} + X_{i;k}) = X^i{}_{;i} + X^k{}_{;k} = 2 X^i{}_{;i} = 2 \operatorname{div} X \\ &\Rightarrow \operatorname{div} X = 0 \end{aligned}$$

(No this does not surprise me since if $\operatorname{div} X \neq 0$ net stuff will flow out of an volume causing an obvious stretch)

$$(\Delta \varphi, \varphi) = \int_M \langle \Delta \varphi, \varphi \rangle * 1 = \int_M \langle (dd^* + d^*d) \varphi, \varphi \rangle * 1$$

$$= \int_M [\langle dd^* \varphi, \varphi \rangle + \langle d^*d \varphi, \varphi \rangle] * 1$$

$$= \int_M [\langle d^* \varphi, d^* \varphi \rangle + \langle d \varphi, d \varphi \rangle] * 1$$

$$\stackrel{\text{prob 86 (ii)}}{=} \int_M [\langle -\operatorname{div} \varphi, -\operatorname{div} \varphi \rangle + \langle d \varphi, d \varphi \rangle] * 1$$

$$\textcircled{10} \quad = \int_M \langle d \varphi, d \varphi \rangle * 1 = (d \varphi, d \varphi) = \|d \varphi\|^2$$

(v)

Consider $d \varphi = d_j X_i dx^j \wedge dx^i$
 $= X_{ij} dx^j \wedge dx^i$
 $= -X_{ji} dx^i \wedge dx^j$
 $= -2 X_{ij} dx^j \wedge dx^i$

since Γ is symmetric
 $\Rightarrow$ $dx^j \wedge dx^i$ is
 anti-symmetric

$$\therefore (d \varphi)_{ij} = -2 X_{ij}$$

$$\text{Now consider } \|d \varphi\|^2 = \int_M \langle d \varphi, d \varphi \rangle * 1 = \int_M (d \varphi)_{ij} (d \varphi)^{ij} * 1$$

$$= 4 \int_M X_{ij} X^{ij} * 1 = 2 \int_M X_{ij} X^{ij} * 1$$

$$\text{By part (iv)} \quad \|d \varphi\|^2 = (\Delta \varphi, \varphi)$$

$$\stackrel{(P151)}{=} \int_M X_{ij} X^{ij} * 1 + \int_M \operatorname{Ric}(X, X) * 1$$

$$\stackrel{\text{from above}}{=} \frac{1}{2} \|d \varphi\|^2 + \int_M \operatorname{Ric}(X, X) * 1$$

Therefore

$$\frac{1}{2} \|d \varphi\|^2 = \int_M \operatorname{Ric}(X, X) * 1$$

Since M is Riemannian $\frac{1}{2} \|dX\|^2 \geq 0$

By assumption $\text{Ric}(X, X) \leq 0$

therefore $\int_M \text{Ric}(X, X) * 1 \leq 0$

Putting facts together

$$0 \leq \frac{1}{2} \|dX\|^2 = \int_M \text{Ric}(X, X) * 1 \leq 0$$

This implies that

$$\int_M \text{Ric}(X, X) * 1 = 0$$

Since $\text{Ric}(X, X) \leq 0$, $\text{Ric}(X, X) \equiv 0$

By assumption this can only happen if

$$X \equiv 0 \quad \underline{\text{QED}}$$

(10)

A

(101)

$$g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \quad dg = \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} 1 & -x \\ 0 & x \end{pmatrix} \frac{1}{x} = \begin{pmatrix} \frac{1}{x} & -\frac{x}{x} \\ 0 & 1 \end{pmatrix}$$

$$g^{-1} dg = \begin{pmatrix} \frac{1}{x} & -\frac{x}{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{dx}{x} & \frac{dy}{x} \\ 0 & 0 \end{pmatrix}$$

$$dg g^{-1} = \begin{pmatrix} dx & dy \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & -\frac{x}{x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{dx}{x} & -\frac{x}{x} dx + dy \\ 0 & 0 \end{pmatrix}$$

