Part II
Differential Geometry
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Part II : Differential Geometry

Lecture 29. $\mathbb{R}^3$ and Minkowski Space

(1) Curvature of a space curve in $\mathbb{R}^3$. Parameterized curve $\mathcal{C}$; $\underset{\sim}{x} = \underset{\sim}{x}(t)$. Velocity vector

$$\underset{\sim}{v}(t) = \dot{\underset{\sim}{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \text{ is tangent to } \mathcal{C}.$$

Introduce arc length parameter by $\left(\frac{ds}{dt}\right)^2 = |\dot{\underset{\sim}{x}}|^2 = v^2 = \frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt}\,dt^2$

Unit tangent vector

$$\underset{\sim}{T} = \frac{d\underset{\sim}{x}}{ds} = \dot{\underset{\sim}{x}}\frac{dt}{ds} = \frac{\underset{\sim}{v}}{v} \quad \therefore \underset{\sim}{v} = v\underset{\sim}{T} \qquad (29.1)$$

Acceleration vector $\ddot{x}$

$$\underset{\sim}{a}(t) = \dot{\underset{\sim}{v}}(t) = \dot{v}\underset{\sim}{T} + v\frac{d\underset{\sim}{T}}{ds}v$$

$$\underset{\sim}{a}(t) = \dot{v}\underset{\sim}{T} + v^2\frac{d\underset{\sim}{T}}{ds}.$$

$\underset{\sim}{T}\cdot\underset{\sim}{T} = 1$
$2\underset{\sim}{T}\cdot\dot{\underset{\sim}{T}} = 0$ $\therefore \dot{\underset{\sim}{T}} \perp \underset{\sim}{T}$

Since $\underset{\sim}{T}$ is of constant length 1, $\frac{d\underset{\sim}{T}}{ds}$ is orthogonal to $\underset{\sim}{T}$ and so is normal to $\mathcal{C}$. If it is not $\underset{\sim}{0}$ it defines a unit normal $\underset{\sim}{n}$ (the principal normal to $\mathcal{C}$) and

$$\frac{d\underset{\sim}{T}}{ds} = \kappa(s)\,\underset{\sim}{n}(s) \qquad (29.2)$$

where $\kappa(s) \geq 0$ is called the curvature of $\mathcal{C}$. Then

$$\underset{\sim}{a}(t) = \dot{v}\underset{\sim}{T} + v^2\kappa\,\underset{\sim}{n} \qquad (29.3)$$

lies in the "osculating plane", i.e. the plane spanned by $\underset{\sim}{T}$ and $\underset{\sim}{n}$. To compute κ using the original parameter t, note

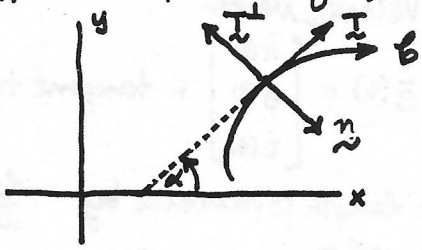
$$\dot{\underset{\sim}{x}} \times \ddot{\underset{\sim}{x}} = v\underset{\sim}{T} \times (\dot{v}\underset{\sim}{T} + v^2\kappa\,\underset{\sim}{n}) = v^3\kappa\,\underset{\sim}{T}\times\underset{\sim}{n}$$

$$\kappa = \frac{\|\dot{\underset{\sim}{x}} \times \ddot{\underset{\sim}{x}}\|}{v^3} \qquad (29.4)$$

- **Problem 48.** Compute the curvature of the helix $x = \cos\omega t$, $y = \sin\omega t$, $z = kt$, ω and k constants.

Remark. The vector $\kappa(s)\,\underset{\sim}{n}(s)$ is called the curvature vector.

Remark. When $\mathcal{C}$ is a plane curve one usually chooses a slightly different definition of κ, allowing κ to be "signed."



One defines $\bar{\kappa}$ by $\frac{d\underset{\sim}{T}}{ds} = \bar{\kappa}\,\underset{\sim}{T}^\perp$, where $\underset{\sim}{T}^\perp$ is the normal to $\mathcal{C}$ obtained by rotating $\underset{\sim}{T}$ thru 90° counterclockwise; thus $\underset{\sim}{T}^\perp = \pm\,\underset{\sim}{N}$

$$\frac{d\underset{\sim}{T}}{ds} = \frac{d}{ds}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} = \begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}\frac{d\alpha}{ds} = \frac{d\alpha}{ds}\,T^\perp$$

giving the familiar

$$\bar{\kappa} = \frac{d\alpha}{ds}.$$

Remark. It is shown in books on differential geometry (see e.g. Struik) that κ and the osculating plane have the following interpretation. To compute $\kappa(s)$ consider the three nearby points $\underset{\sim}{x}(s-\varepsilon)$, $\underset{\sim}{x}(s)$, $\underset{\sim}{x}(s+\varepsilon)$ on $\mathcal{C}$. If they are not collinear they determine a circle of some radius ρ_ε, passing thru $\underset{\sim}{x}(s)$ and lying in a plane P_ε. Under mild conditions

$$\begin{cases}\lim\limits_{\varepsilon\to 0} P_\varepsilon \text{ is the osculating plane spanned by } \underset{\sim}{T} \text{ and } \underset{\sim}{N}\\ \lim\limits_{\varepsilon\to 0} \rho_\varepsilon = \dfrac{1}{\kappa(s)} \text{ is the "radius of curvature" of } \mathcal{C} \text{ at } \underset{\sim}{x}(s).\end{cases}$$

(If $\frac{d\underset{\sim}{T}}{ds}(s) = \underset{\sim}{0}$, we say $\kappa(s) = 0$, $\rho = \infty$, and the osculating plane is undefined). Then (29.3) becomes the classical

$$\boxed{\ddot{\underset{\sim}{x}}(t) = \dot{v}\,\underset{\sim}{T} + \frac{v^2}{\rho}\,\underset{\sim}{n}} \qquad (29.5)$$

giving the tangential and normal components of the acceleration vector.

(2) Minkowski space and Speaal relativity. Minkowski space is $\mathbb{R}^4$ with "pseudo-Riemannian" metric given by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{29.6}$$

in "inertial coordinates" $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

For physical motivation and more details of the following, see Frankel [Fr]. The metric tensor is then

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad (g_{ij}) = \begin{bmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Let $d\underset{\sim}{x} \cdot d\underset{\sim}{x} = dx^2 + dy^2 + dz^2$ be the usual scalar product in $\mathbb{R}^3$.

$$ds^2 = -c^2dt^2 + d\underset{\sim}{x} \cdot d\underset{\sim}{x}$$

where

$$x = \begin{bmatrix} x^0 \\ \underset{\sim}{x} \end{bmatrix} = \begin{bmatrix} t \\ \underset{\sim}{x} \end{bmatrix}.$$

We will denote Minkowski space by M^4. A 4-vector (or tangent vector to M^4)

$$v = \begin{bmatrix} v^0 \\ \underset{\sim}{v} \end{bmatrix} = v^0 \frac{\partial}{\partial t} + \sum v^\alpha \frac{\partial}{\partial x} = v^0 \frac{\partial}{\partial t} + \underset{\sim}{v}$$

is said to be

$$\begin{cases} \text{space-like} & \text{if } \langle v, v \rangle > 0 \\ \text{time-like} & \text{if } \langle v, v \rangle < 0 \\ \text{light-like} & \text{if } \langle v, v \rangle = 0 \end{cases}$$

The world-line (path in M^4) of a massive particle has a tangent

$$\frac{dx}{dt} = \begin{bmatrix} 1 \\ \frac{d\underset{\sim}{x}}{dt} \end{bmatrix}$$

that is time-like since

$$\left\langle \frac{dx}{dt}, \frac{dx}{dt} \right\rangle = -c^2 + \frac{d\underset{\sim}{x}}{dt} \cdot \frac{d\underset{\sim}{x}}{dt} = -c^2 + v^2$$

and $v < c$.

The analogue of arc length in $\mathbb{R}^3$ for the world line of a particle is the "proper time" parameter $d\tau$ defined from

$$\left.\begin{aligned} d\tau^2 &= -\frac{1}{c^2} ds^2 = dt^2 - \frac{1}{c^2} d\underset{\sim}{x} \cdot d\underset{\sim}{x} \\ &= dt^2 - \frac{v^2}{c^2} dt^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \end{aligned}\right\} \tag{29.7}$$

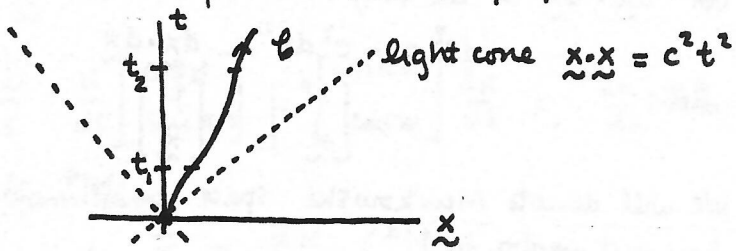
We shall also write

$$\frac{d\tau}{dt} = \gamma^{-1} \equiv \sqrt{1 - v^2/c^2} \qquad (29.8)$$

The analogue of the unit tangent $\underset{\sim}{T}$ in $\mathbb{R}^3$ is the <u>4 velocity</u>

$$u = \frac{dx}{d\tau} = \begin{bmatrix} \frac{dt}{d\tau} \\ \frac{d\underset{\sim}{x}}{d\tau} \end{bmatrix} = \gamma \begin{bmatrix} 1 \\ \underset{\sim}{v} \end{bmatrix} \qquad (29.9)$$

where $\underset{\sim}{v} = \frac{d\underset{\sim}{x}}{dt}$ is the classical velocity vector. For physical interpretation we postulate that the proper time



$$\int_{\mathcal{C}} d\tau = \int_{t_1}^{t_2} \sqrt{1 - v^2/c^2}\, dt$$

is the time kept by an <u>atomic clock</u> moving with the particle along the world-line (thus t is the proper time kept by an atomic clock fixed at the spatial origin $\underset{\sim}{x} = \underset{\sim}{0}$ of the inertial coordinate system)

Associated with a particle is its <u>rest mass</u> m_0; this is an invariant, independent of coordinates.

The (linear) <u>momentum</u> of the particle is the 4 vector

$$P = m_0 u = \begin{bmatrix} m_0\gamma \\ m_0\gamma \underset{\sim}{v} \end{bmatrix} = \begin{bmatrix} m \\ m\underset{\sim}{v} \end{bmatrix} \quad \ldots . \qquad (29.10)$$

where

$$m = m_0\gamma = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad (29.11)$$

is the <u>(relativistic) mass</u> (interpreted as the mass of the moving particle as viewed from the "fixed" inertial system). Note that since

$$\langle u, u \rangle = -c^2\left(\frac{dt}{d\tau}^2\right) + \frac{d\underset{\sim}{x}}{d\tau} \cdot \frac{d\underset{\sim}{x}}{d\tau} = -c^2$$

we have

$$\langle P, P \rangle = -m_0^2 c^2$$

The spatial vector $m\underset{\sim}{v}$ is the classical momentum vector $\underset{\sim}{p}$ (note that mass m is variable)

$$P = \begin{bmatrix} m \\ \underset{\sim}{p} \end{bmatrix}$$

and $-c^2 m_0^2 = \langle P, P \rangle = -c^2 m^2 + \underset{\sim}{p} \cdot \underset{\sim}{p}$, ie

$$m^2 c^2 = m_0^2 c^2 + \underset{\sim}{p} \cdot \underset{\sim}{p} \tag{29.12}$$

The analogue of the curvature vector $d\underset{\sim}{T}/ds$ in $\mathbb{R}^3$ is again called the curvature vector or <u>acceleration</u> 4 vector $\frac{du}{d\tau}$

The Minkowski <u>force</u> 4 vector is defined by

$$f = \frac{d}{d\tau} P = \frac{d}{d\tau}(m_0 u) \tag{29.13}$$

Thus

$$f = \frac{d}{d\tau}\begin{bmatrix} m \\ \underset{\sim}{p} \end{bmatrix} = \begin{bmatrix} \frac{dm}{d\tau} \\ \gamma \frac{d\underset{\sim}{p}}{dt} \end{bmatrix} = \begin{bmatrix} f^0 \\ \gamma \underset{\sim}{f}_c \end{bmatrix} \tag{29.14}$$

where $\underset{\sim}{f}_c = \frac{d\underset{\sim}{p}}{dt}$ is the <u>classical force</u> vector in $\mathbb{R}^3$. Since P has constant "length" $m_0 c$, f must be orthogonal to P (i.e u) in the Minkowski metric

$$0 = \langle f, u \rangle = -c^2 f^0 \gamma + \gamma \underset{\sim}{f}_c \cdot \gamma \underset{\sim}{v}$$

i.e

$$f^0 = \gamma \frac{\underset{\sim}{f}_c \cdot \underset{\sim}{v}}{c^2} \tag{29.15}$$

is (except for γc^{-2}) the classical "power", and finally

$$f = \begin{bmatrix} \gamma \frac{\underset{\sim}{f}_c \cdot \underset{\sim}{v}}{c^2} \\ \gamma \underset{\sim}{f}_c \end{bmatrix} \tag{29.16}$$

Finally $f^0 = \frac{dm}{d\tau} = \gamma \frac{dm}{dt}$ shows that $\frac{dm}{dt} = \frac{1}{c^2} \underset{\sim}{f}_c \cdot \underset{\sim}{v}$ and so

$$c^2 dm = \underset{\sim}{f}_c \cdot \underset{\sim}{v}\, dt = \underset{\sim}{f}_c \cdot d\underset{\sim}{x}$$

is the element of work done by the classical force. Classically this is the <u>energy</u> imparted to the particle. This leads us to associate to a mass m an energy $E = mc^2$ and a rest energy $m_0 c^2$.

29.12 becomes

$$E^2 = E_0^2 + \underset{\sim}{p} \cdot \underset{\sim}{p}\, c^2 \tag{29.17}$$

Lecture 30. E & M in Minkowski Space

(1) The electromagnetic field tensor of Minkowski. The Heaviside - Lorentz force law (16.5) becomes

$$\underset{\sim}{f}_L = e\left(\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\right) \tag{30.1}$$

when units where c needn't be 1 are used. This spatial force vector can be completed to a 4 vector f in Minkowski space by using the prescription (29.16)

$$f = \gamma e\begin{bmatrix}\frac{\underset{\sim}{f}_L\cdot\underset{\sim}{v}}{c^2}\\ \underset{\sim}{f}_L\end{bmatrix} = \gamma e\begin{bmatrix}\frac{\underset{\sim}{E}\cdot\underset{\sim}{v}}{c^2}\\ \underset{\sim}{E}+\frac{1}{c}\underset{\sim}{v}\times\underset{\sim}{B}\end{bmatrix} \tag{30.2}$$

The covariant expression for f, i.e. the associated 1-form $\mathfrak{f}$ in Minkowski space is, from $g_{00} = -c^2$ and from (16.8)

$$\mathfrak{f} = -\gamma e\,(i_{\underset{\sim}{v}}\mathcal{E})dt + \gamma e\left(\mathcal{E} - \tfrac{1}{c}i_{\underset{\sim}{v}}\mathcal{B}\right) \tag{30.3}$$

Recall the velocity 4 vector $u = \begin{bmatrix}\gamma\\ \gamma\underset{\sim}{v}\end{bmatrix} \equiv \begin{bmatrix}\gamma\\ \underset{\sim}{u}\end{bmatrix}$, i.e

$$u = \gamma\frac{\partial}{\partial t} + \gamma\underset{\sim}{v} \tag{30.4}$$

- Problem 49. Show that (30.3) can be written

$$\boxed{\begin{aligned}\mathfrak{f} &= -e\,i_u\mathcal{F}\\ \mathcal{F} &= \mathcal{E}\wedge dt + \tfrac{1}{c}\mathcal{B}\end{aligned}} \tag{30.5}$$

We shall now revert to units where $c = 1$. Written in full

$$\mathcal{F} = (\mathcal{E}_1\,dx + \mathcal{E}_2\,dy + \mathcal{E}_3\,dz)\wedge dt + \\ + \mathcal{B}_1\,dy\wedge dz + \mathcal{B}_2\,dz\wedge dx + \mathcal{B}_3\,dx\wedge dy$$

Writing as usual $\mathcal{F} = \sum_{i<j} F_{ij}\,dx^i\wedge dx^j$ we see

$$(F_{ij}) = \begin{bmatrix}0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{30.6}$$

and (30.5) yields (see Problem 24)

$$\boxed{f_i = e\, F_{ij}\, u^j} \qquad (30.7)$$

The skew symmetric tensor F is the Minkowski electromagnetic field tensor (for a discussion of the tensorial nature of F see [Fl], p. 102).

Consider a Lorentz transformation relating the x system to a system x' moving along the x^1 axis with speed v (see [Fl], p. 5)

$$x = \gamma(x' + vt') \qquad y = y' \qquad \gamma = \text{constant}$$
$$t = \gamma(t' + vx') \qquad z = z'$$

- Problem 50. Show that under this coordinate transformation

$$\mathcal{F} = \mathcal{E} \wedge dt + \mathcal{B} = \mathcal{E}' \wedge dt' + \mathcal{B}'$$

yields the transformation of field components

$$E'_1 = E_1 \qquad B'_1 = B_1$$
$$E'_2 = \gamma(E_2 - vB_3) \qquad B'_2 = \gamma(B_2 + vE_3)$$
$$E'_3 = \gamma(E_3 + vB_2) \qquad B'_3 = \gamma(B_3 - vE_2)$$

(2) Maxwell's equations. In Minkowski space (which, except for its metric, is simply $\mathbb{R}^4$ with coordinates t, x, y, z) we have

$$d = \underset{\sim}{d} + dt \wedge \partial/\partial t$$

(see (19.9)). Then for $\mathcal{F} = \mathcal{E}(t, \underset{\sim}{x}) \wedge dt + \mathcal{B}(t, \underset{\sim}{x})$ we have

$$d\mathcal{F} = \underset{\sim}{d}\mathcal{E} \wedge dt + \underset{\sim}{d}\mathcal{B} + dt \wedge \frac{\partial \mathcal{B}}{\partial t}$$

$$= (\underset{\sim}{d}\mathcal{E} + \frac{\partial \mathcal{B}}{\partial t}) \wedge dt + \underset{\sim}{d}\mathcal{B}$$

and so we see

$$d\mathcal{F} = 0 \iff \begin{cases} \operatorname{curl} \underset{\sim}{E} = -\dfrac{\partial \underset{\sim}{B}}{\partial t} \\ \operatorname{div} \underset{\sim}{B} = 0 \end{cases} \qquad (30.8)$$

i.e.

$$\boxed{d\mathcal{F} = 0} \qquad (30.9)$$

and so $\mathcal{F} = \mathcal{E} \wedge dt + \mathcal{B}$ is a <u>closed</u> form in M^4. If there are no singularities, the converse to the Poincaré Lemma says that $\mathcal{F}$ is exact; there is a 1-form

$$A = \varphi\, dt + \mathcal{A}_1 dx + \mathcal{A}_2 dy + \mathcal{A}_3 dz$$
$$= \varphi\, dt + \mathcal{A}$$

such that

$$\mathcal{F} = d A. \qquad (30.10).$$

In full

$$\mathcal{E} \wedge dt + \mathcal{B} = (\underset{\sim}{d} + dt \wedge \frac{\partial}{\partial t})(\varphi\, dt + \mathcal{A})$$

$$= \underset{\sim}{d}\varphi \wedge dt + \underset{\sim}{d}\mathcal{A} + dt \wedge \frac{\partial \mathcal{A}}{\partial t}$$

and so

$$\boxed{\begin{aligned} \mathcal{E} &= \underset{\sim}{d}\varphi - \frac{\partial \mathcal{A}}{\partial t} \\ \mathcal{B} &= \underset{\sim}{d}\mathcal{A} \end{aligned}} \qquad (30.11)$$

In vector analysis terms $\underset{\sim}{B} =$ curl $\underset{\sim}{A}$ and $\underset{\sim}{E} = \text{grad}\,\varphi - \frac{\partial \underset{\sim}{A}}{\partial t}$. φ is the "scalar" potential, $\underset{\sim}{A}$ is the "vector" potential.