S

(102)

$$f \in \pi^*(U \cap V) \quad f = e_u g_u \quad e_v = e_u C_{uv} \quad f = e_v g_v$$

$$(1) \quad w_v = C_{uv}^{-1} w_u C_{uv} + C_{uv}^{-1} dC_{uv} \quad \left. \begin{array}{l} e_u g_u = f = e_v g_v = e_u C_{uv} g_v \\ \Rightarrow g_u = C_{uv} g_v \text{ or } g_v = C_{vu} g_u \end{array} \right\}$$

$$(2) \quad \tilde{w}_u(x, g_u) = g_u^{-1} \pi^* w_u(x) g_u + g_u^{-1} dg_u$$

$$(3) \quad g_v = C_{vu} g_u \quad (g_v^{-1} = g_u^{-1} C_{vu}^{-1} = g_u^{-1} C_{uv})$$

$$\text{or } g_v = C_{uv}^{-1} g_u$$

$$\text{now,} \quad \tilde{w}_v = g_v^{-1} \pi^* w_v g_v + g_v^{-1} dg_v = g_v^{-1} \pi^* [C_{uv}^{-1} w_u C_{uv} + C_{uv}^{-1} dC_{uv}] g_v + g_v^{-1} dg_v$$

$$= g_v^{-1} C_{uv}^{-1} \pi^* w_u C_{uv} g_v + g_v^{-1} C_{uv}^{-1} dC_{uv} g_v + g_v^{-1} dg_v$$

$$= g_u^{-1} C_{uv} C_{uv}^{-1} \pi^* w_u C_{uv} C_{uv}^{-1} g_u + g_u^{-1} C_{uv} C_{uv}^{-1} dC_{uv} C_{uv}^{-1} g_u + g_u^{-1} C_{uv} d(C_{uv}^{-1} g_u)$$

$$= g_u^{-1} \pi^* w_u g_u + g_u^{-1} dC_{uv} C_{uv}^{-1} g_u + g_u^{-1} C_{uv} dC_{uv} g_u + g_u^{-1} C_{uv} C_{uv}^{-1} dg_u$$

$$= g_u^{-1} \pi^* w_u g_u + g_u^{-1} \underbrace{d(C_{uv} C_{uv}^{-1})}_{=0} g_u + g_u^{-1} dg_u = g_u^{-1} \pi^* w_u g_u + g_u^{-1} dg_u = \tilde{w}_u \quad \text{QED}$$

103

$$\tilde{\Theta}_a \equiv d\tilde{\omega}_a + \frac{1}{2} [\tilde{\omega}_a, \tilde{\omega}_a] = d\tilde{\omega}_a + \tilde{\omega}_a \wedge \tilde{\omega}_a \quad (\text{Eq 64.5})$$

$$\tilde{\omega}_a = g^{-1}_a \pi^* \omega_a g_a + g^{-1}_a dg_a \quad (\text{Eq 64.3})$$

$$\tilde{\Theta} = d(g^{-1} \pi^* \omega g + g^{-1} dg) + (g^{-1} \pi^* \omega g + g^{-1} dg) \wedge (g^{-1} \pi^* \omega g + g^{-1} dg)$$

$$= dg^{-1} \wedge \pi^* \omega g + g^{-1} d\pi^* \omega g - g^{-1} \pi^* \omega dg + dg^{-1} \wedge dg \\ + g^{-1} (\pi^* \omega \wedge \pi^* \omega) g + g^{-1} \pi^* \omega \wedge dg + g^{-1} dg g^{-1} \wedge \pi^* \omega g + g^{-1} dg \wedge g^{-1} dg$$

$$\left[0 = d(g^{-1} g) = g^{-1} dg + dg^{-1} g \Rightarrow dg^{-1} = -g^{-1} dg g^{-1} \right]$$

$$= g^{-1} \pi^* (d\omega + \omega \wedge \omega) g - g^{-1} dg g^{-1} \wedge \pi^* \omega g - g^{-1} dg g^{-1} \wedge dg \\ + g^{-1} dg g^{-1} \wedge \pi^* \omega g + g^{-1} dg \wedge g^{-1} dg$$

$$= g^{-1} (\pi^* \Theta) g$$

$$= g^{-1} \Theta g$$

5

GP #4

M.E. Glinisky

(104) (i) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1 \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_2 \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_3 \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\sigma_1 \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \sigma_3$$

$$\sigma_2 \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i \sigma_3$$

$$\sigma_1 \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i \sigma_2$$

$$\sigma_3 \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \sigma_2$$

$$\sigma_2 \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \sigma_1$$

$$\sigma_3 \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \sigma_1$$

$$\sigma_1 \sigma_1 = \sigma_2 \sigma_2 = \sigma_3 \sigma_3 = I$$

$$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i \sigma_3$$

$$-\sigma_1 \sigma_3 = \sigma_3 \sigma_1 = i \sigma_2$$

$$\sigma_2 \sigma_3 = -\sigma_3 \sigma_2 = i \sigma_1$$

$$\{\sigma_1, \sigma_1\} = \sigma_1 \sigma_1 + \sigma_1 \sigma_1 = 2 \sigma_1 \sigma_1 = 2I$$