(3) <u>Current</u>. Consider in $\mathbb{R}^3$ a charged "fluid" with charge density σ (called ρ in lecture 16) and local velocity $\underset{\sim}{v}$. The <u>current</u> vector $\underset{\sim}{j}$ is simply $\sigma \underset{\sim}{v}$. Now σ is the charge density as measured in the inertial system x; if σ_0 is the charge density as measured by an observer moving instantaneously with the fluid then $\sigma = \gamma \sigma_0$ because the charge contained in a volume must be an invariant (independent of observer) while volumes are decreased by a factor γ when viewed from a system in (relative) motion with speed v (see [Fr] p. 112). Thus

$$\underset{\sim}{j} = \sigma_0 \gamma \underset{\sim}{v}.$$

Since σ_0 is an invariant (by construction), it is clear from (29.9) that we may construct a 4-vector from $\underset{\sim}{j}$ by defining the <u>current</u> 4-vector

$$J = \sigma_0 u = \begin{bmatrix} \sigma_0 \gamma \\ \sigma_0 \gamma \underset{\sim}{v} \end{bmatrix} = \begin{bmatrix} \sigma \\ \sigma \underset{\sim}{v} \end{bmatrix} = \begin{bmatrix} \sigma \\ \underset{\sim}{j} \end{bmatrix} \qquad (30.12)$$

The volume 4-form for Minkowski space is

$$vol^4 = dt \wedge dx \wedge dy \wedge dz$$

and we may then construct the <u>current 3-form</u> $\mathcal{J}$

$$\left.\begin{aligned} \mathcal{J}^3 &= i_J vol^4 \\ &= i_{(\sigma \frac{\partial}{\partial t} + \underset{\sim}{j})} \, dt \wedge dx \wedge dy \wedge dz \\ &= \sigma \, dx \wedge dy \wedge dz - (j_1 dy \wedge dz + j_2 dz \wedge dx + j_3 dx \wedge dy) \wedge dt \end{aligned}\right\} (30.13)$$

(4) <u>Maxwell's equations again</u>. Define the form $*\mathcal{F}$ as follows (the reason for this notation will be taken up shortly)

$$*\mathcal{F} = -\mathcal{H}' \wedge dt + \mathcal{D}^2$$

(see (16.12) and (16.13)). Then note

$$\begin{aligned} d*\mathcal{F} &= (\underset{\sim}{d} + dt \wedge \frac{\partial}{\partial t})(-\mathcal{H} \wedge dt + \mathcal{D}^2) \\ &= -\underset{\sim}{d}\mathcal{H} \wedge dt + \underset{\sim}{d}\mathcal{D}^2 + dt \wedge \frac{\partial \mathcal{D}}{\partial t} \\ &= \underset{\sim}{d}\mathcal{D}^2 - (d\mathcal{H} - \frac{\partial \mathcal{D}}{\partial t}) \wedge dt \end{aligned}$$

which from Gauss' law and Ampere-Maxwell (see (16.16'), (16.17)) gives

$$d*\mathcal{F} = 4\pi\sigma \, dx \wedge dy \wedge dz - 4\pi (j_1 dy \wedge dz + j_2 dz \wedge dx + j_3 dx \wedge dy) \wedge dt$$

Consequently the second set of Maxwell's equations are equivalent to

$$\boxed{d*\mathcal{F} = d(-\mathcal{H} \wedge dt + \mathcal{D}) = 4\pi \mathcal{J}^3} \qquad (30.14)$$

In particular

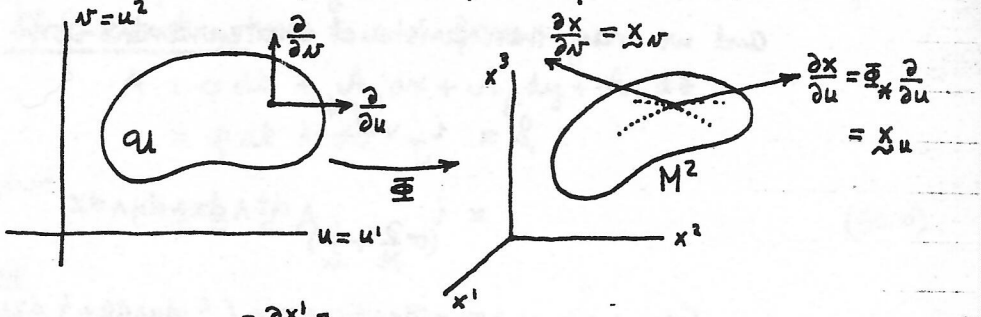
$$d\mathcal{J}^3 = 0 \qquad (30.15)$$

and this is a reflection of "<u>conservation of charge</u>" (see [Fr] p. 111).

Lecture 31. Surfaces in $\mathbb{R}^3$

(1) The first fundamental form, or metric tensor

Let $M^2 \subset \mathbb{R}^3$ be a parameterized surface in $\mathbb{R}^3$; $\Phi : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$



$$\Phi_* \frac{\partial}{\partial u} = \frac{\partial \underset{\sim}{x}}{\partial u} = \begin{bmatrix} \frac{\partial x^1}{\partial u} \\ \frac{\partial x^2}{\partial u} \\ \frac{\partial x^3}{\partial u} \end{bmatrix} \quad \text{etc.}$$

A curve $\underset{\sim}{x} = \underset{\sim}{x}(t)$ that lies on M^2 has tangent (velocity) vector given as follows. The given curve is the image of some curve $u^\alpha = u^\alpha(t)$ in $\mathcal{U}$, so $\underset{\sim}{x}(t) = \underset{\sim}{x}(u(t))$ and then

$$\frac{d\underset{\sim}{x}}{dt} = \frac{\partial \underset{\sim}{x}}{\partial u^\alpha} \frac{du^\alpha}{dt} \equiv \underset{\sim}{x}_\alpha \frac{du^\alpha}{dt}.$$

$\underset{\sim}{x}_\alpha = \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}$, $\alpha = 1, 2$ give a basis for the tangent space to M^2 at the point in question. Any vector A that is tangent to M^2 has such a local expansion $\underset{\sim}{A} = \underset{\sim}{x}_\alpha A^\alpha = \sum_{\alpha=1}^2 \underset{\sim}{x}_\alpha A^\alpha$.
A pair of tangent vectors has Euclidean scalar product

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \underset{\sim}{x}_\alpha A^\alpha \cdot \underset{\sim}{x}_\beta B^\beta = \underset{\sim}{x}_\alpha \cdot \underset{\sim}{x}_\beta A^\alpha B^\beta$$

$$= g_{\alpha\beta} A^\alpha B^\beta$$

where as usual

$$g_{\alpha\beta} \equiv \underset{\sim}{x}_\alpha \cdot \underset{\sim}{x}_\beta = \sum_{i=1}^3 \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} \qquad (31.1)$$

- **Problem 51.** Except for the poles, the sphere of radius a can be parameterized by colatitude $\theta = u^1$ and longitude $\varphi = u^2$. Show that

$$g_{\alpha\beta} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

If we put, as usual

$$ds^2 = g_{\alpha\beta}\, du^\alpha \otimes du^\beta$$

$$= g_{\alpha\beta}\, du^\alpha du^\beta$$

we have

$$\langle A, B \rangle = A \cdot B = ds^2(A,B) = g_{\alpha\beta} A^\alpha B^\beta$$

As usual $\|A\|^2 = \langle A, A \rangle$.

We define the length of a parameterized curve $u^\alpha = u^\alpha(t)$ by

$$L = \int_a^b \left\| \frac{d\underset{\sim}{x}}{dt} \right\| dt = \int_a^b \left\langle \frac{du^\alpha}{dt} \underset{\sim}{x}_\alpha, \frac{du^\beta}{dt} \underset{\sim}{x}_\beta \right\rangle dt$$

$$= \int_a^b \left[g_{\alpha\beta}(u(t)) \frac{du^\alpha}{dt} \frac{du^\beta}{dt} \right]^{\frac{1}{2}} dt$$

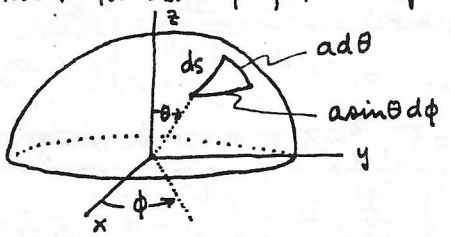
The angle between tangent vectors A and B is given by

$$\cos\chi = \frac{\langle A, B \rangle}{\|A\| \, \|B\|}$$

and the angle between intersecting curves is defined to be the angle between their tangents.

• Problem 52. A "loxodrome" on a sphere of radius a is a curve that makes a constant angle with each meridian of longitude. Usually it will eventually wind around both poles. Compute the length of such a loxodrome that starts at the equator (Hint: use θ or φ as parameter t for the curve).

We interpret $ds^2 = g_{\alpha\beta} du^\alpha du^\beta$ as being an "infinitesimal Pythagoras" rule; sometimes this allows us to replace calculations by a geometric picture. For example, for the sphere of radius a



$ds^2 = (ad\theta)^2 + (a\sin\theta\, d\phi)^2$, in agreement with Problem 51.

For this simple procedure to work it is important that the coordinate curves $u^1 = \text{const.}$ and $u^2 = \text{const}$ be orthogonal (as they are for latitudes and longitudes). In general of course they are not; the angle between a u-curve and a v-curve is given by

$$\cos\omega = \frac{g_{uv}}{\sqrt{g_{uu} g_{vv}}}$$

and so $g_{uv} = 0$ characterizes orthogonal coordinate curves.

For area element, (using $\partial/\partial u$, $\partial/\partial v$ for defining positive orientation)

$$= \| \frac{\partial \underset{\sim}{x}}{\partial u} du \times \frac{\partial \underset{\sim}{x}}{\partial v} dv \| = \| \underset{\sim}{x}_u \times \underset{\sim}{x}_v \| \, du\, dv$$

$$= \|\underset{\sim}{x}_u\| \, \|\underset{\sim}{x}_v\| \sin\omega \; du\, dv$$

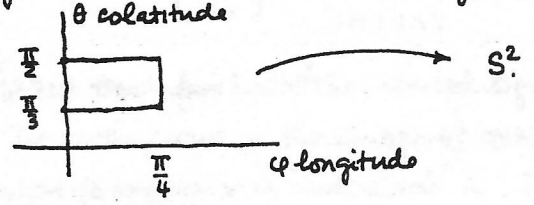
$$= \|\underset{\sim}{x}_u\| \, \|\underset{\sim}{x}_v\| \sqrt{(1-\cos^2\omega)} \, du\, dv$$

$$= \|\underset{\sim}{x}_u\| \, \|\underset{\sim}{x}_v\| \left(1 - \frac{\langle \underset{\sim}{x}_u, \underset{\sim}{x}_v \rangle^2}{\|x_u\|^2 \|\underset{\sim}{x}_v\|^2}\right)^{1/2} du\, dv$$

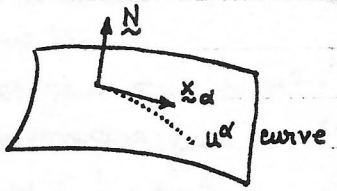
$$dA = (g_{uu} g_{vv} - g_{uv}^2)^{1/2} du\, dv = \sqrt{g}\, du\, dv \qquad (31.2)$$

in agreement with (11.5).

- Problem 53 Compute the area of the region on the Earth's surface bounded by latitudes 0° and 30° and longitudes 0° and 45°



(2) The second fundamental form.



Let $\underset{\sim}{N}(u,v)$ be the unit normal to the surface M^2 in the u,v coordinate patch. We assume that our parameterization is regular (our surface is "immersed"), i.e. Φ_* is 1:1, and so

$$\underset{\sim}{N}(u,v) = \frac{\underset{\sim}{x}_u \times \underset{\sim}{x}_v}{\|\underset{\sim}{x}_u \times \underset{\sim}{x}_v\|} = \begin{bmatrix} N^1(u,v) \\ N^2(u,v) \\ N^3(u,v) \end{bmatrix}.$$

Since $\|\underset{\sim}{N}\| = 1$, $\partial \underset{\sim}{N}/\partial u^\alpha$ must be tangent to the surface, and so

$$\boxed{\frac{\partial \underset{\sim}{N}}{\partial u^\beta} = - b^\alpha{}_\beta \, \underset{\sim}{x}_\alpha = - \underset{\sim}{x}_\alpha \, b^\alpha{}_\beta} \qquad (31.3)$$

for some coefficients $b^\alpha{}_\beta$ (the − sign is introduced for convenience). Equation (31.3) is called Weingarten's equation, and the linear transformation whose matrix is $(b^\alpha{}_\beta)$

$$b : M^2_x \longrightarrow M^2_x$$

is the linear transformation associated to the "2nd fundamental form" (see below; the "1st fundamental form" is the metric tensor $ds^2 = g_{\alpha\beta} \, du^\alpha \, du^\beta$). Recall

(see (7.3)) that to any linear transformation, say $b : M_x^2 \to M_x^2$ of an inner product space we may associate a bilinear form B by

$$B(X,Y) = \langle X, b(Y)\rangle \tag{31.4}$$

where

$$B(X,Y) = b_{\alpha\beta} X^\alpha Y^\beta \,, \quad b_{\alpha\beta} \equiv g_{\alpha\gamma} b^\gamma{}_\beta \tag{31.5}$$

This is the 2nd fundamental form. Note that

$$b_{\alpha\beta} = B(\underset{\sim}{x}_\alpha, \underset{\sim}{x}_\beta) = \langle \underset{\sim}{x}_\alpha, b(\underset{\sim}{x}_\beta)\rangle = -\langle \underset{\sim}{x}_\alpha, \frac{\partial \underset{\sim}{N}}{\partial u^\beta}\rangle$$

$$b_{\alpha\beta} = -\langle \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}, \frac{\partial \underset{\sim}{N}}{\partial u^\beta}\rangle \tag{31.6}$$

Now if A is the bilinear form associated with some linear transformation a, and if a^* is the adjoint of a defined by $\langle a^* X, Y\rangle = \langle X, aY\rangle$, then to say that a is self adjoint, $a^* = a$, is to say

$$A(X,Y) = \langle X, aY\rangle = \langle aX, Y\rangle = \langle Y, aX\rangle = A(Y,X)$$

and so

$$\boxed{a = a^* \text{ iff } A \text{ is symmetric}, \; A(X,Y) = A(Y,X).}$$

It is easy to see that the 2nd fundamental form B is symmetric. Since $\underset{\sim}{N}$ is a normal vector

$$0 = \frac{\partial}{\partial u^\beta}\langle \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}, N\rangle = \langle \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha}, \underset{\sim}{N}\rangle + \langle \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}, \frac{\partial \underset{\sim}{N}}{\partial u^\beta}\rangle$$

$$= \langle \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha}, \underset{\sim}{N}\rangle - b_{\alpha\beta} \,, \quad \therefore$$

$$\boxed{b_{\alpha\beta} = \langle \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha}, \underset{\sim}{N}\rangle = b_{\beta\alpha}} \tag{31.7}$$

showing that b is self adjoint, $b^* = b$! (31.7) is a useful tool for computation

$$b_{\alpha\beta} = \begin{bmatrix} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} \\ \frac{\partial^2 y}{\partial u^\alpha \partial u^\beta} \\ \frac{\partial^2 z}{\partial u^\alpha \partial u^\beta} \end{bmatrix} \cdot \begin{bmatrix} N^1 \\ N^2 \\ N^3 \end{bmatrix}$$

Then we may compute b from

$$b^\rho{}_\gamma = g^{\rho\alpha} b_{\alpha\gamma} \tag{31.8}$$

- **Problem 54.** Consider the surface $z = x^2 - 2y^2$ near the origin. Use $x = u^1$, $y = u^2$ for local coordinates. Compute the matrices $(g_{\alpha\beta})$ and $(b^{\alpha}{}_{\beta})$ at $(0,0)$.

- **Problem 55.** Let M^2 be a surface in $\mathbb{R}^3$ and $\underset{\sim}{x}_0$ a point on this surface. Choose new cartesian coordinates for $\mathbb{R}^3$ having $\underset{\sim}{x}_0$ as origin and such that the new x-y plane is the tangent plane to M^2 at $\underset{\sim}{x}_0$. Use $u^1 = x^1$ and $u^2 = x^2$ as local coordinates for M^2 near $\underset{\sim}{x}_0$. Show that M^2 near $\underset{\sim}{x}_0$ is described by the equations

$$\boxed{x^3 = z(x^1, x^2) = \frac{1}{2} \sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}(0)\, x^{\alpha} x^{\beta} + \text{higher order terms in } x^1, x^2.}$$

where $b_{\alpha\beta}(0)$ is the matrix of the 2nd fundamental form B at $\underset{\sim}{x}_0$.

Lecture 32. Gaussian and Mean Curvature

(1) Principal Normal Curvatures. Let $\mathcal{C}$ be a curve on $M^2 \subset \mathbb{R}^3$. From (31.3) we see

$$\frac{d\underset{\sim}{N}}{dt} = \frac{\partial \underset{\sim}{N}}{\partial u^\alpha}\frac{du^\alpha}{dt} = -\frac{\partial \underset{\sim}{x}}{\partial u^\beta} b^\beta{}_\alpha \frac{du^\alpha}{dt}$$

$$\therefore \quad \frac{d\underset{\sim}{N}}{dt} = -b\left(\frac{d\underset{\sim}{x}}{dt}\right).$$

If we use arc length s for parameter along $\mathcal{C}$ and let $\underset{\sim}{T} = d\underset{\sim}{x}/ds$ be the unit tangent, then

$$\underset{\sim}{T} = \frac{d\underset{\sim}{x}}{ds} = \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}\frac{du^\alpha}{ds}$$

$$\frac{d\underset{\sim}{T}}{ds} = \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}\frac{d^2u^\alpha}{ds^2}.$$

But as a space curve, $\mathcal{C}$, we have $\frac{d\underset{\sim}{T}}{ds} = \kappa \underset{\sim}{n}$ where κ is the curvature and $\underset{\sim}{n}$ is the principal normal to $\mathcal{C}$. From (31.7)

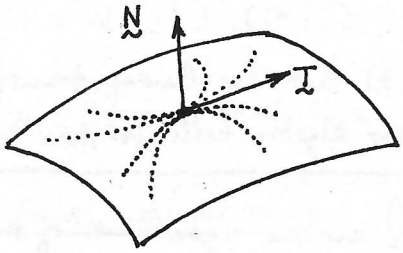
$$\langle \kappa \underset{\sim}{n}, \underset{\sim}{N} \rangle = \left\langle \frac{d\underset{\sim}{T}}{ds}, \underset{\sim}{N} \right\rangle = \left\langle \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha}, \underset{\sim}{N} \right\rangle \frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

$$\langle \kappa \underset{\sim}{n}, \underset{\sim}{N} \rangle = B(\underset{\sim}{T}, \underset{\sim}{T}) \qquad (32.1)$$

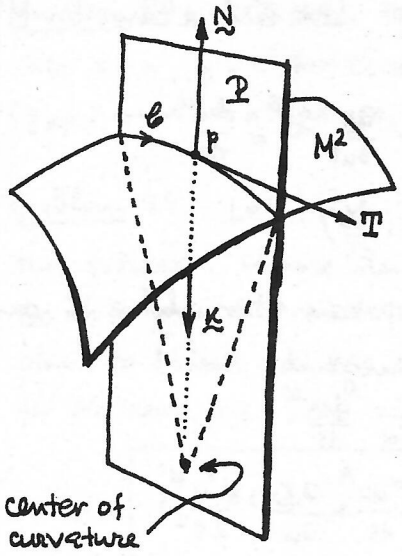
i.e., the (spatial) curvature vector $\kappa \underset{\sim}{n}$ for the curve $\mathcal{C}$ has a component along the surface normal $\underset{\sim}{N}$ given by

$$B(\underset{\sim}{T}, \underset{\sim}{T}) = b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

and this "normal curvature" depends only on the tangent to the curve at the point in question! (Of course



there are many curves tangent to $\underset{\sim}{T}$ at the given point). This is the geometric significance of the 2nd fundamental form.



In particular, let $\underset{\sim}{T}$ be a unit tangent vector to M^2 at p and let $\mathcal{P}$ be the plane spanned by $\underset{\sim}{T}$ and the surface normal $\underset{\sim}{N}$ at p. $\mathcal{P}$ cuts out on M a curve $\mathcal{C}$ whose unit tangent is $\underset{\sim}{T}$. $\mathcal{C}$ is called a "normal section" of M and of course it is a plane curve (lying as it does in $\mathcal{P}$). Its curvature vector $\underset{\sim}{\kappa} = \kappa \underset{\sim}{n}$ (as a space curve) points from p towards the center of curvature. Thus for this normal section

$$B(\underset{\sim}{T},\underset{\sim}{T}) = \pm \kappa$$

where the + sign is used only if the curve $\mathcal{C}$ is curving towards the surface normal chosen; in our figure $B(\underset{\sim}{T},\underset{\sim}{T})$ is negative.

Now keep $p \in M^2$ fixed but rotate $\underset{\sim}{T}$ in the tangent plane M_p; the curvatures $B(\underset{\sim}{T},\underset{\sim}{T})$ will change in general. We define the <u>principal (normal) curvatures</u> of M at p by

$$\left.\begin{aligned} \kappa_1(p) &= \max_{\substack{\underset{\sim}{T}\in M_p \\ \|\underset{\sim}{T}\|=1}} B(\underset{\sim}{T},\underset{\sim}{T}) \\ \kappa_2(p) &= \min_{\substack{\underset{\sim}{T}\in M_p \\ \|\underset{\sim}{T}\|=1}} B(\underset{\sim}{T},\underset{\sim}{T}) \end{aligned}\right\} \qquad (32.2)$$

We have seen in (31.7) that the linear transformation $b: M_p \to M_p$ is self adjoint. Linear algebra tells us then that the following holds.

> κ_1 and κ_2 are the eigenvalues of the self adjoint linear transformation b. There are eigenvectors $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$
>
> $$b(\underset{\sim}{T}_\alpha) = \kappa_\alpha \underset{\sim}{T}_\alpha \ , \quad \alpha = 1,2. \qquad (32.3)$$
>
> If $\kappa_1 \neq \kappa_2$ then automatically $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ are orthogonal; they define the <u>principal directions</u> of M at p.
>
> (Euler)

Of course if $\kappa_1 = \kappa_2$ then all of the normal curvatures at p coincide (p is then called an "umbilic" point).

(2) Gaussian curvature and Mean curvature. We now define two types of curvature for M at p.

$$\left.\begin{aligned} \underline{\text{Gauss curvature}} = K = \det b = \frac{\det(b_{\alpha\beta})}{\det(g_{\alpha\beta})} &= \kappa_1\kappa_2 \\ \underline{\text{Mean curvature}} = H = \operatorname{tr}(b) = \textstyle\sum b^{\alpha}_{\alpha} &= \kappa_1 + \kappa_2 \end{aligned}\right\} \quad (32.4)$$

(Note; some authors define H to be $\frac{1}{2}(\kappa_1+\kappa_2)$).

- Problem 56. Compute K and H at origin in Problem 54. What is the normal curvature for direction $y=x$ at origin?
- Problem 57 Show that the normal curvature for a direction on an M^2 that makes an angle θ with the principal direction $\underset{\sim}{T}_{(1)}$ is

$$\kappa = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

- Problem 58. For a surface M^2 given in "non-parametric" form, i.e

$$z = f(x,y)$$

we can of course introduce $x=u$ and $y=v$ as coordinates. Show that

$$K = \frac{\det f_{\alpha\beta}}{W^2}$$

and

$$H = W^{-3/2} \cdot \left[f_{uu}(1+f_v^2) - 2f_{uv}f_uf_v + f_{vv}(1+f_u^2) \right]$$

where $f_{\alpha\beta} = \dfrac{\partial^2 f}{\partial u^\alpha \partial u^\beta}$ and $W \equiv 1 + f_u^2 + f_v^2$

(3) K and the Gauss normal map. Consider the following purely algebraic fact.

Lemma. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation and let ω^m be an m-form on $\mathbb{R}^m$. Then for any basis $e_1, \dots, e_m$ of $\mathbb{R}^m$

$$\omega^m(Ae_1, \dots, Ae_m) = \det A \;\; \omega^m(e_1, \dots, e_m) \quad (32.5)$$

Proof. $\omega^m(Ae_1,\dots,Ae_m) = \omega^m(e_{i_1}A^{i_1}{}_1,\dots,e_{i_m}A^{i_m}{}_m)$

$= \omega^m(e_{i_1},\dots,e_{i_m})\,A^{i_1}{}_1\cdots A^{i_m}{}_m$

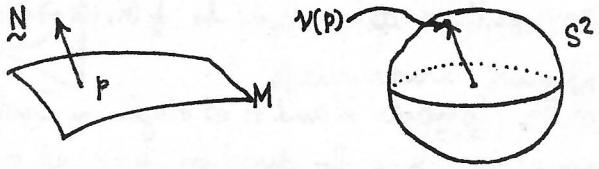
$= \omega^m(e_1,\dots,e_m)\,\delta^{1\cdots m}_{i_1\cdots i_m}\,A^{i_1}{}_1\cdots A^{i_m}{}_m$

$= \omega^m(e_1,\dots,e_m)\det A.$ □

If $M^2 \subset \mathbb{R}^3$ is a surface, define the Gauss (normal) map

$\nu: M^2 \longrightarrow$ unit sphere S^2

by $\nu(p) = \underset{\sim}{N}(p)$, the unit normal to M^2 at p.



Let $\mathrm{vol}^2_M = i_{\underset{\sim}{N}}\mathrm{vol}^3$ and $\mathrm{vol}^2_S = i_\nu \mathrm{vol}^3$ be the area forms for M^2 and S^2 respectively. Then for the pull back $\nu^*\mathrm{vol}^2_S$ we have

$$(\nu^*\mathrm{vol}^2_S)\left(\frac{\partial X}{\partial u},\frac{\partial X}{\partial v}\right) = \mathrm{vol}^2_S\left(\nu_*\frac{\partial X}{\partial u},\nu_*\frac{\partial X}{\partial v}\right)$$

$$= (\text{see problem 44}) = \mathrm{vol}^2_S\left(\frac{\partial \underset{\sim}{N}}{\partial u},\frac{\partial \underset{\sim}{N}}{\partial v}\right)$$

$$= \mathrm{vol}^2_S\left(-b\left(\frac{\partial X}{\partial u}\right),-b\left(\frac{\partial X}{\partial v}\right)\right)$$

$$= \det b\cdot \mathrm{vol}^2_S\left(\frac{\partial X}{\partial u},\frac{\partial X}{\partial v}\right)$$

$$= K\cdot i_{\underset{\sim}{N}}\mathrm{vol}^3\left(\frac{\partial X}{\partial u},\frac{\partial X}{\partial v}\right) = K\cdot \mathrm{vol}^2_M\left(\frac{\partial X}{\partial u},\frac{\partial X}{\partial v}\right),$$

i.e.

$$\boxed{\nu^*\mathrm{vol}^2_S = K\cdot \mathrm{vol}^2_M} \qquad (32.6)$$

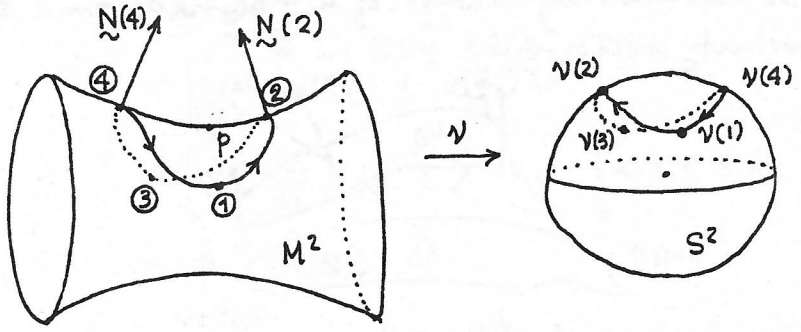
showing that K is the "magnification factor" for areas under the normal map $\nu: M^2 \longrightarrow S^2$. Using true (rather than pseudo) forms

$$\text{Signed area } \nu(\mathcal{U}) = \int_{\nu(\mathcal{U})} \mathrm{vol}^2_S = \int_{\mathcal{U}} \nu^*\mathrm{vol}^2_S = \int_{\mathcal{U}} K\,\mathrm{vol}^2_M$$

and so

$$\lim_{\substack{\mathcal{U}\text{ shrinks}\\ \text{to point } p\in M}} \frac{\text{Signed area } \nu(\mathcal{U})}{\text{area}(\mathcal{U})} = K(p)$$

which was Gauss' original definition of $K(p)$. If $K(p) < 0$, i.e if $\kappa_1(p)$ and $\kappa_2(p)$ have different signs ("saddle" shaped at p), then ν reverses orientation



Remark 1. $H = \kappa_1 + \kappa_2$ depends on which of the two unit normals to M is chosen (a change $\underset{\sim}{N} \longrightarrow -\underset{\sim}{N}$ changes the sign of H) but $K = \kappa_1 \kappa_2$ is independent of the choice!

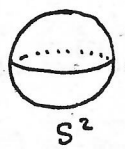
- Problem 59. The Brouwer degree (Lecture 24) of the Gauss map $\nu : M^2 \longrightarrow S^2$ of a closed surface is (why?)

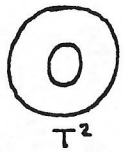
$$\deg(\nu) = \frac{1}{4\pi} \int_{M^2} K \, \mathrm{vol}^2_M = \frac{1}{4\pi} \int_{M^2} K \, dA$$

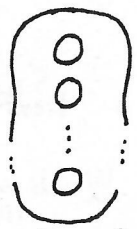
Use (24.2) to write down

$$\frac{1}{4\pi} \int_{M^2} K \, dA$$

for

(i) $M^2 =$  S^2

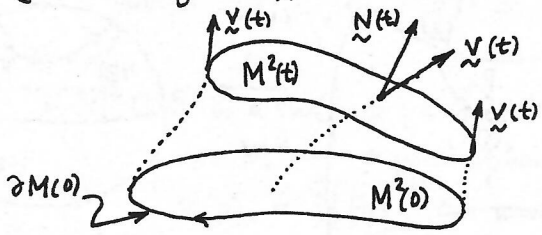
(ii) $M^2 =$  T^2

(iii) $M^2 =$  surface of genus g.

The integral $\int_{M^2} K \, dA$ is called the total curvature of M^2

Lecture 33. H and Variation of Area

(1) First variation of area. Consider a 1-parameter family of surfaces $M^2(t)$ in $\mathbb{R}^3$, a "variation of $M^2(0)$". We describe the variation by means of a (perhaps time dependent) velocity vector field $\underset{\sim}{v}(t)$ in $\mathbb{R}^3$.



$N(t)$ is really defined on $M^2(t) \times t$ of $\mathbb{R}^3 \times \mathbb{R}$ for small t. Extend to a vector field on all of $\mathbb{R}^3 \times \mathbb{R}$. Do the same for $\underset{\sim}{v}(t)$.

Then for area of $M^2(t)$ we have

$$A(t) \equiv \text{area of } M(t) = \int_{M(t)} i_{\underset{\sim}{N}(t)} \, vol^3$$

where $\underset{\sim}{N}(t)$ is the (time dependent) unit normal field to $M^2(t)$. Then from (19.11)

$$A'(t) = \int_{M(t)} \frac{\partial}{\partial t} i_{\underset{\sim}{N}(t)} vol^3 + \int_{M(t)} i_{\underset{\sim}{v}} \underset{\sim}{d}\, i_{\underset{\sim}{N}} vol^3 + \int_{\partial M(t)} i_{\underset{\sim}{v}} i_{\underset{\sim}{N}} vol^3 \qquad (33.1)$$

Look at each integral separately. First, since $\frac{\partial \underset{\sim}{N}}{\partial t}$ is tangent to $M^2(t)$

$$\text{(i)} \quad \int_{M^2(t)} \frac{\partial}{\partial t} i_{\underset{\sim}{N}(t)} vol^3 = \int_{M^2(t)} i_{\frac{\partial \underset{\sim}{N}}{\partial t}} vol^3 = 0.$$

Next, $\underset{\sim}{d}\, i_{\underset{\sim}{N}} vol^3 = (\operatorname{div} \underset{\sim}{N}(t)) \cdot vol^3$ and second integral is

$$\text{(ii)} \quad \int_{M(t)} i_{\underset{\sim}{v}} \underset{\sim}{d}\, i_{\underset{\sim}{N}} vol^3 = \int_{M(t)} \operatorname{div} \underset{\sim}{N}(t)\, i_{\underset{\sim}{v}} vol^3 = \int_{M(t)} \langle \underset{\sim}{v}, \underset{\sim}{N} \rangle \operatorname{div} \underset{\sim}{N} \, vol^2_M$$

Finally, in last integral, use arc length s for parameter along $\partial M(t)$ and let $\underset{\sim}{n}$ be the unit vector field along $\partial M(t)$ that is tangent to $M^2(t)$, normal to $\partial M(t)$, and pointing out of $M(t)$.

$$\text{(iii)} \quad \int_{\partial M^2} i_{\underset{\sim}{V}} i_{\underset{\sim}{N}} \, vol^3 = \int_0^l (i_{\underset{\sim}{V}} i_{\underset{\sim}{N}} vol^3)\left(\frac{d\underset{\sim}{x}}{ds}\right) ds$$

$$= \int_0^l i_{\underset{\sim}{N}} vol^3 \left(\underset{\sim}{V}, \frac{d\underset{\sim}{x}}{ds}\right) ds$$

$$= \int_0^l vol^3 \left(\underset{\sim}{N}, \underset{\sim}{V}, \frac{d\underset{\sim}{x}}{ds}\right) ds$$

$$= \int_0^l \left\langle \frac{d\underset{\sim}{x}}{ds} \times \underset{\sim}{N}, \underset{\sim}{V} \right\rangle ds = \int_0^l \langle \underset{\sim}{m}, \underset{\sim}{V} \rangle \, ds$$

$$= \oint_{\partial M^2(t)} \langle \underset{\sim}{m}, \underset{\sim}{V} \rangle \, ds .$$

We then have

$$A'(t) = \int_{M^2(t)} \langle \underset{\sim}{V}, \underset{\sim}{N} \rangle \operatorname{div} \underset{\sim}{N} \, vol^2_M + \oint_{\partial M(t)} \langle \underset{\sim}{V}, \underset{\sim}{m} \rangle \, ds \qquad (33.2).$$

We now show how $\operatorname{div} \underset{\sim}{N}$ can be evaluated in terms of the surface $M^2(t)$. We first claim that at any point, for any vector field $\underset{\sim}{W}$

$\operatorname{div} \underset{\sim}{W}$ is the trace of the linear transformation L that sends the vector $\underset{\sim}{X}$ at the given point $p \in \mathbb{R}^3$ into the vector $D_{\underset{\sim}{X}}(\underset{\sim}{W})$.

$$\underset{\sim}{X} \in \mathbb{R}^3_p \xrightarrow{\;L\;} D_{\underset{\sim}{X}}(\underset{\sim}{W}) \in \mathbb{R}^3_p ,$$

for, using cartesian coordinates

$$\operatorname{tr} L = \sum_{i=1}^3 \left\langle L\left(\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^i} \right\rangle = \sum_i \left\langle \frac{\partial}{\partial x^i} \underset{\sim}{W}, \frac{\partial}{\partial x^i} \right\rangle$$

$$= \sum_i \sum_j \left\langle \frac{\partial W^j}{\partial x^i} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right\rangle = \sum_i \frac{\partial W^i}{\partial x^i} = \operatorname{div} \underset{\sim}{W} \qquad \square.$$

To compute $\operatorname{div} \underset{\sim}{N}$ we must compute $\operatorname{tr} \underset{\sim}{X} \longmapsto D_{\underset{\sim}{X}} \underset{\sim}{N}$. We may use any basis of $\mathbb{R}^3_p$ to compute the trace; we choose, in terms of the u, v parameterization of $M^2(t)$

$$\underset{\sim}{e}_1 = \frac{\partial \underset{\sim}{x}}{\partial u} = \underset{\sim}{x}_1 , \quad \underset{\sim}{e}_2 = \frac{\partial \underset{\sim}{x}}{\partial v} = \underset{\sim}{x}_2 , \quad \underset{\sim}{e}_3 = \underset{\sim}{N} .$$

Then

$\underset{\sim}{N}(t)$ is defined on $M(t)$. Extend $\partial\, M(t)$ arbitrarily to get a vector field in $\mathbb{R}^3$

$$D_{\underset{\sim}{X}_1} \underset{\sim}{N} = \frac{\partial \underset{\sim}{N}}{\partial u^1} = - \underset{\sim}{X}_1 b^1{}_1 - \underset{\sim}{X}_2 b^2{}_1$$

$$D_{\underset{\sim}{X}_2} \underset{\sim}{N} = \frac{\partial \underset{\sim}{N}}{\partial u^2} = - \underset{\sim}{X}_1 b^1{}_2 - \underset{\sim}{X}_2 b^2{}_2$$

$D_{\underset{\sim}{N}} \underset{\sim}{N}$ is orthogonal to $\underset{\sim}{N}$.

Then

$$\operatorname{div} \underset{\sim}{N} = \operatorname{tr}: \underset{\sim}{X} \to D_{\underset{\sim}{X}} \underset{\sim}{N} = - b^1{}_1 - b^2{}_2$$

$$\boxed{\operatorname{div} \underset{\sim}{N} = - H} \qquad (33.3)$$

and we finally have, from (33.2),

Gauss' formula of first variation of area

$$A'(t) = - \int_{M^2(t)} \langle \underset{\sim}{V}, \underset{\sim}{N} \rangle H \, \mathrm{vol}^2_M + \oint_{\partial M^2(t)} \langle \underset{\sim}{X}, \underset{\sim}{n} \rangle \, ds \qquad (33.4)$$

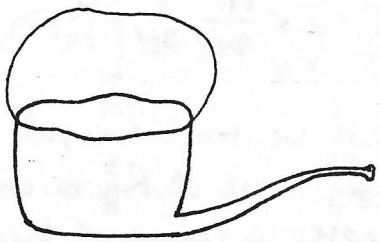
Classically this would be written as a first variation equation

$$\delta A = - \int_{M^2} H \, \delta \underset{\sim}{X}_N \, dS + \oint_{\partial M} \delta \underset{\sim}{X}_n \, ds$$

Here $\delta \underset{\sim}{X}_N$ is the surface normal component of the "variation vector" $\delta \underset{\sim}{X}$, $\delta \underset{\sim}{X}_n$ is the component of the variation vector along the normal to ∂M tangent to M.

(2) Soap Bubbles and Minimal Surfaces.