$$\{\sigma_2, \sigma_2\} = \sigma_2 \sigma_2 + \sigma_2 \sigma_2 = 2 \sigma_2 \sigma_2 = 2I$$

$$\{\sigma_3, \sigma_3\} = \sigma_3 \sigma_3 + \sigma_3 \sigma_3 = 2 \sigma_3 \sigma_3 = 2I$$

$$\{\sigma_1, \sigma_2\} = \sigma_1 \sigma_2 + \sigma_2 \sigma_1 = \sigma_1 \sigma_2 - \sigma_1 \sigma_2 = 0$$

$$\{\sigma_1, \sigma_3\} = \sigma_1 \sigma_3 + \sigma_3 \sigma_1 = \sigma_1 \sigma_3 - \sigma_1 \sigma_3 = 0$$

$$\{\sigma_2, \sigma_3\} = \sigma_2 \sigma_3 + \sigma_3 \sigma_2 = \sigma_2 \sigma_3 - \sigma_2 \sigma_3 = 0$$

$$\{\sigma_i, \sigma_j\} = 2 \delta_{ij}$$

(ii) $(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = A^i \sigma_i B^j \sigma_j = A^i B^j \sigma_i \sigma_j$

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i \epsilon_{ijk} \sigma_k$$

$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} I$$

$$[\sigma_i, \sigma_j] + \{\sigma_i, \sigma_j\} = 2\sigma_i \sigma_j = 2i \epsilon_{ijk} \sigma_k + 2\delta_{ij} I$$

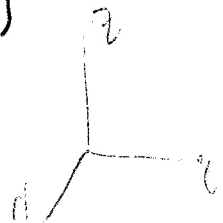
$$\begin{aligned}
 &= A^i B^j (\epsilon_{ijk} \sigma_k + \delta_{ij} I) \\
 &= (\delta_{ij} A^i B^j) I + i (\epsilon_{ijk} A^i B^j) \sigma_k \\
 &= (\vec{A} \cdot \vec{B}) I + i (\vec{A} \times \vec{B})_k \sigma_k \\
 &= (\vec{A} \cdot \vec{B}) I + i (\vec{A} \times \vec{B}) \cdot \vec{\sigma}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad e^{-i \frac{\theta}{2} \vec{A} \cdot \vec{\sigma}} &= \sum_{n=0}^{\infty} (-i \frac{\theta}{2} \vec{A} \cdot \vec{\sigma})^n \frac{1}{n!} \\
 &= \sum_{n=0}^{\infty} (-i \frac{\theta}{2} \vec{A} \cdot \vec{\sigma})^{2n} \frac{1}{(2n)!} + \sum_{n=0}^{\infty} (-i \frac{\theta}{2} \vec{A} \cdot \vec{\sigma})^{2n+1} \frac{1}{(2n+1)!} \\
 &= \sum_{n=0}^{\infty} [(-i)^2 (\vec{A} \cdot \vec{\sigma})^2]^n \left(\frac{\theta}{2}\right)^{2n} \frac{1}{(2n)!} + (-i)(\vec{A} \cdot \vec{\sigma}) \sum_{n=0}^{\infty} \left(\frac{\theta}{2}\right)^{2n+1} \frac{1}{(2n+1)!} \\
 &\quad [(-i)^2 (\vec{A} \cdot \vec{\sigma})^2]^n \\
 &= \sum_{n=0}^{\infty} [(-) I]^n \left(\frac{\theta}{2}\right)^{2n} \frac{1}{(2n)!} + (-i)(\vec{A} \cdot \vec{\sigma}) \sum_{n=0}^{\infty} \left(\frac{\theta}{2}\right)^{2n+1} \frac{1}{(2n+1)!} [(-) I]^n \\
 &= I \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} \right] - i(\vec{A} \cdot \vec{\sigma}) I \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1} \right] \\
 &= I \cos \frac{\theta}{2} - i(\vec{A} \cdot \vec{\sigma}) \sin \frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad R_1 R_2 &= e^{-i \frac{\theta}{2} (\vec{A} \cdot \vec{\sigma})} e^{-i \frac{\phi}{2} (\vec{B} \cdot \vec{\sigma})} \\
 &= [I \cos \frac{\theta}{2} - i(\vec{A} \cdot \vec{\sigma}) \sin \frac{\theta}{2}] [I \cos \frac{\phi}{2} - i(\vec{B} \cdot \vec{\sigma}) \sin \frac{\phi}{2}] \\
 &= I I \cos \frac{\theta}{2} \cos \frac{\phi}{2} - (\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) \sin \frac{\theta}{2} \sin \frac{\phi}{2} \\
 &\quad - i I (\vec{B} \cdot \vec{\sigma}) \cos \frac{\theta}{2} \sin \frac{\phi}{2} - i (\vec{A} \cdot \vec{\sigma}) I \sin \frac{\theta}{2} \cos \frac{\phi}{2} \\
 &= I \cos \frac{\theta}{2} \cos \frac{\phi}{2} - [(\vec{A} \cdot \vec{B}) I + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}] \sin \frac{\theta}{2} \sin \frac{\phi}{2} \\
 &\quad - i (\vec{B} \cdot \vec{\sigma}) \cos \frac{\theta}{2} \sin \frac{\phi}{2} - i (\vec{A} \cdot \vec{\sigma}) \sin \frac{\theta}{2} \cos \frac{\phi}{2} \\
 &= I [\cos \frac{\theta}{2} \cos \frac{\phi}{2} - (\vec{A} \cdot \vec{B}) \sin \frac{\theta}{2} \sin \frac{\phi}{2}] \\
 &\quad - i [(\vec{A} \cdot \vec{\sigma}) \sin \frac{\theta}{2} \cos \frac{\phi}{2} + (\vec{B} \cdot \vec{\sigma}) \cos \frac{\theta}{2} \sin \frac{\phi}{2} + \vec{\sigma} \cdot (\vec{A} \times \vec{B}) \sin \frac{\theta}{2} \sin \frac{\phi}{2}]
 \end{aligned}$$

(v) $R_1 = \text{rotation about } z\text{-axis of } 90^\circ = e^{-i \frac{\theta}{2} (\vec{A} \cdot \vec{\sigma})}$
 $R_2 = \text{rotation about } y\text{-axis of } 90^\circ = e^{-i \frac{\theta}{2} (\vec{B} \cdot \vec{\sigma})}$

$\psi = \theta = \frac{\pi}{2} \quad \vec{A} = \hat{z} = e_3 \quad \vec{B} = \hat{y} = e_2$



$$R_1 R_2 = I \left[\cos \frac{\pi}{4} \cos \frac{\pi}{4} - (\hat{z} \cdot \hat{y}) \sin \frac{\pi}{4} \sin \frac{\pi}{4} \right] \\ - i \left[(\hat{z} \cdot \vec{\sigma}) \sin \frac{\pi}{4} \cos \frac{\pi}{4} + (\hat{y} \cdot \vec{\sigma}) \cos \frac{\pi}{4} \sin \frac{\pi}{4} + \vec{\sigma} \cdot (\hat{z} \times \hat{y}) \sin \frac{\pi}{4} \sin \frac{\pi}{4} \right]$$

$$= \frac{I}{2} - i \left(\frac{\sigma_3}{2} + \frac{\sigma_2}{2} - \frac{\sigma_1}{2} \right)$$

$$= \left(\frac{1}{2} \right) I - i \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \vec{\sigma} = \frac{1}{2} I - i \frac{\sqrt{3}}{2} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \vec{\sigma}$$

$$= \left(\cos \frac{\pi}{3} \right) I - \left(i \sin \frac{\pi}{3} \right) \underbrace{\left(\frac{1}{\sqrt{3}} \hat{y} + \hat{z} - \hat{x} \right)}_{\vec{C}} \cdot \vec{\sigma}$$

$$= e^{-i \frac{\pi}{3} (\vec{C} \cdot \vec{\sigma})}$$

= rotation about $\frac{1}{\sqrt{3}} (\hat{y} + \hat{z} - \hat{x})$ axis of $\frac{2\pi}{3}$ rad or 120°