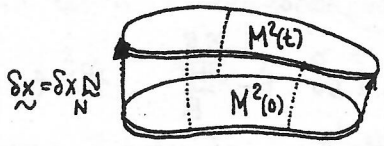
Consider a soap bubble blown with a pipe with perhaps irregular rim



By blowing air in very slowly (so that air pressure inside is spatially constant), the rate at which work is done is given by

$$\frac{\delta W}{\delta t} = p \frac{dV}{dt} \qquad (33.5)$$

Consider a small piece of the soap film as it sweeps out a "cylinder" while being blown up. The pressure will force a normal displacement of the film of small amount δx_N. The small volume swept out will be, in classical notation (putting $\delta x_N = \langle \delta \underset{\sim}{x}_N, \underset{\sim}{N} \rangle$)



$$\delta V = \int_{M^2(0)} \delta x_N \, dS$$

and consequently

$$\delta W = \int_{M^2(0)} p \, \delta x_N \, dS$$

On the other hand, the work done against surface tension during the stretching of the soap film is given by

$$\delta W = 2\sigma \, \delta A = -2\sigma \int_{M^2(0)} H \, \delta x_N \, dS$$

Here σ is the surface tension constant, the factor 2 arises because the soap film has 2 sides (inside & outside) and we have used Gauss' formula with $\langle \delta \underset{\sim}{x}, \underset{\sim}{n} \rangle = 0$ since displacements are normal to the surface. We then have

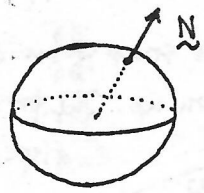
$$\int_{M^2(0)} (p + 2\sigma H) \, \delta x_N \, dS = 0 \qquad (33.6)$$

Since this must hold for each arbitrarily small portion of the film, and since (presumably) $\delta x_N > 0$, we are led to conclude

$$\boxed{p = -2\sigma H} \qquad (33.7)$$

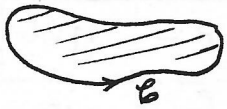
Laplace's formula

A bubble in equilibrium has a constant pressure inside; we see that a soap bubble in equilibrium describes a surface of constant mean curvature! For a spherical bubble of radius R



$H = \kappa_1 + \kappa_2 = -\frac{2}{R}$, and so $p = \frac{4\sigma}{R}$; the larger the bubble the smaller the pressure!

A soap film spanning a wire frame



has the same pressure on both sides of the film, and so $p = 0$. A soap film spanning a given curve $\mathcal{C}$ describes a surface with mean curvature $H = 0$! Such a surface is called a minimal surface (i.e., any surface with $H \equiv 0$). From Gauss' formula, we see that we can characterize minimal surfaces as follows.

> Suppose the surface $M^2 = M^2(0)$ spans the curve $\mathcal{C} = \partial M^2$ and suppose that for any variation $M^2(t)$ of M that leaves the boundary fixed (ie $\delta x_m = 0$ on $\mathcal{C}$) we have that first variation of area vanishes
>
> $$A'(0) = 0.$$
>
> Then we must have that $H \equiv 0$ on M^2, i.e M^2 is a minimal surface.

This variational problem was first investigated by Lagrange. Experimental studies of it using soap films were carried out by the physicist Plateau.

Remarks : (i) We have not investigated the second variation $A''(0)$, consequently we do not claim that a minimal surface spanning a curve $\mathcal{C}$ gives minimum area, even locally!

(ii) $H = \kappa_1 + \kappa_2 = 0 \Rightarrow K = \kappa_1 \kappa_2 \leq 0$, i.e minimal surfaces are surfaces of non-positive Gauss curvature (they are locally "saddle" shaped

Lecture 34: The Fundamental Equations of Surface Theory

(1) The Gauss and Codazzi Equations. Let $M^2 \subset \mathbb{R}^3$ be a surface in space and let u^1, u^2 be local coordinates. Recall that

$$\underset{\sim}{x}_\alpha = \frac{\partial \underset{\sim}{x}}{\partial u^\alpha}, \quad \alpha = 1, 2$$

give a basis for the tangent plane at each point of M^2 in the given coordinate patch. Of course

$$\underset{\sim}{x}_{\alpha\beta} = \frac{\partial^2 \underset{\sim}{x}}{\partial u^\beta \partial u^\alpha} = \underset{\sim}{x}_{\beta\alpha}$$

need not be tangent to M^2. Decompose into tangential and normal parts;

$$\underset{\sim}{x}_{\alpha\beta} = \partial_\beta \partial_\alpha \underset{\sim}{x} = \underset{\sim}{x}_\gamma \Gamma^\gamma_{\beta\alpha} + \langle \underset{\sim}{x}_{\alpha\beta}, \underset{\sim}{N} \rangle \underset{\sim}{N} \tag{34.1}$$

$$\boxed{\underset{\sim}{x}_{\alpha\beta} = \underset{\sim}{x}_\gamma \Gamma^\gamma_{\beta\alpha} + b_{\alpha\beta} \underset{\sim}{N}} \tag{34.1}$$

where the coefficients $\Gamma^\gamma_{\beta\alpha}$ are to be determined. $\Gamma^\gamma_{\beta\alpha} = \Gamma^\gamma_{\alpha\beta}$.

$$\langle \underset{\sim}{x}_{\alpha\beta}, \underset{\sim}{x}_\mu \rangle = \langle \underset{\sim}{x}_\gamma, \underset{\sim}{x}_\mu \rangle \Gamma^\gamma_{\beta\alpha} = g_{\gamma\mu} \Gamma^\gamma_{\beta\alpha} \equiv \Gamma_{\beta\alpha,\mu}.$$

Then note

$$\frac{\partial}{\partial u^\beta} g_{\alpha\mu} = \frac{\partial}{\partial u^\beta} \langle \underset{\sim}{x}_\alpha, \underset{\sim}{x}_\mu \rangle = \langle \underset{\sim}{x}_{\alpha\beta}, \underset{\sim}{x}_\mu \rangle + \langle \underset{\sim}{x}_\alpha, \underset{\sim}{x}_{\mu\beta} \rangle.$$

$$= \Gamma_{\beta\alpha,\mu} + \Gamma_{\mu\beta,\alpha}$$

Cyclically permuting

$$\begin{cases} \dfrac{\partial g_{\alpha\mu}}{\partial u^\beta} = \Gamma_{\beta\alpha,\mu} + \Gamma_{\mu\beta,\alpha} \\ \dfrac{\partial g_{\beta\alpha}}{\partial u^\mu} = \Gamma_{\mu\beta,\alpha} + \Gamma_{\alpha\mu,\beta} \\ \dfrac{\partial g_{\mu\beta}}{\partial u^\alpha} = \Gamma_{\alpha\mu,\beta} + \Gamma_{\beta\alpha,\mu} \end{cases}$$

Using symmetry of $g_{\mu\nu}$ and $\Gamma_{\mu\nu,\gamma}$ in μ,ν we conclude

$$\frac{\partial g_{\alpha\mu}}{\partial u^\beta} + \frac{\partial g_{\beta\alpha}}{\partial u^\mu} - \frac{\partial g_{\mu\beta}}{\partial u^\alpha} = 2\Gamma_{\mu\beta,\alpha} = 2\Gamma^\rho_{\mu\beta} g_{\rho\alpha}$$

and so we have

$$\Gamma^{\tau}_{\mu\beta} = \frac{1}{2} g^{\alpha\tau}\left(\frac{\partial g_{\mu\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\mu}} - \frac{\partial g_{\mu\beta}}{\partial u^{\alpha}}\right) \qquad (34.2)$$

Christoffel symbols (of the 2nd kind)

and all the coefficients in the surface equations (34.1) have been determined!

Gauss now took a further step by calculating what

$$\underset{\sim}{x}_{\alpha\beta\gamma} = \frac{\partial^3 \underset{\sim}{x}}{\partial u^{\gamma}\partial u^{\beta}\partial u^{\alpha}} = \underset{\sim}{x}_{\alpha\gamma\beta}$$

implies.

- Problem 60 Using the surface equations (34.1) and the Weingarten equations (31.3) show that

$$\underset{\sim}{x}_{\alpha\beta\gamma} - \underset{\sim}{x}_{\alpha\gamma\beta} = \left(R^{\tau}{}_{\alpha\gamma\beta} - U^{\tau}_{\alpha\beta\gamma}\right)\underset{\sim}{x}_{\tau} + V_{\alpha\beta\gamma}\underset{\sim}{N}$$

where

$$R^{\tau}{}_{\alpha\gamma\beta} = \frac{\partial \Gamma^{\tau}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\tau}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\tau}_{\sigma\gamma} - \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\tau}_{\sigma\beta} \qquad (34.3)$$

is the "Riemann tensor", and

$$U^{\tau}_{\alpha\beta\gamma} = b^{\tau}{}_{\gamma}\, b_{\alpha\beta} - b^{\tau}{}_{\beta}\, b_{\alpha\gamma}$$

$$V_{\alpha\beta\gamma} = \Gamma^{\sigma}_{\alpha\beta} b_{\sigma\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \Gamma^{\sigma}_{\alpha\gamma} b_{\sigma\beta} - \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}}.$$

We then conclude, with Gauss, Codazzi & Peterson

$$R^{\tau}_{\alpha\gamma\beta} = b^{\tau}{}_{\gamma}\, b_{\alpha\beta} - b^{\tau}{}_{\beta}\, b_{\alpha\gamma} \quad \text{Gauss' equations}$$

$$\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \Gamma^{\sigma}_{\alpha\gamma} b_{\sigma\beta} = \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} - \Gamma^{\sigma}_{\alpha\beta} b_{\sigma\gamma} \quad \text{Codazzi equations} \qquad (34.4)$$

These are "integrability conditions", i.e. conditions that must be satisfied by $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$ in order for these forms to be the first and second fundamental forms of a surface. Bonnet showed in fact that these conditions are also sufficient for the (local) existence of a surface with forms $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$.

was this his motivation?

(2) Gauss' THEOREMA EGREGIUM Gauss' calculation of the first equation in (34.4) led him to the greatest discovery in differential geometry. First some background.

A geographical map is a mapping φ: a portion of the Earth's surface $\longrightarrow$ a portion of the plane. Crudely speaking $\varphi: S^2 \longrightarrow \mathbb{R}^2$. Ideally one would want a map that preserves distances (up to a constant scale factor). The length of a curve $\underset{\sim}{x}(t)$ on the Earth's surface is

$$\int_0^1 \left\| \frac{d\underset{\sim}{x}}{dt} \right\| dt = \int_0^1 \left\langle \frac{d\underset{\sim}{x}}{dt}, \frac{d\underset{\sim}{x}}{dt} \right\rangle^{\frac{1}{2}} dt$$

while its image in $\mathbb{R}^2$ has length

$$\int_0^1 \left\langle \varphi_* \frac{d\underset{\sim}{x}}{dt}, \varphi_* \frac{d\underset{\sim}{x}}{dt} \right\rangle^{\frac{1}{2}} dt .$$

We say that a local map $\varphi: M^m \longrightarrow V^m$ is a <u>local isometry</u> if

$$\langle \varphi_* X, \varphi_* X \rangle_V = \langle X, X \rangle_M$$

for all vectors X tangent to M (in the domain of φ). Note then that φ_* will also automatically preserve all scalar products, for $\langle\ ,\ \rangle$ can be expressed in terms of $\|\ \|$

$$\langle X, Y \rangle = \frac{1}{2} \left\{ \|X+Y\|^2 - \|X\|^2 - \|Y\|^2 \right\}.$$

If $\varphi: S^2 \longrightarrow \mathbb{R}^2$ is a local isometry then the lengths of all curves will be preserved (as well as angles and areas). Of course a local isometry will also be a local diffeomorphism because $\varphi_*: M^m_p \longrightarrow V^m_{\varphi(p)}$ will be an isomorphism, and the inverse fctn theorem of Lecture 3 assures us that φ is a local diffeomorphism.

A familiar example of a local isometry is when a flat sheet of paper is rolled up into a cylinder; though the paper is "bent" there is basically no "stretching".

If $\varphi: M^m \longrightarrow V^m$ is a local isometry (and hence a local diffeomorphism) one may transplant a local coordinate system y near $\varphi(p_0)$ back to a coordinate system x near p_0

$$x^i(p) = y^i(\varphi(p)) = y^i \circ \varphi\ (p).$$

Then, in terms of these coordinates φ is given simply by $y^i = x^i$. Thus in these coordinates $\varphi_* \partial/\partial x^i = \partial/\partial y^i$. Since φ_* is a local isometry

$$g^V_{ij}(y) = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle_V = \left\langle \varphi_* \frac{\partial}{\partial x^i}, \varphi_* \frac{\partial}{\partial x^j} \right\rangle_V$$

$$= \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_M = g^M_{ij}(x)$$

i.e, in these associated coordinates the metric tensors of M and V are identical at associated points! But then the Christoffel symbols and Riemann tensor components (which are defined in n dimensions just as for surfaces), by (34.2) and (34.3) are also identical! But look carefully (with Gauss) at the Gauss equations (34.4)

$$R^{12}{}_{12} \equiv g^{2\alpha} R^1{}_{\alpha 12} = g^{2\alpha}(b^1{}_1 b_{\alpha 2} - b^1{}_2 b_{\alpha 1})$$

$$= (b^1{}_1 b^2{}_2 - b^1{}_2 b^2{}_1) = \det(b) = K$$

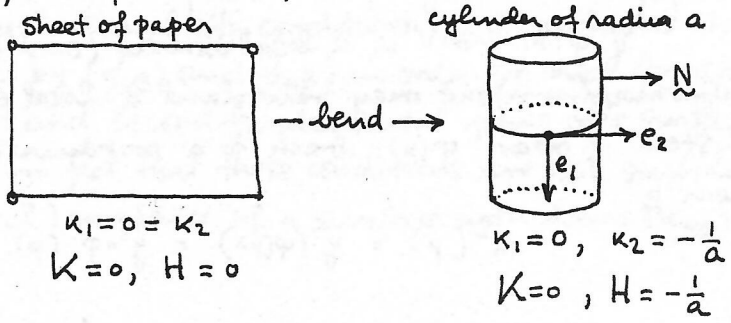
$\therefore$

$$\boxed{\kappa_1 \kappa_2 = K = R^{12}{}_{12}} \qquad (34.5)$$

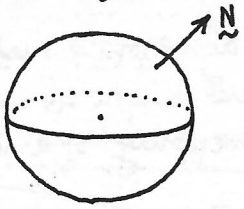
But since $R^{12}{}_{12}$ is expressible entirely by means of the metric tensor we see that

The Gauss curvature $K = \kappa_1 \kappa_2$ is an isometry invariant
(Gauss' THEOREMA EGREGIUM)

Thus, in particular, if a surface M^2 is bent, without stretching or tearing, then although the principal curvatures κ_1, κ_2 will change, their product $\kappa_1 \kappa_2 = K$ will not!



Also, for a sphere of radius a



$$\kappa_1 = \kappa_2 = -\frac{1}{a}$$

$$K = \frac{1}{a^2}, \quad H = -\frac{2}{a}$$

Since $K = \frac{1}{a^2} \neq 0$ we see that there is no local isometry $\varphi: S^2 \longrightarrow \mathbb{R}^2$. Thus every map of the Earth must distort distances!

Note that the mean curvature H is not an isometry invariant.

Gauss' "theorema egregium" is the most important theorem in all of differential geometry. It is the greatest discovery in all of geometry since the "Pythagorean" theorem! General relativity would be unthinkable without Riemann's generalization of Gauss' theorem (as Einstein found out in 1912).

Gauss is telling us that certain geometric properties of a surface $M^2 \subset \mathbb{R}^3$, e.g. the product of the principal curvatures at a point, are in fact computable without use of the 2nd fundamental form, i.e computable in terms of the metric tensor alone!

Sharpen up your pencils! With patience you can do

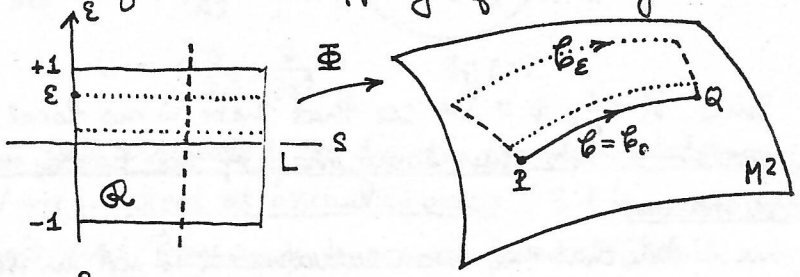
- Problem 61. The metric on the sphere of radius a is of course

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Using (34.3) show that indeed $R^{12}{}_{12} = 1/a^2$.

Lecture 35: Geodesics and Parallel Displacement

(1) Geodesics. Let $\mathcal{C}$ be a curve on a surface M^2. We shall consider the first variation of arc length as we vary the curve. A variation of $\mathcal{C}$ is a mapping of a rectangle into M^2



$$\Phi: R^2 \longrightarrow M^2 \qquad R^2 = [0,L]\times(-1,1) \;;\; \underset{\sim}{x} = \underset{\sim}{x}(s,\varepsilon).$$

Here L is the length of the curve $\mathcal{C} = \mathcal{C}_0$; $\underset{\sim}{x} = \underset{\sim}{x}(s,0)$ and we shall assume that s is arc length along $\mathcal{C}_0$; we shall not assume that s is arc length along all $\mathcal{C}_\varepsilon$; $\underset{\sim}{x} = \underset{\sim}{x}(s,\varepsilon)$ for that would force all $\mathcal{C}_\varepsilon$ to have the same length L ! The length of the curve $\mathcal{C}_\varepsilon$ is

$$\ell(\varepsilon) = \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial s}(s,\varepsilon), \frac{\partial \underset{\sim}{x}}{\partial s}(s,\varepsilon)\right\rangle^{\frac{1}{2}} ds$$

and so

$$\ell'(\varepsilon) = \int_0^L \frac{\partial}{\partial \varepsilon}\left\langle \frac{\partial \underset{\sim}{x}}{\partial s}, \frac{\partial \underset{\sim}{x}}{\partial s}\right\rangle^{\frac{1}{2}} ds = \int_0^L \left\| \frac{\partial \underset{\sim}{x}}{\partial s}\right\|^{-1} \cdot \left\langle \frac{\partial^2 \underset{\sim}{x}}{\partial \varepsilon \partial s}, \frac{\partial \underset{\sim}{x}}{\partial s}\right\rangle ds$$

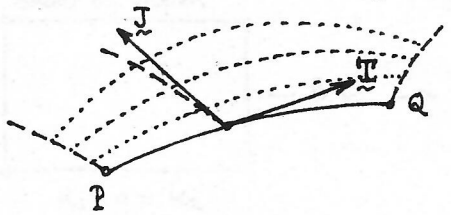
Since s is arc length when $\varepsilon = 0$, $\left\|\frac{\partial \underset{\sim}{x}}{\partial s}(s,0)\right\| = 1$ and

$$\ell'(0) = \int_0^L \left\langle \frac{\partial^2 \underset{\sim}{x}}{\partial \varepsilon \partial s}, \frac{\partial \underset{\sim}{x}}{\partial s}\right\rangle ds = \int_0^L \left\langle \frac{\partial^2 \underset{\sim}{x}}{\partial s \partial \varepsilon}, \frac{\partial \underset{\sim}{x}}{\partial s}\right\rangle ds$$

$$= \int_0^L \frac{\partial}{\partial s}\left\langle \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}, \frac{\partial \underset{\sim}{x}}{\partial s}\right\rangle ds - \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}, \frac{\partial}{\partial s}\underset{\sim}{T}\right\rangle ds$$

$$\ell'(0) = \left\langle \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}, \underset{\sim}{T}\right\rangle \Big|_P^Q - \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}, \frac{\partial}{\partial s}\underset{\sim}{T}\right\rangle ds \qquad (35.1)$$

where $\underset{\sim}{T} = \frac{\partial \underset{\sim}{x}}{\partial s}(s,0)$ is the unit tangent to $\mathcal{C}_0$. Also, $\underset{\sim}{J} \equiv \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}(s,0)$



is called the variation vector along $\mathcal{C}$. Both $\underset{\sim}{T}$ and $\underset{\sim}{J}$ are of course tangent to M^2

Rewriting

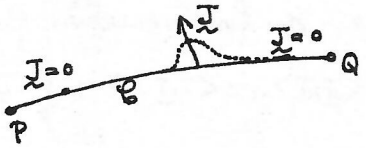
$$\ell'(0) = \langle \underset{\sim}{J}, \underset{\sim}{T} \rangle \Big|_P^Q - \int_0^L \langle \underset{\sim}{J}, \frac{\partial}{\partial s} \underset{\sim}{T} \rangle \, ds \qquad (35.2)$$