5

GP # 6 M E Glinster

(105) $\alpha^p, \beta^q, \gamma^r$

$$\textcircled{1} \quad (-)^{pr} [[\alpha, \beta], \gamma] + (-)^{qr} [[\gamma, \alpha], \beta] + (-)^{pq} [[\beta, \gamma], \alpha] = 0$$

$$\theta = d\omega + \frac{1}{2} [\omega, \omega]$$

$$D\theta = d\theta + [\omega, \theta] = d(d\omega + \frac{1}{2} [\omega, \omega]) + [\omega, d\omega + \frac{1}{2} [\omega, \omega]]$$

$$\text{Eq. 63.8} \quad = d(d\omega) + \frac{1}{2} d[\omega, \omega] + [\omega, d\omega] + \frac{1}{2} [\omega, [\omega, \omega]]$$

$$\text{Eq. 63.7} \quad = 0 + \frac{1}{2} ([d\omega, \omega] - [\omega, d\omega]) + [\omega, d\omega] + \frac{1}{2} [\omega, [\omega, \omega]]$$

$$= 0 + \frac{1}{2} (-[\omega, d\omega] - [\omega, d\omega]) + [\omega, d\omega] + \frac{1}{2} [\omega, [\omega, \omega]]$$

$$= -[\omega, d\omega] + [\omega, d\omega] + \frac{1}{2} [\omega, [\omega, \omega]]$$

$$= \frac{1}{2} [\omega, [\omega, \omega]]$$

$$\text{but by } \textcircled{1} \quad 0 = (-)^{11} [[\omega, \omega], \gamma] + (-)^{11} [[\gamma, \omega], \omega] + (-)^{11} [[\omega, \omega], \omega]$$

$$= -3 [[\omega, \omega], \omega] \Rightarrow [[\omega, \omega], \omega] = 0$$

$$\text{by 63.7} \quad [\omega, [\omega, \omega]] = 0$$

$$\therefore D\theta = 0$$

5

(106)

$$\langle X, Y \rangle_{\mathfrak{so}} \equiv -\text{tr } XY = -X_{ij} Y_{ji}$$

$$X, Y \in \mathfrak{so} = \mathfrak{u}(n) \Rightarrow \bar{X}^T = -X, \bar{Y}^T = -Y$$

or $-\bar{X}_{ij} = X_{ji} \quad -\bar{Y}_{ij} = Y_{ji}$

$$\langle X, Y \rangle = -X_{ij} Y_{ji} \xrightarrow{\text{can interchange } i \leftrightarrow j} -X_{ji} Y_{ij} = -Y_{ij} X_{ji} = \langle Y, X \rangle$$

can interchange $i \leftrightarrow j$
since dummy indices
summed over.

$$\begin{aligned} \overline{\langle X, Y \rangle} &= -\overline{X_{ij} Y_{ji}} = -\bar{X}_{ij} \bar{Y}_{ji} = -(-X_{ji})(-Y_{ij}) = -X_{ji} Y_{ij} \\ &= -Y_{ij} X_{ji} = \langle Y, X \rangle = \langle X, Y \rangle \Rightarrow \langle X, Y \rangle \text{ is real} \end{aligned}$$

$$\langle X, X \rangle = -X_{ij} X_{ji} = \bar{X}_{ij} X_{ij} = \sum_{ij} |X_{ij}|^2 \geq 0$$

will equal zero iff $X_{ij} = 0 \forall ij$, since $|X_{ij}|^2 \geq 0 \forall ij$
(=0) $\Leftrightarrow X_{ij} = 0$

5

GP#7

M. E. Glinka

(107)

$$0 = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} + i g \{ [A_\alpha, F_{\beta\gamma}] + [A_\beta, F_{\gamma\alpha}] + [A_\gamma, F_{\alpha\beta}] \}$$

(Bianchi identity $D\theta = 0$)

$$F^{0i} = E^i = -F_{0i}$$

$$B^k = \frac{1}{2} \epsilon^{ijk} F_{ij}$$

Let $\alpha=1, \beta=2, \gamma=3$

$$0 = \partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} + i g \{ [A_1, F_{23}] + [A_3, F_{12}] + [A_2, F_{31}] \}$$

$$= \partial_1 (B^1) + \partial_3 (B^2) + \partial_2 (B^3) + i g \{ [A_1, B^1] + [A_2, B^2] + [A_3, B^3] \}$$

$$= \partial_i B^i + i g [A_i, B^i]$$

$$\Rightarrow \partial_i B^i = -i g (A_i B^i - B^i A_i) \Rightarrow \boxed{\nabla \cdot \vec{B} = -i g (\vec{A} \cdot \vec{E} - \vec{E} \cdot \vec{A})}$$