"first variation of arc length" formula

$\mathcal{C}$ is said to be a geodesic if $\ell'(0) = 0$ for all variations that vanish at the end points P and Q, i.e. for all variations $\underset{\sim}{x}(s,\varepsilon)$ such that $\underset{\sim}{x}(0,\varepsilon) = P$ and $\underset{\sim}{x}(L,\varepsilon) = Q$. For such variations $J = 0$ at P and Q and the first variation vanishing yields

$$0 = \int_0^L \langle \underset{\sim}{J}, \frac{\partial \underset{\sim}{T}}{\partial s} \rangle \, ds .$$

Since the variations are very general except at $s = 0$ and $s = L$



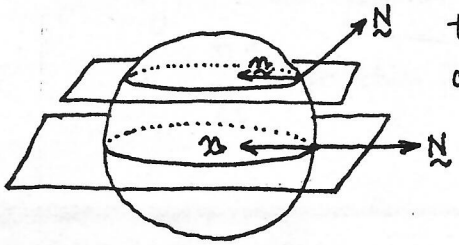
we conclude that if $\mathcal{C}$ is a geodesic then $\langle \underset{\sim}{J}, \frac{\partial \underset{\sim}{T}}{\partial s} \rangle = 0$ for every tangent vector $\underset{\sim}{J}$ at each parameter value $0 \le s \le L$. Thus $\frac{\partial \underset{\sim}{T}}{\partial s}$ must be normal to the surface M^2 along $\mathcal{C}$. Since

$$\frac{\partial \underset{\sim}{T}}{\partial s} = \kappa \underset{\sim}{n}$$

is the curvature vector of $\mathcal{C}$ considered as a space curve, we recover the theorem of John Bernoulli (1697)

Theorem *$\mathcal{C}$ on M^2 is a geodesic iff the principal normal $\underset{\sim}{n}$ to $\mathcal{C}$ is also normal to the surface M^2 along $\mathcal{C}$.*

Thus if we cut out a curve on S^2 by slicing S^2 with a plane



the resulting circle will be a geodesic on S^2 only when it is a great circle

Note that if J is a tangent vector to M^2, then

$$\langle J, \frac{\partial}{\partial s} T \rangle = \langle J, \frac{\nabla}{\partial s} T \rangle$$

where

$$\frac{\nabla}{\partial s} T = \frac{\partial T}{\partial s} - \langle \frac{\partial T}{\partial s}, N \rangle N \qquad (35.3)$$

is the tangential part of $\frac{\partial T}{\partial s}$

$\frac{\nabla T}{\partial s}$ is called the intrinsic or covariant derivative of T and then a geodesic C is characterized by

$$\frac{\nabla}{\partial s} T = 0 \quad ; \text{ geodesic equation} \qquad (35.4)$$

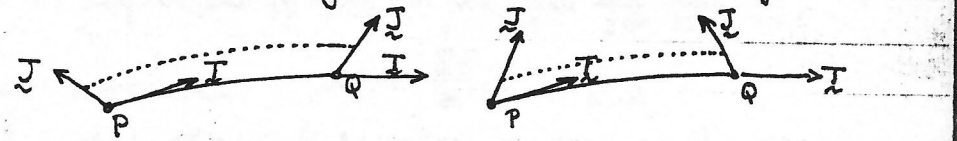
Since $\frac{\partial T}{\partial s} = k n$ is orthogonal to T, so is $\frac{\nabla T}{\partial s}$ ($\frac{\nabla T}{\partial s}$ is the orthogonal projection of kn into the tangent plane of M^2 at the point in question. This vector

$$\frac{\nabla T}{\partial s} = k_g \qquad (35.5)$$

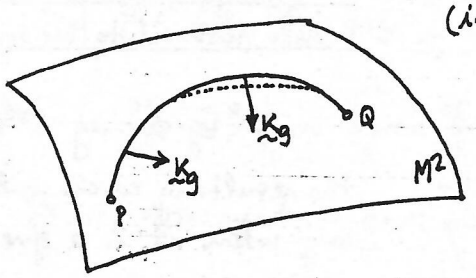
is also called the geodesic curvature vector, k_g, and its length is the geodesic curvature k_g. A geodesic has vanishing geodesic curvature. The first variation formula (35.2)

$$l'(0) = \langle J, T \rangle_Q - \langle J, T \rangle_P - \int_0^L \langle J, k_g \rangle \, ds$$

gives two contributions to the first variation of arc length for a general curve C on M^2. First there is a contribution from how the end points are being moved (even when C is a geodesic)



and secondly one tends to shorten a (non-geodesic) curve by moving it in the direction of its geodesic curvature vector k_g (i.e, make $\langle J, k_g \rangle \geq 0$).



(2) Covariant differentiation and parallel displacement. Let us write out the geodesic equations (35.4) in terms of local coordinates, via (34.1).

$$\underset{\sim}{T} = \frac{\partial \underset{\sim}{x}}{\partial s} = \frac{\partial \underset{\sim}{x}}{\partial u^\beta}\frac{\partial u^\beta}{\partial s}$$

$$\frac{\partial^2 \underset{\sim}{x}}{\partial s^2} = \frac{\partial^2 \underset{\sim}{x}}{\partial u^\alpha \partial u^\beta}\frac{\partial u^\alpha}{\partial s}\frac{\partial u^\beta}{\partial s} + \frac{\partial \underset{\sim}{x}}{\partial u^\beta}\frac{\partial^2 u^\beta}{\partial s^2}$$

$$= \left(\frac{\partial \underset{\sim}{x}}{\partial u^\gamma}\Gamma^\gamma_{\alpha\beta} + b_{\alpha\beta}\underset{\sim}{N}\right)\frac{\partial u^\alpha}{\partial s}\frac{\partial u^\beta}{\partial s} + \frac{\partial \underset{\sim}{x}}{\partial u^\gamma}\frac{\partial^2 u^\gamma}{\partial s^2}$$

$$\frac{\nabla \underset{\sim}{T}}{\partial s} = \frac{\partial \underset{\sim}{x}}{\partial u^\gamma}\cdot\left(\frac{\partial^2 u^\gamma}{\partial s^2} + \Gamma^\gamma_{\alpha\beta}\frac{\partial u^\alpha}{\partial s}\frac{\partial u^\beta}{\partial s}\right).$$

Thus a curve $\underset{\sim}{x} = \underset{\sim}{x}(u(s))$ parameterized by arc length s is a geodesic on M^2 iff u satisfies the system of ordinary differential equations

$$\frac{d^2u^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = 0 \qquad (35.6)$$

The fundamental theorem for systems of ordinary differential equations (Lecture 4, § 1) tells us that this system, i.e

$$\begin{cases} \dfrac{du^\gamma}{ds} = T^\gamma \\ \dfrac{dT^\gamma}{ds} = -\Gamma^\gamma_{\alpha\beta}T^\alpha T^\beta \end{cases}$$

has a unique solution $u^\gamma = u^\gamma(s)$ for given initial data $u^\gamma(0) = u^\gamma_0$ and $\frac{du^\gamma}{ds}(0) = T^\gamma_0$. Thus there is a unique geodesic starting at a given point u_0 with initial tangent (unit) vector T^γ_0. Since the system is non-linear we may not insist that solution exist for all parameter values s.

It is clear that the intrinsic derivative $\nabla \underset{\sim}{T}/ds$ is playing an important role. We extend its definition as follows.

If a vector field $\underset{\sim}{\xi}(t)$ tangent to M^2 is defined along a parameterized curve $\mathcal{C}: u = u(t)$ on M^2 we define the covariant derivative of $\underset{\sim}{\xi}$ along $\mathcal{C}$ to again be the tangential part of the usual derivative

$$\frac{\nabla \underset{\sim}{\xi}}{dt} = \text{projection of } \frac{d\underset{\sim}{\xi}}{dt} \text{ into the tangent plane of } M. \qquad (35.7)$$

Writing $\underset{\sim}{\xi} = \frac{\partial x}{\partial u^\alpha}\,\xi^\alpha(u(t))$ we get as before

$$\boxed{\frac{\nabla \xi^\gamma}{dt} = \frac{d\xi^\gamma}{dt} + \frac{du^\beta}{dt}\,\Gamma^\gamma_{\beta\alpha}\,\xi^\alpha} \tag{35.8}$$

Again, given a parameterized curve $\mathcal{C}$, $u = u(t)$ there exists a unique vector field $\underset{\sim}{\xi} = \underset{\sim}{\xi}(t)$ defined along $\mathcal{C}$ such that $\underset{\sim}{\xi}(0) = \underset{\sim}{\xi}_0$ is given and $\underset{\sim}{\xi}$ satisfies

$$\frac{\nabla \xi^\gamma}{dt} = 0. \tag{35.9}$$

Since the system (35.8) is a <u>linear</u> system for ξ we see that the solution of (35.8) exist for all parameter values t. This unique vector field $\underset{\sim}{\xi}$ is said to be <u>parallel displaced</u> along $\mathcal{C}$.

Note in particular that if $\mathcal{C}$ is a <u>geodesic</u> parameterized by arc length s, then its unit tangent $\underset{\sim}{\xi} = \underset{\sim}{T}$ is parallel displaced along $\mathcal{C}$.

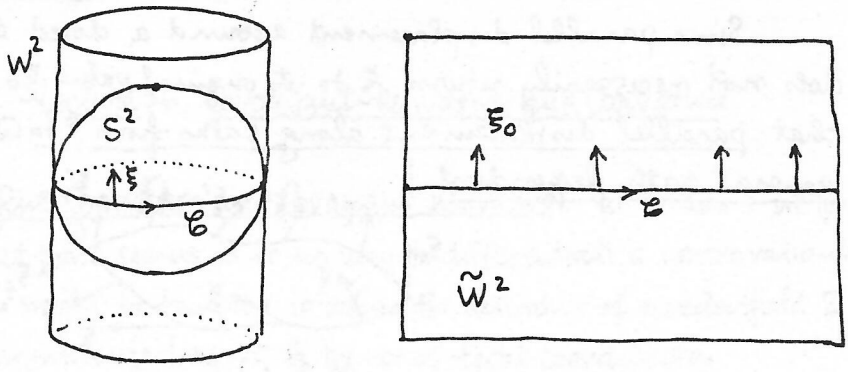
Note that (35.9) merely says that $\frac{d\underset{\sim}{\xi}}{dt}$ <u>is always normal to the surface along the curve</u> $\mathcal{C}$. This notion of parallel displacement was first described by Levi-Civita in about 1917.

The notion of covariant derivative is seen from (35.8) to involve only the metric tensor, <u>not</u> the 2nd fundamental form. In particular if $\varphi: M^2 \longrightarrow V^2$ is an <u>isometry</u> (e.g. a bending), and if $\underset{\sim}{\xi}$ is parallel displaced along $\mathcal{C}$ of M^2, then $\varphi_* \underset{\sim}{\xi}$ is parallel displaced along $\varphi(\mathcal{C})$ of V^2.

$$\boxed{\begin{array}{l}\text{Parallel displacement is an}\\ \text{isometry invariant}\end{array}} \tag{35.10}$$

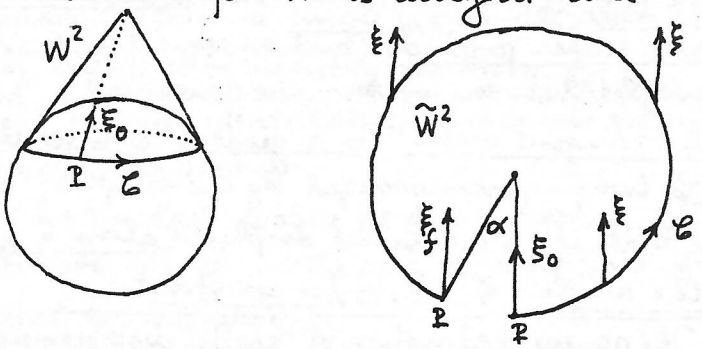
Furthermore, if $M^2 \subset R^3$ and $W^2 \subset R^3$ are two surfaces in R^3 that are <u>tangent</u> along a common curve $\mathcal{C}$, then from (35.7) we see that if ξ is parallel displaced along $\mathcal{C}$ as a vector field on M^2 then ξ is also parallel displaced as a vector field on W^2!

For example, let $M^2 = S^2$ be a 2-sphere, and let W^2 be a right circular cylinder tangent to S^2 along an equator.

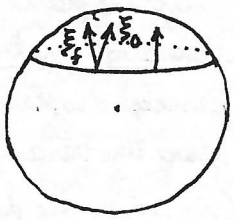


Let ξ_0 be orthogonal to $\mathcal{C}$ and let ξ be parallel displaced on S^2 along $\mathcal{C}$. It is then parallel displaced on W^2. But W^2 can be considered as a bent version of a flat sheet $\tilde{W}^2$. In $\tilde{W}^2$ one can introduce cartesian coordinates; in particular $\Gamma^\gamma_{\alpha\beta} = 0$ in $\tilde{W}^2$. It is then clear from (35.8) that parallel displacement in $\tilde{W}^2$ with cartesian coordinates is the usual parallel displacement in R^2. Thus parallel displacement of ξ in S^2 along the equator results in a vector field that is always orthogonal to the equator and of constant length.

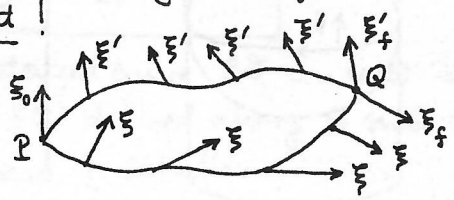
To consider parallel displacement along a "small circle" $\mathcal{C}$ on S^2 we take for W^2 a tangent cone.



W^2 is now a bent version of a flat $\tilde{W}^2$ with "opening angle" α dependent on the latitude of $\mathcal{C}$ on S^2. Thus when ξ is parallel displaced on S^2 with initial ξ_0 at P it returns with final value ξ_f that has been rotated counterclockwise then the angle α!



Since parallel displacement around a closed curve does not necessarily return $\underset{\sim}{\xi}$ to its original value $\underset{\sim}{\xi}_0$, we see that <u>parallel displacement along paths from P to Q is in general path dependent</u> !



Let us now consider parallel displacement of a <u>pair</u> of vectors along a given curve $\mathcal{C}$. First an important fact. Let $\underset{\sim}{\xi}(t)$ and $\underset{\sim}{\eta}(t)$ be any pair of vectorfields defined along the parameterized curve $\mathcal{C}$. Then, using $\langle\ ,\ \rangle$ from $\mathbb{R}^3$

$$\frac{d}{dt}\langle \underset{\sim}{\xi}(t), \underset{\sim}{\eta}(t)\rangle = \langle \frac{d\underset{\sim}{\xi}}{dt}, \underset{\sim}{\eta}\rangle + \langle \underset{\sim}{\xi}, \frac{d\underset{\sim}{\eta}}{dt}\rangle$$

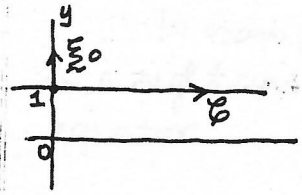
and so (why?) reverts in plane see §5.3

$$\boxed{\frac{d}{dt}\langle \underset{\sim}{\xi}, \underset{\sim}{\eta}\rangle = \langle \frac{\nabla\underset{\sim}{\xi}}{dt}, \underset{\sim}{\eta}\rangle + \langle \underset{\sim}{\xi}, \frac{\nabla\underset{\sim}{\eta}}{dt}\rangle} \qquad (35.11)$$

In particular, if $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are both parallel displaced along $\mathcal{C}$ we have $\langle \underset{\sim}{\xi}, \underset{\sim}{\eta}\rangle$ is constant! Thus <u>parallel translation preserves scalar products</u> (and consequently lengths of vectors and angles between vectors are conserved). In particular <u>since the tangent vector T to a geodesic $\mathcal{C}$ is parallel displaced along $\mathcal{C}$</u> (when parameterized by arc-lengths, see (35.4)) we see that <u>if $\underset{\sim}{\xi}$ is parallel displaced along a geodesic then the angle $\measuredangle(\underset{\sim}{\xi}, \underset{\sim}{T})$ is constant.</u>

The generalized notion of parallel displacement is defined in <u>any</u> Riemannian M^2 via (35.8) and (35.9), though Levi-Civita's interpretation (35.7) is lost if M^2 is not in $\mathbb{R}^3$. Covariant differentiation & parallel displacement are metric, i.e. Riemannian concepts!

- <u>Problem 62</u> The upper half plane $y>0$ can be endowed with the "<u>Poincaré metric</u>" $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. Parallel displace the initial vector $\underset{\sim}{\xi}_0 = \frac{\partial}{\partial y}$ at $(0,1)$ along the parameterized curve $\mathcal{C}$; $x(t)=t$, $y(t)\equiv 1$ (i.e. solve the differential equations (35.9)).



Lecture 36. Covariant Derivatives and Curvature

(1) **Affine Connection and Covariant Derivative.** In Lecture 7 we pointed out that there seems to be no way to differentiate a contravariant vector field. One might be tempted to define the derivative of a vector field Z along a parameterized curve ξ by use of local coordinates

$$\frac{DZ^i}{dt} \stackrel{?}{=} \frac{\partial Z^i}{\partial x^k}\frac{dx^k}{dt} = \frac{dZ^i}{dt},$$

but since $(\partial Z^i/\partial x^k)$ do not form the components of a mixed tensor (see Prob. 15) we find

$$\frac{D\bar{Z}^i}{dt} \neq \frac{\partial \bar{x}^i}{\partial x^j}\frac{DZ^j}{dt}$$

in an overlap and we have thus failed to define a vector field along ξ. Basically differention involves subtraction of vectors at different points followed by a limiting procedure

$$\frac{dZ^i}{dt} = \lim_{t\to 0} \frac{Z^i(x(t)) - Z^i(x(0))}{t}.$$

This makes sense in a single coordinate patch because a coordinate system allows us to identify the vector $\frac{\partial}{\partial x^i}\big|_{x(t)}$ with the vector $\frac{\partial}{\partial x^i}\big|_{x(0)}$, but this identification is of course prejudiced by by the coordinate system.

In the last lecture we did succeed in defining a notion of derivative $\nabla Z/dt$ in the case of a surface $M^2 \subset R^3$. This succeeded for two reasons.

(i) We accepted the classical notion of derivative in R^3, dZ/dt, using Cartesian coordinates in R^3; i.e. R^3 is a special manifold that comes equipped with a notion of differentiation of vector fields.

(ii) Again using Euclidean geometry we could project dZ/dt at $x(0)$ orthogonally into the tangent space $M^2_{x(0)}$ to M^2, yielding the tangent vector $\nabla Z/dt$.

It should be clear from this that we could define $\nabla Z/dt$ for any $M^m \subset R^N$, but it should also be clear that the answer would depend on how M^m is embedded in R^N (although it would be independent of bending without stretching, see (35.12)).

Can we define a notion of $\nabla Z/dt$ for a manifold M^n that is not given as a submanifold of R^N (for example, the real space M^3 in which we live)?

A general procedure is as follows. Try to find, in each coordinate patch, "connection components" Γ^i_{jk} having the property that if we define

$$\frac{\nabla Z^i}{dt} \equiv \frac{dZ^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}Z^k \tag{36.1}$$

then in an overlap of coordinate patches we have

$$\frac{\nabla \bar{Z}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^l}\frac{\nabla Z^l}{dt}. \tag{36.2}$$

This will clearly require the Γ's to have a special transformation rule, to be discussed later. Such a collection of Γ's (one set for each patch) will define what is called an "affine connection", and the associated "covariant differentiation" operator ∇ will have the familiar properties

$$\left.\begin{aligned} &\text{(i) linearity: } \frac{\nabla(Z+W)^i}{dt} = \frac{\nabla Z^i}{dt} + \frac{\nabla W^i}{dt}, \\ &\text{(ii) Leibnitz: } \frac{\nabla}{dt}(fZ^i) = \frac{df}{dt}Z^i + f\frac{\nabla Z^i}{dt} \end{aligned}\right\} \tag{36.3}$$

for differentiable functions f.

For a Riemannian manifold, such Γ's were first exhibited by Christoffel; in his notation (I believe)

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} = \frac{1}{2} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \tag{36.4}$$

just as in (34.2) in the case of a Riemannian surface M^2. The behavior in an overlap of patches is rather messy to verify, but in the case of an $M^n \subset R^N$ it is immediate because of Levi-Civita's geometric interpretation; for $M^n \subset R^N$, $\nabla Z/dt$ is simply orthogonal projection of dZ/dt into the tangent space!

Once we have ∇ the notion of parallel displacement or translation of a vector Z along a curve ξ is defined, as before, by solving

$$\frac{\nabla \tilde{Z}^i}{dt} = \frac{d\tilde{Z}^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}\tilde{Z}^k = 0 \tag{36.5}$$

for $\tilde{Z}(t)$, given $\tilde{Z}(0)$.

Conversely, knowledge of parallel translation along $\mathcal{C}$ allows us to recover the covariant derivative. Let $Z(t)$ be a vector field along $\mathcal{C}$ and let $\tilde{Z}(t)$ be the result of parallel translating $Z(0)$ along $\mathcal{C}$. Then

$$\frac{d}{dt}\left[Z^i(t) - \tilde{Z}^i(t)\right]_{t=0} = \frac{dZ^i}{dt}(0) - \frac{d\tilde{Z}^i}{dt}(0)$$

$$= \frac{dZ^i}{dt}(0) + \Gamma^i_{jk}(0)\frac{dx^j}{dt}(0)\tilde{Z}^k(0) = \left(\frac{dZ^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}Z^k\right)_{t=0}$$

$$= \frac{\nabla Z^i}{dt}(0)$$

(2) <u>The Riemannian or Levi-Civita Connection</u>. In a general "affine connection", the connection coefficients Γ^i_{jk} need not be symmetric in jk; this will be discussed more in Lect. 38. For the present we shall merely say that an affine connection is "<u>symmetric</u>" or "<u>torsion free</u>" if

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

If Γ is a connection on a Riemannian M^n, this connection need have no relation to the metric tensor, i.e. Γ need not be the Christoffel symbols $\{\,\}$. These Christoffel connections can be characterized as follows

<u>Proposition</u> (Levi-Civita). The Christoffel (or Levi-Civita) connection on a Riemannian M^n is the <u>unique</u> affine connection that satisfies

$$\left.\begin{aligned} &\text{(i) it is symmetric, } \Gamma^i_{jk} = \Gamma^i_{kj} \\ &\text{(ii) } \frac{d}{dt}\langle Y(t), Z(t)\rangle = \left\langle \frac{\nabla Y}{dt}, Z\right\rangle + \left\langle Y, \frac{\nabla Z}{dt}\right\rangle \end{aligned}\right\} \quad (36.6)$$

along each curve $\mathcal{C}$.

Proof. Consider one of the coordinate curves x^j, parameterized by $t = x^j$. Along this curve $\mathcal{C}$ $dx^\ell/dt = \delta^\ell_j$ and so

$$\frac{\partial}{\partial x^j}\langle Y, Z\rangle = \frac{\partial}{\partial x^j}(g_{ik}Y^iZ^k)$$

$$= \frac{\partial g_{ik}}{\partial x^j}Y^iZ^k + g_{ik}\left(\frac{\partial Y^i}{\partial x^j}Z^k + Y^i\frac{\partial Z^k}{\partial x^j}\right).$$

On the other hand

$$\left\langle \frac{\nabla Y}{\partial x^j}, Z \right\rangle + \left\langle Y, \frac{\nabla Z}{\partial x^j} \right\rangle = g_{rs}\left(\frac{\partial Y^r}{\partial x^j} + \Gamma^r_{\ell i}\,\delta^\ell_j\, Y^i\right) Z^s +$$

$$+ g_{rs}\, Y^r \left(\frac{\partial Z^s}{\partial x^j} + \Gamma^s_{\ell i}\,\delta^\ell_j\, Z^i\right)$$

$$= g_{rs}\left(\frac{\partial Y^r}{\partial x^j} + \Gamma^r_{ji} Y^i\right) Z^s + g_{rs}\; Y^r \left(\frac{\partial Z^s}{\partial x^j} + \Gamma^s_{ji} Z^i\right)$$

and so (ii) implies

$$\frac{\partial g_{ik}}{\partial x^j} Y^i Z^k = g_{rk}\,\Gamma^r_{ji}\, Y^i Z^k + g_{is}\,\Gamma^s_{jk}\, Y^i Z^k$$

for all Y^i and Z^k. Thus we must have

$$\frac{\partial g_{ik}}{\partial x^j} = g_{kr}\,\Gamma^r_{ji} + g_{ir}\,\Gamma^r_{jk}$$

$$= \Gamma_{ji,k} + \Gamma_{jk,i}$$

where, just as in the expression below (34.1), $\Gamma_{ji,k} \equiv g_{k\ell}\,\Gamma^\ell_{ji}$, etc. Cyclically permuting, as in lecture 34, yields that (using symmetry)

$$\Gamma^r_{ji} = \left\{ {r \atop ji} \right\}$$

are indeed the Christoffel symbols. □

In the above computations the expression

$$\boxed{\nabla_j Y^k \equiv Y^k_{|j} \equiv \frac{\partial Y^k}{\partial x^j} + \Gamma^k_{ji} Y^i} \qquad (36.7)$$

occurs repeatedly; in some books it is written $Y^k_{;j}$. Since, along a curve $\mathcal{C}$, $\nabla_j Y^k \frac{dx^j}{dt} = Y^k_{|j} \frac{dx^j}{dt}$ is the covariant derivative along $\mathcal{C}$, $Y^k_{|j}$ must form the components of a mixed tensor, the <u>covariant differential</u> of Y, written ∇Y

$$\nabla Y = (\nabla_j Y^k)\, \frac{\partial}{\partial x^k} \otimes dx^j = \frac{\partial}{\partial x^k} \otimes Y^k_{|j}\, dx^j \qquad (36.8)$$

We may consider any mixed tensor, once contravariant and once covariant as a "<u>vector valued 1-form</u>"; for example the value of ∇Y on a vector X is the vector

$$\nabla_X Y \equiv \nabla Y(X) = \frac{\partial}{\partial x^k} \otimes Y^k_{|j}\, dx^j(X)$$

$$= Y^k_{|j} X^j \frac{\partial}{\partial x^k}, \qquad (36.9)$$

the "covariant derivative" of Y with respect to the vector X. Note that while X need only be defined at a point Y must be a vector field.

• Problem 63. Show that if both X and Y are vector fields and if the connection is symmetric, then for Lie bracket we have

$$[X, Y]^i = X^j Y^i_{|j} - Y^j X^i_{|j}, \text{ i.e}$$

$$[X, Y] = \nabla_X Y - \nabla_Y X \qquad (36.10)$$

(3) Curvature of a Connection. Let us compare what we've done so far with the basic surface calculations of Lect. 34.

Let $\partial/\partial x^i$ be the i^{th} coordinate vector on M^n in a patch. Since for purposes of linear algebra we like to write our basis elements to the left, e.g. $v = \frac{\partial}{\partial x^i} v^i$ and since we do not wish this ever to be confused with differentiating v^i, $\partial v^i/\partial x^i$ we shall always try to use the notation

$$\partial_i \equiv \partial/\partial x^i \quad \text{as a basis vector, etc.}$$

For a surface $M^2 \subset \mathbb{R}^3$ we have

$$\partial_\alpha = \frac{\partial \underset{\sim}{x}}{\partial u^\alpha} = \underset{\sim}{x}_\alpha.$$

Then the surface equations (34.1)

$$\frac{\partial}{\partial u^\beta}\left(\frac{\partial \underset{\sim}{x}}{\partial u^\alpha}\right) = \underset{\sim}{x}_{\alpha\beta} = \underset{\sim}{x}_\gamma \Gamma^\gamma_{\beta\alpha} + b_{\alpha\beta} \underset{\sim}{N}$$

become, after taking tangential part of usual derivative $\frac{\partial}{\partial u^\beta}(\underset{\sim}{x}_\alpha)$

$$\nabla_{\partial_\beta} \partial_\alpha = \partial_\gamma \Gamma^\gamma_{\beta\alpha} \qquad (36.11)$$

Warning. ∇_{∂_β} must not be confused with ∇_β of (36.7)

∇_{∂_β} is only applied to vector fields, the answer being a vector field.
∇_β is only applied to components of a vector field, the answer being the component of a mixed tensor.

It should be clear from Prob. 60 that $R^\tau{}_{\alpha\gamma\beta}$ is at least related to the tangential part of

$$\frac{\partial}{\partial u^\gamma}\frac{\partial}{\partial u^\beta}\left(\underset{\sim}{x}_\alpha\right) - \frac{\partial}{\partial u^\beta}\frac{\partial}{\partial u^\gamma}\left(\underset{\sim}{x}_\alpha\right)$$

- Problem 64. Show, using (36.11) that

$$\nabla_{\partial_\gamma}\nabla_{\partial_\beta}\partial_\alpha - \nabla_{\partial_\beta}\nabla_{\partial_\gamma}\partial_\alpha = \partial_\tau R^\tau{}_{\alpha\gamma\beta} \qquad (36.12)$$

Thus the "Riemann curvature tensor" (34.3) arises from the non-commutativity of covariant differentiations.

Suppose now that instead of considering coordinate basis vectors ∂_α, ∂_β and ∂_γ we consider 3 vector fields

$$X = \partial_\gamma X^\gamma, \quad Y = \partial_\beta Y^\beta, \text{ and } Z = \partial_\alpha Z^\alpha$$

then repeated use of (36.3), e.g.

$$\nabla_X Z = \nabla_X\left(\partial_\alpha Z^\alpha\right) = \nabla_X(\partial_\alpha)\, Z^\alpha + \partial_\alpha\, X(Z^\alpha)$$

and

$$\nabla_X Z = \nabla_{\partial_\gamma X^\gamma} Z = \left(\nabla_{\partial_\gamma} Z\right) X^\gamma$$

will give the following

- Problem 65. Show

$$\left.\begin{aligned}&\left(\nabla_X\nabla_Y - \nabla_Y\nabla_X\right)Z - \nabla_{[X,Y]}Z = R(X,Y)Z\\ &\text{where}\quad \left(R(X,Y)Z\right)^\tau = R^\tau{}_{\alpha\gamma\beta}\, X^\gamma Y^\beta Z^\alpha\end{aligned}\right\} \qquad (36.13)$$

We abbreviate this by

$$\boxed{\left[\nabla_X, \nabla_Y\right] - \nabla_{[X,Y]} = R(X,Y)}$$

It is clear that the right hand side of (36.13) makes sense for any triple of vectors X_p, Y_p, Z_p defined at a single point p, and so, remarkably must the left hand side! That is, given X_p, Y_p, Z_p, extend these vectors to vectorfields in any way you wish. Then the vector

$$\nabla_{X_p}\left(\nabla_Y Z\right) - \nabla_{Y_p}\left(\nabla_X Z\right) - \nabla_{\nabla_{X_p}Y - \nabla_{Y_p}X}(Z)$$

at p is in fact independent of how the extensions were made, and equals the vector $R(X_p, Y_p)Z_p$. In local coordinates this vector is simply

$$\partial_\tau R^\tau{}_{\alpha\gamma\beta} X_p^\gamma Y_p^\beta Z_p^\alpha .$$

In other words, the multilinear function on $M_p^* \times M_p \times M_p \times M_p$ defined locally by

$$(\mu_p, Z_p, X_p, Y_p) \longmapsto \mu_\tau R^\tau{}_{\alpha\gamma\beta} X_p^\gamma Y_p^\beta Z_p^\alpha$$

is in fact a 4th rank tensor, the Riemann curvature tensor

$$\partial_\tau R^\tau{}_{\alpha\gamma\beta}\, dx^\alpha \otimes dx^\gamma \otimes dx^\beta$$

(1) Covariant Differentiation of Tensors. We have defined, in (36.8), the covariant differential of a vector field Y to be the mixed tensor ∇Y with components

$$\nabla_j Y^k = Y^k_{|j} = \frac{\partial Y^k}{\partial x^j} + \Gamma^k_{jl} Y^l.$$

It has the property that if f is a function then the components of $\nabla(fY)$ are

$$(fY)^k_{|j} = f\, Y^k_{|j} + Y^k \frac{\partial f}{\partial x^j}, \quad \text{i.e.}$$

$$\nabla(fY) = f\,\nabla Y + Y \otimes df$$

(recall that we prefer to write Yf but this notation makes it appear that Y is acting on f)

If f is a function we shall define its <u>covariant differential</u> to be its usual differential

$$\nabla f = df = \frac{\partial f}{\partial x^j} dx^j = f_{|j}\, dx^j.$$

If μ is a covariant vector, i.e., a 1-form we shall define its <u>covariant differential</u> $\nabla\mu$ to be the twice covariant tensor (<u>not</u> skew symmetric)

$$\nabla\mu = \mu_{k|j}\, dx^j \otimes dx^k = (\nabla_j \mu_k)\, dx^j \otimes dx^k$$

uniquely defined by the requirement, for each vector field Y

$$d(\mu(Y)) = (\mu_{k|j} Y^k + \mu_k Y^k_{|j})\, dx^j,$$

i.e

$$\frac{\partial}{\partial x^j}(\mu_k Y^k) = (\mu_k Y^k)_{|j} = \mu_{k|j} Y^k + \mu_k Y^k_{|j} \qquad (37.1)$$

(a "Leibnitz-rule"). Thus

$$\frac{\partial \mu_k}{\partial x^j} Y^k + \mu_k \frac{\partial Y^k}{\partial x^j} = \mu_{k|j} Y^k + \mu_k \left(\frac{\partial Y^k}{\partial x^j} + \Gamma^k_{jl} Y^l\right)$$

and so

$$\boxed{\mu_{k|j} = \nabla_j \mu_k = \frac{\partial \mu_k}{\partial x^j} - \mu_l \Gamma^l_{jk}} \qquad (37.2)$$

Note that $\mu_{k|j}$ is not skew symmetric and so does not define a 2-form; however if the connection is symmetric we clearly do have $\mu_{k|j} - \mu_{j|k} = \partial\mu_k/\partial x^j - \partial\mu_j/\partial x^k$ and in this case

$$d(\mu_k dx^k) = \sum_{j<k} (\mu_{k|j} - \mu_{j|k}) dx^j \wedge dx^k \qquad (37.3)$$

Symmetric connection

We have now defined ∇ on both vector fields and covariant vector fields, i.e, on all first rank tensors. It is now reasonable to define ∇ on all tensor fields so that a type of product rule is satisfied. Thus we would demand, e.g., for a 1-form μ and vector field Y

$$\nabla(Y \otimes \mu) = (\nabla Y) \otimes \mu + Y \otimes \nabla\mu \qquad (37.4)$$

i.e.

$$(Y \otimes \mu)^k{}_{i|j} = (Y^k \mu_i)_{|j} = Y^k{}_{|j}\,\mu_i + Y^k \mu_{i|j}$$

(a third rank tensor, once contravariant, twice covariant). We shall describe this construction later; for the moment let us simply state that if T is a tensor, p times contravariant and q times covariant, then ∇T will be a tensor p times contravariant again, but $(q+1)$ times covariant (thus the term <u>covariant</u> differential), written

$$T^{k_1 \cdots k_p}{}_{i_1 \cdots i_q | j}.$$

(2) <u>The Exterior Covariant Differential</u>. We shall be concerned here with tensors of a very special type, the <u>vector valued p-forms</u>

$$T = \partial_k \otimes T^k{}_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

i.e tensors once contravariant, p times covariant, and skew symmetric in the covariant indices. If Y is a vector and α^p is a p-form, then $Y \otimes \alpha^p$ is the vector valued p-form

$$Y \otimes \alpha^p(X_1, \ldots, X_p) = \alpha^p(X_1, \ldots, X_p)\, Y$$

with components

$$(Y \otimes \alpha^p)^k{}_{i_1 \cdots i_p} = Y^k \alpha_{i_1 \cdots i_p}$$

For example, in $\mathbb{R}^3$ with <u>cartesian coordinates</u>

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \partial_i \otimes dx^i = \partial_k \otimes \delta^k{}_i \, dx^i$$

is the vector valued 1-form $\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} (A^1\partial_1 + A^2\partial_2 + A^3\partial_3)$

$$= \begin{pmatrix} dx\,(A^1\partial_1 + A^2\partial_2 + A^3\partial_3) \\ dy\,(\quad " \quad) \\ dz\,(\quad " \quad) \end{pmatrix} = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} = \underset{\sim}{A},$$

i.e $\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ is the identity linear transformation. Cartan called it

$$dp = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

(p stands for position vector) and it confused generations of mathematicians*. Another good one in $\mathbb{R}^3$ is

$$\begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix},$$

the vector valued 2-form whose value on $(\underset{\sim}{A}, \underset{\sim}{B})$ is the vector whose x component, e.g., $A_2B_3 - A_3B_2$ is $\pm$ the area of the parallelogram spanned by the projection of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ into the yz plane.

Note that the most general vector valued p-form is locally of the form

$$\Pi = \sum_k \partial_k \otimes \left\{ \sum_{i_1 < \cdots < i_p} T^k{}_{i_1 \cdots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right\}$$

i.e., a sum of terms of the form $Y \otimes \alpha^p$ where Y is a vector field and α^p is a p-form.

We define a product $\Pi \circledwedge \beta^q$ of a vector valued p-form Π and an ordinary exterior q-form β by using linearity and the local expressions (warning, $\circledwedge$ is my own notation)

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* It is not confusing in $\mathbb{R}^3$ but can be when one writes dp on a general manifold, as we shall.

$$(Y\otimes\alpha^p)\otimes\beta^q = Y\otimes(\alpha^p\wedge\beta^q)$$
$$= (-1)^{pq}\,\beta^q\otimes(Y\otimes\alpha^p) \qquad (37.5)$$

The covariant differential of $Y\otimes\alpha^p$, considered simply as a mixed tensor, is a mixed tensor $\nabla(Y\otimes\alpha^p)$ but it is not a vector valued $(p+1)$ form. We define, with E. Cartan, a new differential d^∇ that will take vector valued p-forms into vector valued $(p+1)$ forms by means of linearity and the local expression

$$d^\nabla(Y\otimes\alpha^p) = \nabla Y\otimes\alpha^p + Y\otimes d\alpha^p \qquad (37.6)$$

This is the <u>exterior covariant differential</u>. In components

$$d^\nabla(Y\otimes\alpha^p) = d^\nabla\left(\partial_k Y^k\otimes a_{\underline{L}}\,dx^{\underline{L}}\right)$$
$$= \partial_k\otimes Y^k_{\ ij}\ a_{\underline{L}}\,dx^j\wedge dx^{\underline{L}} + \partial_k Y^k\otimes d(a_{\underline{L}}\,dx^{\underline{L}})$$
$$= \partial_k\otimes\left\{Y^k_{\ ij}\ a_{\underline{L}}\,dx^j\wedge dx^{\underline{L}} + Y^k\,\delta^{j\underline{L}}_{\underline{I}}\,(\partial_j a_{\underline{L}})\,dx^{\underline{I}}\right\} \qquad (37.7)$$

With Cartan, <u>we shall usually omit the symbol</u> ∇ in d^∇, it being understood that one cannot differentiate a vector field without a connection being given. Thus with Cartan we write instead of (37.6)

$$d(Y\otimes\alpha^p) = \nabla Y\otimes\alpha^p + Y\otimes d\alpha^p \qquad (37.8)$$

Cartan carried this notation several steps further. Instead of ∇Y he wrote simply dY, where now

$$\nabla Y = dY = d(\partial_k Y^k) = \partial_k\otimes Y^k_{\ ij}\,dx^j$$

Further, Cartan eliminated the symbols $\otimes$ and $\otimes$; he would write

$$d(Y\alpha^p) = dY\wedge\alpha^p + Y\,d\alpha^p.$$

I won't go this far, but I will write, instead of (37.8)

$$\boxed{d(Y\otimes\alpha^p) = dY\otimes\alpha^p + Y\otimes d\alpha^p} \qquad (37.9)$$

Final remarks. Note from (37.7) that in general

$$d^2(Y\otimes\alpha^p) \neq 0$$

We note also that when the connection is symmetric it is possible to write down $d^\nabla T$ in terms of ∇ by completely skew symmetrizing the covariant part of ∇T (again we shall discuss this in more detail when needed). Thus the 2nd term in (37.7) can be replaced by

$$\partial_k \otimes Y^k \delta^{jL}_{I} (\nabla_j a_L)\, dx^I$$

the reasoning being very much like in (37.3).