Let $\alpha=0, \beta=i, \gamma=j$

$$0 = \partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} + i g \{ [A_0, F_{ij}] + [A_j, F_{0i}] + [A_i, F_{j0}] \}$$

$$= \partial_0 F_{ij} + (\partial_j F_{0i} - \partial_i F_{0j}) + i g \{ [A_0, F_{ij}] + (A_j F_{0i} - A_i F_{0j}) + (F_{0j} A_i - F_{0i} A_j) \}$$

$$= \partial_0 F_{ij} + i g [A_0, F_{ij}] + \underbrace{(\delta_j^m \delta_i^l - \delta_i^m \delta_j^l)}_{\epsilon^{mln} \epsilon_{jin}} \{ \partial_m F_{0l} + i g (A_m F_{0l} + F_{0m} A_l) \}$$

(multiply both sides by ϵ^{ijk}) $\epsilon^{mln} \epsilon_{jin}$

$$0 = \partial_0 (\epsilon^{ijk} F_{ij}) + i g [A_0, \epsilon^{ijk} F_{ij}] - \underbrace{\epsilon^{ijk} \epsilon_{jin}}_{2\delta_n^k} \epsilon^{mln} \{ \partial_m F_{0l} + i g (A_m F_{0l} + F_{0m} A_l) \}$$

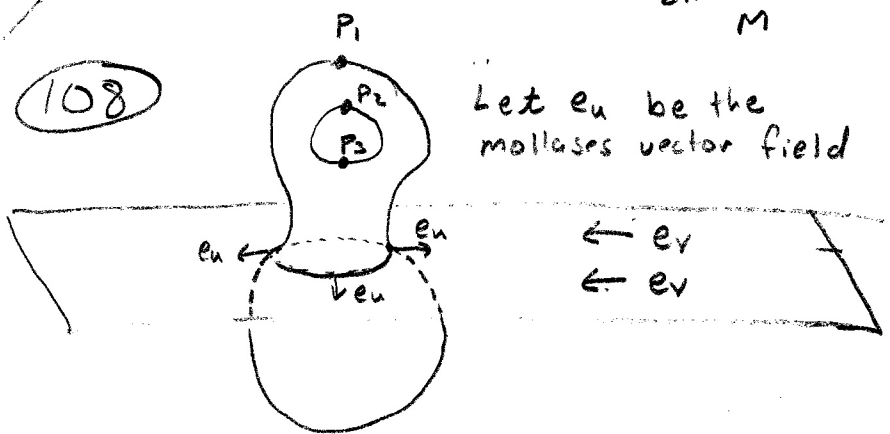
$$\Rightarrow 0 = \partial_0 B^k + i g [A_0, B^k] - \epsilon^{mlk} \{ \partial_m F_{0l} + i g (A_m F_{0l} + F_{0m} A_l) \}$$

$$\Rightarrow 0 = \partial_0 \vec{B} + i g [A_0, \vec{B}] - \vec{\nabla} \times (-\vec{E}) - i g \{ \vec{A} \times (-\vec{E}) + (-\vec{E}) \times \vec{A} \}$$

$$\Rightarrow \boxed{0 = \partial_0 \vec{B} + \nabla \times \vec{E} + i g (A_0 \vec{B} - \vec{B} A_0) + i g (\vec{A} \times \vec{E} + \vec{E} \times \vec{A})}$$

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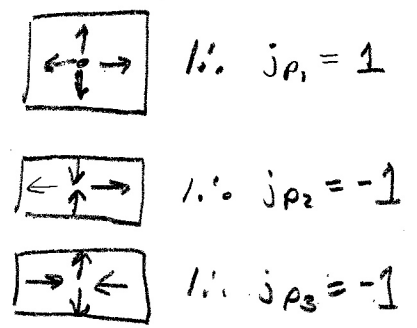
$$\frac{1}{2\pi} \iint_M K ds = \sum j_p(e_u) - \frac{1}{2\pi} \oint d\chi(e_v, e_u)$$