Finally we remark that one has a formula like (18.10) for vector valued p-forms

$$d^\nabla T(u_0,\dots,u_p) = \sum(-1)^r \nabla_{u_r}\big(T(u_0,\dots,\hat{u}_r,\dots u_p)\big) + \sum(-1)^{r+s} T\big([u_r,u_s],u_0,\dots,\hat{u}_r,\dots,\hat{u}_s,\dots u_p\big) \qquad (37.10)$$

Note that since $T(u_0,\dots,\hat{u}_r\dots u_p)$ is a vector field we must take a covariant derivative ∇_{u_r} of it.

Lecture 38. Cartan's Moving Frames

(1) Frames or "n-beins". Let $e = (e_1, \dots, e_n)$ be a "frame" or "n-bein", i.e a collection of n linearly independent vector fields defined in some open set $\mathcal{U} \subset M^n$. Any vector v based in $\mathcal{U}$ can be expressed (see Lecture 5, § 1)

$$v = e\,v = (e_1 \cdots e_n)\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = e_i\, v^i \qquad (38.1).$$

Let σ denote the matrix of the dual basis of 1-forms

$$\sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \vdots \\ \sigma^n \end{pmatrix} \quad ; \quad \sigma^i(e_j) = \delta^i_j \qquad (38.2)$$

Then for any function $f : \mathcal{U} \to \mathbb{R}$

$$df = df(e_i)\,\sigma^i = e_i(f)\,\sigma^i \equiv f_{,i}\,\sigma^i \qquad (38.3)$$

($f_{,i} = e_i(f)$ becomes $\partial f/\partial x^i$ when $e_i = \partial_i$ in a "coordinate frame" $\partial_i = \partial/\partial x^i$).

A first natural question is whether or not the frame $e = (e_1 \cdots e_n)$ is necessarily a coordinate frame. The answer is clearly no, since if $e_i = \partial_i$, for some coordinate system $x^1, \dots, x^n$ then we must have $[e_i, e_j] = [\partial_i, \partial_j] = 0$! We can also see that this is locally also a sufficient condition.

Proposition. A frame e is a coordinate frame locally iff $[e_i, e_j] = 0$, all i, j.

Proof. We need only show that if $[e_i, e_j] = 0$ then there are n "linearly independent" functions $x^1, \dots, x^n$ such that $e_i = \partial/\partial x^i$ (functions are "linearly independent" if their differentials are). This in turn will follow if we can show that $\sigma^i = dx^i$ for n such functions. But the duals σ to e satisfy (see (18.8))

$$d\sigma^i(e_j, e_k) = e_j(\sigma^i(e_k)) - e_k(\sigma^i(e_j)) - \sigma^i([e_j, e_k])$$
$$= -\sigma^i([e_j, e_k]) = 0,$$

since $[e_j, e_k] = 0$ by assumption. Hence $d\sigma^i = 0$ in $\mathcal{U}$

Then locally there are n functions $x^1, \dots, x^n$ such that $\sigma^i = dx^i$. Since $dx^1 \wedge \cdots \wedge dx^n = \sigma^1 \wedge \cdots \wedge \sigma^n \neq 0$ those functions do form a coordinate system and $e_i = \partial_i$ □.

(2) The Connection Forms. Assume that M^n carries some affine connection ∇. Then certainly there are some coefficients Γ_{ij}^k such that

$$\boxed{\nabla_{e_i} e_j = e_k \Gamma_{ij}^k} \tag{38.4}$$

If $e_i = \partial_i$ is a coordinate frame, then the Γ_{ij}^k are simply the components of the affine connection, as in lecture 36, as can easily be verified (we did this, for example, in (36.11) in the case of a Riemannian surface in $\mathbb{R}^3$). As far as we are concerned, we take (38.4) as defining the connection coefficients with respect to the given frame e. Even when the connection is Riemannian the coefficients Γ_{ij}^k need not reduce to the Christoffel symbols $\left\{ {k \atop ij} \right\}$ when the frame is not coordinate; they may not even be symmetric in ij!

Since $\nabla_{e_i} e_j$ is the value of the vector valued 1-form ∇e_j on the vector e_i (see (36.9)) we can write

$$\nabla e_j = e_k \otimes \Gamma_{ij}^k \sigma^i$$

since then $\nabla e_j (e_\ell) = e_k \Gamma_{ij}^k \sigma^i(e_\ell) = e_k \Gamma_{\ell j}^k = \nabla_{e_\ell} e_j$.

We write

$$\left.\begin{aligned} de_j \equiv \nabla e_j &= e_k \otimes \omega^k{}_j \quad \text{or} \quad de = e \otimes \omega \\ &\text{where} \\ \omega = (\omega^k{}_j) &= \Gamma_{ij}^k \sigma^i \end{aligned}\right\} \tag{38.5}$$

The matrix $\omega = (\omega^k{}_j) = \Gamma_{ij}^k \sigma^i$ is called the matrix of connection forms, or the connection matrix.

Look now at the exterior differentials of the forms σ^i. Certainly

$$d\sigma^i = \tfrac{1}{2} C^i{}_{jk} \sigma^j \wedge \sigma^k .$$

for some local functions (C^i_{jk}). Well

$$\left.\begin{aligned} d\sigma^i(e_j, e_k) &= -\sigma^i([e_j, e_k]) \\ \text{and so} \quad C^i_{jk} &= -\sigma^i([e_j, e_k]) \end{aligned}\right\} \tag{38.6}$$

We shall now assume that the connection is symmetric, or torsion free (see Lect 36, § 2). If e were a coordinate frame we would then have $\Gamma^k_{ij} = \Gamma^k_{ji}$ but this need not be so for a general frame. We do know from (36.10) that for a symmetric connection ∇ that we have

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

In fact this property can serve as the definition of a symmetric connection! For a symmetric connection we now have

$$\begin{aligned} d\sigma^i(e_j, e_k) &= -\sigma^i([e_j, e_k]) \\ &= -\sigma^i(\nabla_{e_j} e_k - \nabla_{e_k} e_j) \\ &= -\sigma^i(e_\ell \Gamma^\ell_{jk} - e_\ell \Gamma^\ell_{kj}) \\ (d\sigma^i_{jk} &= \Gamma^i_{kj} - \Gamma^i_{jk} \end{aligned} \tag{38.7}$$

and so

$$\begin{aligned} d\sigma^i &= \tfrac{1}{2}\, d\sigma^i(e_j, e_k)\, \sigma^j \wedge \sigma^k \\ &= \tfrac{1}{2}(\Gamma^i_{kj} - \Gamma^i_{jk})\, \sigma^j \wedge \sigma^k \\ &= -\tfrac{1}{2}\, \omega^i{}_k \wedge \sigma^k - \tfrac{1}{2}\, \omega^i{}_j \wedge \sigma^j \end{aligned}$$

Thus

$$d\sigma^i = -\omega^i{}_j \wedge \sigma^j \quad \text{or} \quad d\sigma = -\omega \wedge \sigma \tag{38.8}$$

symmetric connection.

In summary we have

$$\boxed{\begin{aligned} &de = e \otimes \omega \text{ , always, } \omega^k_j = \Gamma^k_{ij}\sigma^i \\ &d\sigma = -\omega \wedge \sigma \text{ when } \nabla \text{ symmetric} \end{aligned}} \tag{38.9}$$

Of course if $v = eV$ is a vector field, then for $\nabla v = dv$ we have from (37.8)

$$dv = d(eV) = de\,V + e \otimes dV = e \otimes \omega V + e \otimes dV$$

i.e

$$\left.\begin{aligned} dv &= e \otimes (dV + \omega V) \\ &= e_k \otimes (dV^k + \omega^k{}_j V^j) \end{aligned}\right\} \qquad (38.10)$$

We define

$$\left.\begin{aligned} \mathcal{D}V &= dV + \omega V \\ \text{i.e } \mathcal{D}V^k &= dV^k + \omega^k{}_j V^j \end{aligned}\right\} \qquad (38.11)$$

and then we can say

$$dv = d(eV) = e \otimes \mathcal{D}V \qquad (38.12).$$

(3) Second Differentials and Curvature. We define the matrix of curvature 2-forms ϑ with respect to the frame e by

$$\boxed{\begin{aligned} \vartheta &= d\omega + \omega \wedge \omega \\ \text{i.e } \vartheta^k{}_j &= d\omega^k{}_j + \omega^k{}_i \wedge \omega^i{}_j \\ \text{and expand } \vartheta^k{}_j &= \tfrac{1}{2} R^k{}_{jrs}\, \sigma^r \wedge \sigma^s \end{aligned}} \qquad (38.13)$$

for some coefficients $R^k{}_{jrs}$.

- Problem 66. Verify that when $e_i = \partial_i$ is a coordinate frame then $R^k{}_{jrs}$ as defined by (38.13) coincides with the Riemann tensor (34.3).

For any given pair of vectors X and Y, $\vartheta^k{}_j(X,Y)$ is a matrix of functions and so represents a linear transformation. It is not difficult to show, and should not be surprising in view of (36.12) and (36.13) that in fact this linear transformation is $R(X,Y)$

$$\vartheta(X,Y) = R(X,Y) \qquad (38.14)$$

Now consider 2nd differentials! From (37.8)

$$d^2e = d(de) = d(e\otimes\omega) = de\otimes\omega + e\otimes d\omega$$
$$= e\otimes\omega\wedge\omega + e\otimes d\omega$$
$$= e\otimes(d\omega + \omega\wedge\omega) = e\otimes\vartheta$$

i.e

$$d^2e_j = e_k\otimes\vartheta^k{}_j = \tfrac{1}{2}e_k\otimes R^k{}_{jrs}\sigma^r\wedge\sigma^s \qquad (38.15)$$

Then for any vector field $v = eN$, $dv = e\otimes(dN + \omega N)$

$$d^2v = e\otimes\omega\wedge(dN+\omega N) + e\otimes(d^2N + d\omega N - \omega\wedge dN)$$
$$= e\otimes(\omega\wedge\omega + d\omega)N \quad (\text{since } d^2N = 0 \text{ for functions})$$

i.e

$$\left.\begin{aligned} d^2v &= e\otimes\vartheta N \\ &= e_k\otimes\vartheta^k{}_j N^j = \tfrac{1}{2}e_k\otimes R^k{}_{jrs}N^j\sigma^r\wedge\sigma^s \end{aligned}\right\} \qquad (38.16)$$

Note the remarkable fact that d^2v at p depends only on v at p! (compare with (36.13)) and is linear in v_p.

(4) Riemannian Connections. The expressions given in the previous paragraphs are valid for any symmetric connection. We now specialize to the case of a Riemannian connection, i.e we shall now invoke property (ii) of (36.6). In terms of differentials (ii) says

$$d\langle X,Y\rangle = \langle\nabla X, Y\rangle + \langle X, \nabla Y\rangle \qquad (38.17)$$
$$d\langle X,Y\rangle(Z) = Z\langle X,Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

for any pair of vector fields X and Y. In particular for our frame e

$$dg_{ij} = d\langle e_i, e_j\rangle = \langle\nabla e_i, e_j\rangle + \langle e_i, \nabla e_j\rangle$$
$$= \langle e_k\otimes\omega^k{}_i, e_j\rangle + \langle e_i, e_\ell\otimes\omega^\ell{}_j\rangle$$
$$= \langle e_k, e_j\rangle\,\omega^k{}_i + \langle e_i, e_\ell\rangle\,\omega^\ell{}_j$$
$$= g_{jk}\,\omega^k{}_i + g_{i\ell}\,\omega^\ell{}_j .$$

We define

$$\omega_{ji} = g_{jk}\,\omega^k{}_i \qquad (38.18)$$

and then have

$$\boxed{dg_{ij} = \omega_{ji} + \omega_{ij}} \qquad (38.19)$$

In particular, if the frame e is orthonormal the matrix ω_{ij} is skew-symmetric since $g_{ij} = \delta_{ij}$

$$\boxed{\omega_{ij} = -\omega_{ji}} \quad \text{orthonormal frame} \tag{38.20}$$

Orthonormal frames are easy to come by; given any frame on $\mathcal{U} \subset M^n$, e.g. a coordinate frame, we can deform the frame by the Gram-Schmidt process to obtain an orthonormal one. On the other hand, orthonormal coordinate frames are usually non-existent on a Riemannian manifold, as we shall see in Lecture 40.

Note also that (38.20) holds more generally in any frame e for which $g_{ij} = \langle e_i, e_j \rangle = \text{constant}$.

(5) Change of Frame The matrices ω and ϑ (of 1-forms and 2-forms resp.) are dependent on the frame e used, as is evident from (38.4). How do these matrices change when the frame is changed? (In physics, as we shall see next quarter, the ω's are essentially "gauge fields", the ϑ's are "field strengths", and a change of frame or n-bein is a "gauge transformation")

Consider two frame fields defined on an open $\mathcal{U} \subset M^n$ with affine connection ∇.

$$\bar{e} = eP \quad \text{i.e} \quad \bar{e}_j = e_k P^k{}_j \tag{38.21}$$

for some non-singular matrix P of functions on $\mathcal{U}$. Then $v = e\nu = \bar{e}\bar{\nu} = eP\bar{\nu}$ says $P\bar{\nu} = \nu$, or

$$\bar{\nu} = P^{-1}\nu \quad \text{i.e.} \quad \bar{\nu}^k = (\bar{P}^{-1})^k{}_j \nu^j \tag{38.22}$$

Also $\sigma^k = \sigma^k(\bar{e}_j)\,\bar{\sigma}^j = \sigma^k(e_\ell P^\ell{}_j)\,\bar{\sigma}^j$

$$= \delta^k{}_\ell P^\ell{}_j \bar{\sigma}^j = P^k{}_j \bar{\sigma}^j$$

and so

$$\bar{\sigma} = P^{-1}\sigma \quad \text{i.e} \quad \bar{\sigma}^j = (P^{-1})^k{}_j \sigma^j \tag{38.23}$$

Then

$$\begin{cases} \nabla\bar{e} = \bar{e}\otimes\bar{\omega} \quad \text{and} \quad \nabla e = e\otimes\omega \\ \nabla\bar{e} = \nabla(eP) = \nabla e\, P + e\, dP \end{cases}$$

$$= e\otimes\omega P + e\, dP$$

$$= \bar{e}P^{-1}\otimes\omega P + \bar{e}P^{-1}dP$$

shows that

$$\bar{\omega} = P^{-1}\omega P + P^{-1}dP \qquad (38.24)$$

i.e

$$\bar{\omega}^k{}_i = (P^{-1})^k{}_j\,\omega^j{}_\ell\,P^\ell{}_i + (P^{-1})^k{}_j\,dP^j{}_i$$

Finally $\bar{\vartheta} = d\bar{\omega} + \bar{\omega}\wedge\bar{\omega}$ gives

$$\bar{\vartheta} = d\left(P^{-1}\omega P + P^{-1}dP\right) + \left(P^{-1}\omega P + P^{-1}dP\right)\wedge\left(P^{-1}\omega P + P^{-1}dP\right).$$

To complete this calculation, ie to compare with $\vartheta = d\omega + \omega\wedge\omega$, we use $P^{-1}P = I$, $\therefore\ dP^{-1}P + P^{-1}dP = 0$ or

$$dP^{-1} = -P^{-1}dP\,P^{-1} \qquad (38.25)$$

Problem 67 Show that

$$\bar{\vartheta} = P^{-1}\vartheta P \quad \text{i.e.} \quad \bar{\vartheta}^k{}_i = (P^{-1})^k{}_j\,\vartheta^j{}_\ell\,P^\ell{}_i \qquad (38.26)$$

In summary

$$\boxed{\begin{aligned} &\bar{e} = eP\,, \quad \bar{\nu} = P^{-1}\nu \\ &\bar{\sigma} = P^{-1}\sigma \\ &\bar{\omega} = P^{-1}\omega P + P^{-1}dP \\ &\bar{\vartheta} = P^{-1}\vartheta P \end{aligned}} \qquad (38.27)$$

Note that $\bar{\nu} = P^{-1}\nu$ merely says that the n-tuple ν transforms as a vector. (In the case of coordinate frames $\bar{e} = \partial/\partial\bar{x} = \frac{\partial x}{\partial\bar{x}}\,\partial/\partial x$ just tells us that $P = \partial x/\partial\bar{x}$). Likewise $\bar{\nu} = P^{-1}\vartheta P$ says that the matrix of curvature 2-forms transforms like a (mixed) tensor; we say that ϑ is a "tensorial form". On the other hand, ω does not transform tensorially because of the inhomogeneous term $P^{-1}dP$.

Note in particular that if the curvature matrix Θ vanishes in some frame at a point p then $\bar{\Theta}$ vanishes in any other frame at p. On the other hand, clearly if $\omega = 0$ in frame e then $\bar{\omega}$ need not vanish. For example, consider $\mathbb{R}^2$ with usual Riemannian metric $ds^2 = dx^2 + dy^2$, ie $g_{ij} = \delta_{ij}$, in the coordinate frame $e = (\partial/\partial x, \partial/\partial y)$. Of course then $\omega = 0$ and $\Theta = 0$. We can use this to write down $\bar{\omega}$ and $\bar{\Theta}$ when we use the polar coordinate frame $\bar{e}_1 = \partial/\partial r$, $\bar{e}_2 = \partial/\partial\theta$.

$$\bar{e} = \left(\frac{\partial}{\partial r} \quad \frac{\partial}{\partial\theta}\right) = \left(\frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \quad \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}\right)$$

● Problem 68 Show that $P = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ and use $\bar{\omega} = P^{-1}dP$ to get

$$\bar{\omega} = \begin{pmatrix} 0 & -r\,d\theta \\ \frac{d\theta}{r} & \frac{dr}{r} \end{pmatrix}. \text{ Verify } \bar{\Theta} = d\bar{\omega} + \bar{\omega}\wedge\bar{\omega} = 0.$$

Lecture 39. Curvature and Parallel Displacement

(1) **Gaussian Curvature of a Riemannian M^2.** Let M^2 be a Riemannian surface. Just as analytic geometry in R^2 is simpler when one uses Cartesian coordinates, so Riemannian geometry in an M^n is usually simpler if one restricts oneself to orthonormal (o.n) frames. (In the next lecture we shall see that o.n coordinate frames usually do not exist).