First let's calculate $\sum j_p(e_u)$

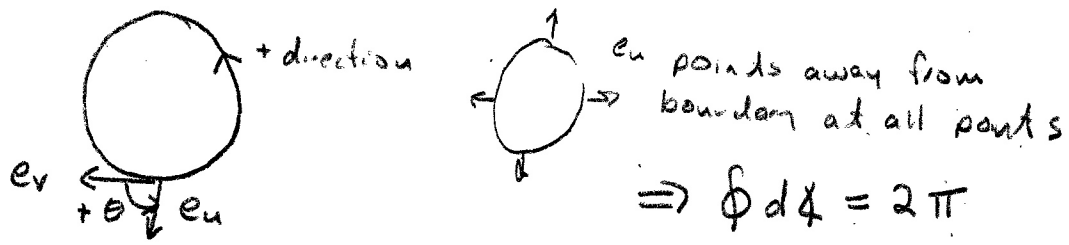
at p_1 the vector field looks like
 at p_2 the " " " "
 at p_3 the " " " "

looking down from above



$$\Rightarrow \sum j_p(e_u) = 1 - 1 - 1 = -1$$

Now let's calculate $\oint d\chi$



$$\Rightarrow \frac{1}{2\pi} \int d\chi = 1$$

For the term $\int_{M^2} K ds$, first construct the chain

$$M' - B + P \text{ where } M' = \text{figure 8} \quad B = \text{circle} \quad P = \text{rectangle}$$

$$\text{Now note } \int_M K ds = \int_{M' - B + P} K ds = \int_{M'} K ds - \int_B K ds + \int_P K ds$$

$$\text{Since } P \text{ is flat } K=0 \Rightarrow \int_P K ds = 0$$

Using the result of problem #59

that $\int_M K dA = 4\pi(1-g)$ where M is a surface of genus g

one finds that $\int_M K ds = 0$
 $M'(g=1)$

$$\int_{B(g=0)} K ds = 4\pi(1-0) = 4\pi$$

$$\therefore \int_M K ds = 0 - (4\pi) + 0 = -4\pi$$

$$\Rightarrow \frac{1}{2\pi} \int_M K ds = \sum j_p(e_a) - \frac{1}{2\pi} \oint d\chi(e_v, e_u)$$

$$\begin{aligned} \frac{1}{2\pi} (-4\pi) &= (-1) - (1) \\ -2 &= -2 \quad \text{as was to be verified} \end{aligned}$$

(109) assume $\langle\langle \cdot \cdot \rangle\rangle_{\mathfrak{g}}$ is invariant under the adjoint action of G
 by 72.4 $\langle\langle \text{ad } Y, Z \rangle\rangle = -\langle\langle Y, \text{ad } Z \rangle\rangle$

$$\text{or } \langle\langle [X, Y], Z \rangle\rangle = -\langle\langle Y, [X, Z] \rangle\rangle$$

also one knows that $[X, Y] = -[Y, X]$

Now let's evaluate properties of $\langle\langle [X, Y], Z \rangle\rangle$

$$\langle\langle [X, Y], Z \rangle\rangle \stackrel{72.4}{=} -\langle\langle Y, [X, Z] \rangle\rangle \stackrel{\text{scalar product symmetric}}{=} -\langle\langle [X, Z], Y \rangle\rangle$$

$$\langle\langle [X, Y], Z \rangle\rangle = \langle\langle [Y, X], Z \rangle\rangle \stackrel{\text{bilinearity of scalar product}}{=} -\langle\langle [Y, X], Z \rangle\rangle$$

$$\begin{aligned} \langle\langle [X, Y], Z \rangle\rangle &= -\langle\langle [Y, X], Z \rangle\rangle = +\langle\langle [Y, Z], X \rangle\rangle \\ &= -\langle\langle [Z, Y], X \rangle\rangle \end{aligned}$$

$\therefore \langle\langle [\cdot, \cdot], \cdot \rangle\rangle$ is skew symmetric in all entries. $\therefore$ it is a three form $\Omega_3(X, Y, Z) = \langle\langle [X, Y], Z \rangle\rangle$

(110) $\mathcal{C}_2 = -\frac{1}{8\pi^2} [\text{tr} \theta \wedge \text{tr} \theta - \text{tr}(\theta \wedge \theta)]$

$$\begin{aligned} -8\pi^2 d\mathcal{C}_2 &= d[\text{tr} \theta \wedge \text{tr} \theta - \text{tr}(\theta \wedge \theta)] \\ &= d(\text{tr} \theta) \wedge \text{tr} \theta + \text{tr} \theta \wedge d(\text{tr} \theta) - d[\text{tr}(\theta \wedge \theta)] \\ &= \text{tr}(d\theta) \wedge \text{tr} \theta + \text{tr} \theta \wedge \text{tr}(d\theta) - \text{tr}(d\theta \wedge \theta + \theta \wedge d\theta) \end{aligned}$$

$D\theta = 0 \stackrel{\text{Eq 7.1.16}}{=} d\theta + \omega \wedge \theta - (-1^2 \theta \wedge \omega) \Rightarrow d\theta = \theta \wedge \omega - \omega \wedge \theta$
 $\stackrel{\text{Bianchi 2}}{\Rightarrow} (d\theta)^i_k = \theta^i_j \wedge \omega^j_k - \omega^i_j \wedge \theta^j_k$
 $\text{tr}(d\theta) = (d\theta)^i_i = \theta^i_j \wedge \omega^j_i - \omega^i_j \wedge \theta^j_i$
 $= \theta^i_j \wedge \omega^j_i - \theta^j_i \wedge \omega^i_j = \theta^i_j \wedge \omega^j_i - \theta^j_i \wedge \omega^i_j = 0$

$$\begin{aligned} (d\theta \wedge \theta)^i_k &= (d\theta)^i_l \wedge \theta^l_k = (\theta^i_j \wedge \omega^j_l - \omega^i_j \wedge \theta^j_l) \wedge \theta^l_k \\ &= \omega^j_l \wedge \theta^i_j \wedge \theta^l_k - \omega^i_j \wedge \theta^j_l \wedge \theta^l_k \\ (\theta \wedge d\theta)^i_k &= \theta^i_j \wedge (d\theta)^j_k = \theta^i_j \wedge (\theta^j_l \wedge \omega^l_k - \omega^j_l \wedge \theta^l_k) \\ &= \omega^l_k \wedge \theta^i_j \wedge \theta^j_l - \omega^j_l \wedge \theta^i_j \wedge \theta^l_k \end{aligned}$$

$$\begin{aligned} \text{tr}(d\theta \wedge \theta + \theta \wedge d\theta) &= (d\theta \wedge \theta)^i_i + (\theta \wedge d\theta)^i_i \\ &= \omega^j_l \wedge \theta^i_j \wedge \theta^l_i - \omega^i_j \wedge \theta^j_l \wedge \theta^l_i + \omega^l_k \wedge \theta^i_j \wedge \theta^j_l - \omega^j_l \wedge \theta^i_j \wedge \theta^l_k \\ &= 0 \end{aligned}$$

$$= 0 \wedge \text{tr} \theta + \text{tr} \theta \wedge 0 - 0 = 0$$

$$\Rightarrow \boxed{d\mathcal{C}_2 = 0}$$

(111)

Consider $(\sum_j \lambda_j)^3 = \sum_{r,s,t} \lambda_r \lambda_s \lambda_t$

$$= \sum_{\substack{r \neq s \\ s \neq t \\ t \neq r}} \lambda_r \lambda_s \lambda_t + 3 \sum_{t \neq r} \lambda_r^2 \lambda_t + \sum_r \lambda_r^3$$

$$= 6 \sum_{r < s < t} \lambda_r \lambda_s \lambda_t + 3 \sum_{r,t} \lambda_r^2 \lambda_t - 2 \sum_r \lambda_r^3$$

$$= 6 \sum_{r < s < t} \lambda_r \lambda_s \lambda_t + 3 \left(\sum_r \lambda_r^2 \right) \left(\sum_t \lambda_t \right) - 2 \sum_r \lambda_r^3$$

$$\Rightarrow (\text{tr } K^1) \wedge (\text{tr } K^1) \wedge (\text{tr } K^1) = 6 \quad \mathcal{C}_3 + 3 (\text{tr } K^2) \wedge (\text{tr } K^1) - 2 (\text{tr } K^3)$$

$$\Rightarrow \mathcal{C}_3 = \frac{1}{6} [(\text{tr } K^1) \wedge (\text{tr } K^1) \wedge (\text{tr } K^1) - 3 (\text{tr } K^2) \wedge (\text{tr } K^1) + 2 (\text{tr } K^3)]$$

$$\text{where } K^1 = \frac{i}{2\pi} \theta \quad K^2 = \frac{-1}{4\pi^2} \theta \wedge \theta \quad K^3 = \frac{-i}{8\pi^3} \theta \wedge \theta \wedge \theta$$

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