Let e be an o.n frame on $U \subset M^2$ and let σ be its dual frame of 1-forms, let ω be the connection matrix and ϑ the curvature matrix. When using o.n. frames, raising and lowering indices is the identity operation, $\omega^i{}_j = \omega_{ij}$, etc., since $g_{ij} = \delta_{ij}$, and one usually writes all indices as covariant. Since (see (38.20)) $\omega_{ij} = -\omega_{ji}$ we see that for an M^2 with an o.n frame

$$\omega_{21} = -\omega_{12} \qquad \omega_{11} = \omega_{22} = 0$$

and so there is basically only one 1-form ω_{12} to worry about. Likewise $\vartheta_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}$ gives

$$\vartheta_{ji} = d\omega_{ji} + \omega_{jk} \wedge \omega_{ki} = -d\omega_{ij} - \omega_{ki} \wedge \omega_{jk}$$
$$= -d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\vartheta_{ij}.$$

Thus in an o.n frame ϑ is skew symmetric (any Riemannian M^n) and for an M^2 we again have only one significant 2 form ϑ_{12},

$$\vartheta^1{}_2 = \vartheta_{12} = d\omega_{12} + \omega_{1\alpha} \wedge \omega_{\alpha 2} \qquad \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22}$$

$$\boxed{\begin{array}{l} \vartheta_{12} = d\omega_{12} \\ M^2, \text{ o.n. frame} \end{array}} \tag{39.1}$$

In an o.n. frame $\sigma^1 \wedge \sigma^2$ is $\pm$ the area form for the Riemannian metric and of course there is some coefficient function K such that $\vartheta_{12} = d\omega_{12} = K\sigma^1 \wedge \sigma^2$. We claim that K is the Gaussian curvature! To prove this, note from (38.13) that

$$\vartheta^1{}_2 = \sum_{i<j} R^1{}_{2ij}\sigma^i\wedge\sigma^j = R^1{}_{212}\sigma^1\wedge\sigma^2 = R^{12}{}_{12}\sigma^1\wedge\sigma^2.$$

Now if we were dealing with a coordinate frame, $\bar{e}_i = \partial_i$, then $\bar{R}^{12}{}_{12}$ would indeed be the Gaussian curvature K (see prob. 65 and (34.5)). Given our o.n frame e and a point $p \in \mathcal{U}$, we can always find a coordinate system which is o.n at the point p and which is such that

$$\bar{e}_1 = \frac{\partial}{\partial x^1} = e_1 \quad \text{and} \quad \bar{e}_2 = \frac{\partial}{\partial x^2} = e_2 \quad \text{at } p.$$

Then $\bar{e} = eP$ where $P = I$ at the point p and so (38.27) yields

$$\bar{\vartheta}^1{}_2 = \vartheta^1{}_2 \quad \text{at } p.$$

But then $R^{12}{}_{12} = \bar{R}^{12}{}_{12}$ = Gauss curvature K at p □.

$$\boxed{\begin{array}{c}\vartheta_{12} = d\omega_{12} = K\sigma^1\wedge\sigma^2 \\ K = \text{Gaussian curvature} \\ \text{o.n frame}\end{array}} \tag{39.2}$$

(2) Computing K for a Surface M^2. The expression (39.2) is frequently useful for actually calculating curvatures. Consider, e.g., a surface where the metric has the form

$$ds^2 = du^2 + G^2(u,v)\,dv^2$$

in some local coordinate system (you should compare the following calculation with what you did in Prob. 61 for the very special case of the 2-sphere). The coordinate frame $\partial/\partial u$, $\partial/\partial v$ is orthogonal but not unit. In an orthonormal frame we would have

$$ds^2 = (\sigma^1)^2 + (\sigma^2)^2.$$

This inspires us to define

$$\sigma^1 = du, \quad \sigma^2 = G(u,v)\,dv$$

and so

$$e_1 = \frac{\partial}{\partial u} \qquad e_2 = \frac{1}{G}\frac{\partial}{\partial v}.$$

We know then that in this frame ω is skew symmetric and we need only compute ω_{12}. We put

$$\omega_{12} = a\sigma^1 + b\sigma^2$$

for some coefficients $a(u,v)$, $b(u,v)$; a and b are really connection coefficients, $\omega^1{}_2 = \Gamma^1_{12}\sigma^1 + \Gamma^1_{22}\sigma^2$, but why should we clutter up the page? From (38.9) and skew symmetry

$$d\sigma^1 = -\omega^1{}_2 \wedge \sigma^2 = -(a\sigma^1 + b\sigma^2)\wedge\sigma^2 = -a\sigma^1\wedge\sigma^2.$$

But $d\sigma^1 = d(du) = 0$, and so $a = 0$ and

$$\omega^1{}_2 = -b\sigma^2.$$

Also

$$d\sigma^2 = -\omega^2{}_1 \wedge \sigma^1 = \omega^1{}_2 \wedge \sigma^1 = -b\sigma^2\wedge\sigma^1 = b\sigma^1\wedge\sigma^2$$

and $d\sigma^2 = d(G\,dv) = G_u\,du\wedge dv = \dfrac{G_u}{G}\sigma^1\wedge\sigma^2$, yields

$$\omega_{12} = -b\sigma^2 = -\frac{G_u}{G}\sigma^2 = -G_u\,dv.$$

Then $\vartheta_{12} = d\omega_{12} = -G_{uu}\,du\wedge dv = -\dfrac{G_{uu}}{G}\sigma^1\wedge\sigma^2$

$$\boxed{K = -\frac{1}{G}\frac{\partial^2 G}{\partial u^2}} \qquad ds^2 = du^2 + G^2(u,v)\,dv^2 \qquad (39.3)$$

- Problem 69. Consider the Poincaré metric on the upper-half plane

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad y > 0$$

Compute the Gauss curvature K of this metric using the above method. Compute it also using (39.3) directly.

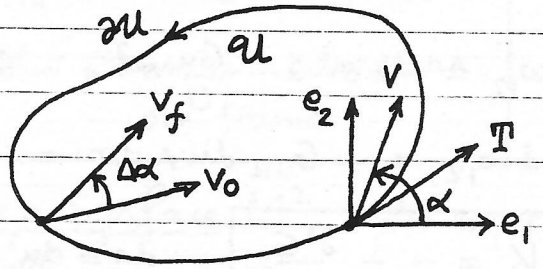
(3) Parallel Displacement and Curvature of a Surface. In Lecture 35, §2 we saw that parallel displacement between two given points is path dependent, i.e, parallel displacement of a vector around a closed path , results in a final vector v_f that disagrees with the original vector v_0. We gave as an explicit example parallel displacement around a small circle on the 2-sphere. There is

a remarkable result, in the case of surfaces, relating their "holonomy" $v_f \neq v_0$ with Gaussian curvature.

Theorem. Let $\mathcal{U}^2 \subset M^2$ be a compact region on a Riemannian surface having a piecewise smooth boundary $\partial\mathcal{U}$. We assume that $\mathcal{U}$ can be covered by a single o.n. frame (e.g. $\mathcal{U}$ is in a coordinate patch). Let a unit vector v be parallel translated around $\partial\mathcal{U}$, starting with v_0 and ending with v_f. Then the angle $\Delta\alpha$ between v_f and v_0 is given by

$$\boxed{\Delta\alpha = \iint_{\mathcal{U}} K\,dA = \iint_{\mathcal{U}} K\sigma^1\wedge\sigma^2} \tag{39.4}$$

Proof.



Let $\alpha = \measuredangle(e_1, v)$, then $\Delta\alpha = \measuredangle(v_0, v_f) = \oint_{\partial\mathcal{U}} d\alpha$. But

$$v = eN = e_1\cos\alpha + e_2\sin\alpha = (e_1\ e_2)\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}.$$

Then, from (38.10)

$$dv = e\otimes(dN + \omega N) = (e_1\ e_2)\otimes\left\{d\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + \begin{pmatrix}0 & \omega^1_2\\ -\omega^1_2 & 0\end{pmatrix}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\right\}$$

$$dv = (e_1\ e_2)\otimes\begin{pmatrix}-\sin\alpha\,d\alpha + \omega^1_2\sin\alpha\\ \cos\alpha\,d\alpha - \omega^1_2\cos\alpha\end{pmatrix} \tag{39.5}$$

To say that v is parallel displaced around $\partial\mathcal{U}$ is to say that $(dN + \omega N)(T) = 0$, where T is the unit tangent vector to $\partial\mathcal{U}$. From (39.5) we must then have

$$d\alpha(T) = \omega'_2(T) \qquad (39.6)$$

(or briefly $d\alpha = \omega'_2$ along $\partial\mathcal{U}$). Then, using arc length for parameter along $\partial\mathcal{U}$

$$\Delta\alpha = \oint_{\partial\mathcal{U}} d\alpha = \int_0^L d\alpha(T)\,ds = \int_0^L \omega'_2(T)\,ds$$

$$= \oint_{\partial\mathcal{U}} \omega'_2 = \int_{\mathcal{U}} d\omega'_2 = \int_{\mathcal{U}} \vartheta'_2 = \int_{\mathcal{U}} K\sigma^1\wedge\sigma^2 \qquad \square.$$

Corollary 1. Parallel translation on a Riemannian surface M^2 is locally independent of path iff $K \equiv 0$

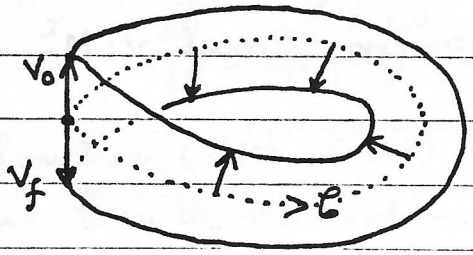
The word locally is important here; we must restrict ourselves to closed curves $\mathcal{C}$ that are first of all boundaries, $\mathcal{C} = \partial\mathcal{U}$, and secondly the entire region $\mathcal{U}$ must be covered by some orthonormal frame (but see remark below!)

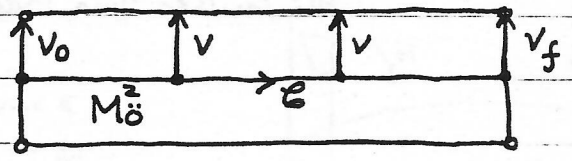
Definition. A connection ∇ on a manifold M^n is "flat" provided the curvature of the connection vanishes

$$R(X,Y) = 0 = \vartheta(X,Y)$$

for all vectors X, Y.

Note that flatness of a connection is a local property. A cylinder or a Möbius band in $\mathbb{R}^3$ is flat in spite of the "curved" appearance because $K = 0$. Parallel translation of a "vertical" v_0





around the central circle $\mathcal{C}$ yields $v_f = -v_0$, even though $K \equiv 0$

This does not contradict our Theorem since the curve $\mathcal{C}$ does not bound any region!

We have used the assumption that $\mathcal{U}$ can be covered by a single o.n frame (since we look at $\measuredangle(e_1, v)$); however it can be shown that this is no restriction since any compact region $\mathcal{U}$ lying in the interior of M^2 can be so covered!

Another remark; even closed manifolds can be flat! For example the 2-torus $T^2 = S^1 \times S^1$ with angular coordinates θ, φ can be endowed with the flat metric

$$ds^2 = d\theta^2 + d\varphi^2.$$

This is certainly not the metric that a torus $T^2 \subset \mathbb{R}^3$ would inherit from $\mathbb{R}^3$, however it is the metric that T^2 would inherit from one embedding in $\mathbb{R}^4$. The embedding $(\theta, \varphi) \longrightarrow (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$, i.e $x = \cos\theta$, $y = \sin\theta$, $z = \cos\varphi$, $w = \sin\varphi$ yields from the Euclidean metric on $\mathbb{R}^4$, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$ the pull back $ds^2 = d\theta^2 + d\varphi^2$. This embedding is the so called "Clifford Torus".

For another corollary of our Theorem we can consider the Gauss normal map $\nu: M^2 \longrightarrow S^2$ for any $M^2 \subset \mathbb{R}^3$. From §2.6),

Corollary If $M^2 \subset \mathbb{R}^3$ is an embedded surface then $\Delta\alpha = \pm$ the area of the spherical image of $\mathcal{U}$ under the Gauss normal map $\nu: M^2 \longrightarrow S^2$.

Lecture 40. The Horizontal Distribution and Riemann's Theorem

(1) **Locally Flat Metrics.** A quadratic form $Q_{ij}dx^i dx^j$ at a single point can always be reduced do a sum of squares by an appropriate linear change of coordinates

$$Q_{ij}\,dx^i dx^j = (dy^1)^2 + \cdots + (dy^m)^2, \qquad \text{at } x_0$$

where $y^i = P^i_j x^j$. This is simply a consequence of the fact that Q_{ij} is a symmetric matrix. Can a variable quadratic form $g_{ij}(x)\,dx^i dx^j$ in some region about x_0 be reduced do such a sum of squares by some (perhaps non-linear) change of coordinates $y^i = y^i(x^1, \cdots, x^m)$? It was precisely for such considerations that Riemann introduced his tensor R^h_{ijk}; we know that if $g_{ij}(x)\,dx^i dx^j = (dy^1)^2 + \cdots + (dy^m)^2$ then the Riemann tensor for the y coordinate system vanishes. Since the Riemann tensor *is* a tensor, the Riemann tensor in the x coordinate system must also vanish (see (38.27)). Riemann also noted that this necessary condition for a metric do be "flat" or "locally Euclidean"

$$ds^2 = (dy^1)^2 + \cdots + (dy^m)^2$$

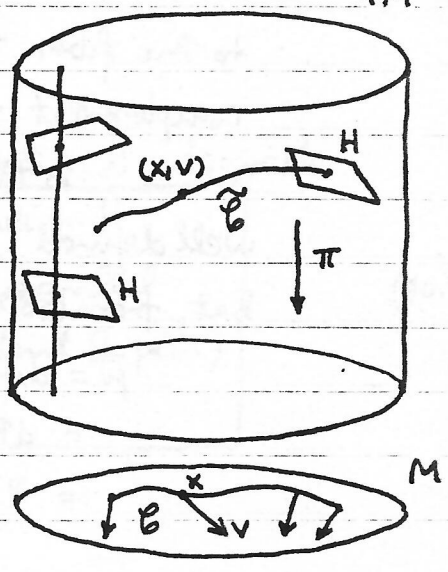
is also sufficient. We shall now discuss this from a more geometrical perspective.

(2) **The Horizontal Distribution of an Affine Connection.**

Parallel displacement of a vector v along a curve $\mathcal{C} \subset M^m$ is characterized by the system of differential equations

$$\frac{dv^i}{dt} + \Gamma^i_{jk} v^k \frac{dx^j}{dt} = 0$$

The functions $(x^i(t), v^i(t))$ define a curve $\tilde{\mathcal{C}}$ in the tangent bundle TM that lies over $\mathcal{C}$, i.e. it defines a



"cross section" of TM over $\mathcal{C}$. (In this lecture we shall denote the fiber coordinates by $v^1, \dots, v^n$ rather than $\dot{x}^1, \dots, \dot{x}^n$). Since the x's are also coordinates in TM, $\omega^i{}_k(x) = \Gamma^i_{jk}\, dx^j$ can also be considered a matrix of 1-forms in TM (we are really considering $\pi^*(\omega^i{}_k)$, we simply omit the π^*). Then parallel displacement just tells us that the 1-forms

$$\mu^i \equiv dv^i + \omega^i{}_k v^k \qquad i = 1, 2, \dots, n \tag{40.1}$$

vanish when restricted to the lifted curve $\tilde{\mathcal{C}}$, i.e., when evaluated on the tangent vector $\frac{dx^k}{dt}\frac{\partial}{\partial x^k} + \frac{dv^k}{dt}\frac{\partial}{\partial v^k}$ to $\tilde{\mathcal{C}}$. In other words

> A curve $\tilde{\mathcal{C}}$ in TM represents parallel translation of a vector v along $\mathcal{C}$ in M iff the curve $\tilde{\mathcal{C}}$ covers $\mathcal{C}$ (i.e $\pi(\tilde{\mathcal{C}}) = \mathcal{C}$) and $\tilde{\mathcal{C}}$ is tangent to the n-plane distribution H in TM defined by
>
> $$\mu^1 = 0, \dots, \mu^n = 0$$

Since no "vertical" vector $\sum a^i \frac{\partial}{\partial v^i} \neq 0$ ever lies in this distribution, H is called the "horizontal distribution" of the connection (note: TM might not be endowed with a Riemannian metric and there is thus no natural notion of a vector X in TM being "orthogonal" to the fiber $\pi^{-1}(x)$, but a connection allows us to exhibit a complement to the fiber directions).

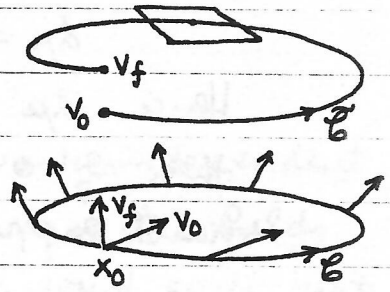
Is the distribution H, described locally by $\mu^i = dv^i + \omega^i{}_k v^k = 0$ well defined? In a second patch we consider $\bar{\mu}^i = d\bar{v}^i + \bar{\omega}^i{}_k \bar{v}^k = 0$. But, from (38.27) and (38.25)

$$\begin{aligned}\bar{\mu} = d\bar{v} + \bar{\omega}\bar{v} &= d(P^{-1}v) + (P^{-1}\omega P + P^{-1}dP)P^{-1}v \\ &= dP^{-1}v + P^{-1}dv + P^{-1}\omega v + P^{-1}dP\,P^{-1}v \\ &= P^{-1}(dv + \omega v) = P^{-1}\mu\end{aligned}$$

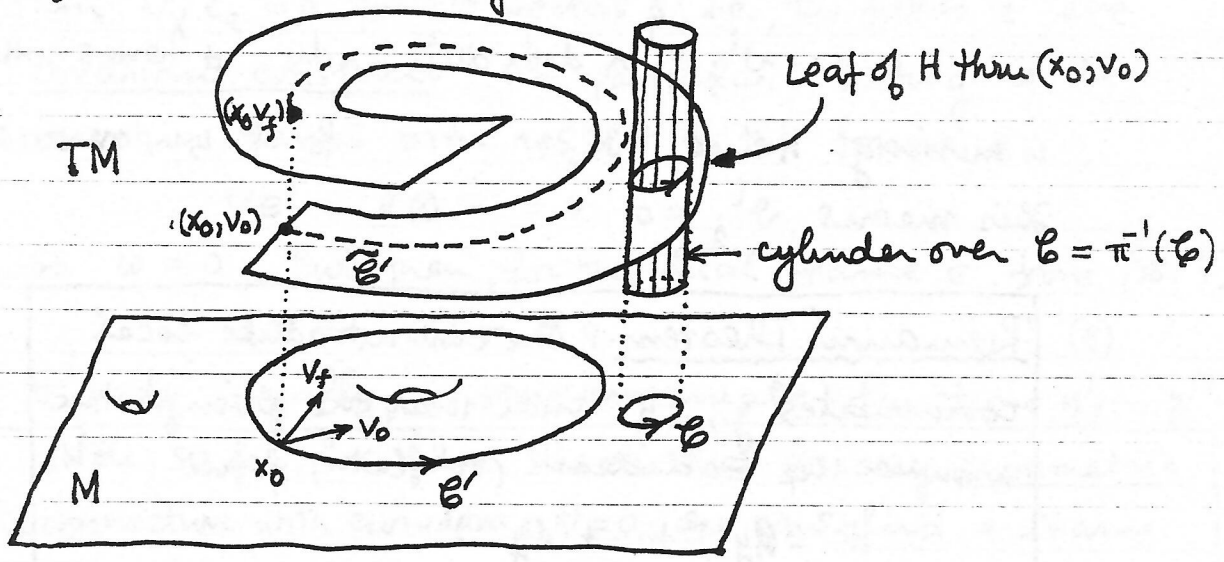
$$\bar{\mu} = P^{-1}\mu \tag{40.2}$$

Thus $\mu = 0$ iff $\bar{\mu} = 0$, and the distribution H is well defined.

To say that v returns to itself after being parallel translated around a closed curve $\mathcal{C}$ in M is to say that the lift $\tilde{\mathcal{C}}$ of $\mathcal{C}$ via v, is itself a closed curve, tangent to H



Now if the distribution H is integrable, and if we choose a closed curve $\mathcal{C}$ that is so small that the lift $\tilde{\mathcal{C}}$ will lie in a Frobenius chart (lect. 26, §1) then $\tilde{\mathcal{C}}$ will also have to be closed because $\tilde{\mathcal{C}}$ will lie on the leaf L_0 thru x_0, v_0. This need not be the case if the curve $\mathcal{C}'$ is long



Question; when is the horizontal distribution H integrable?

Theorem. The horizontal distribution H is integrable (and consequently parallel displacement is locally independent of path) iff the curvature vanishes. (40.3)

Proof. H is defined by $\mu^i = dv^i + \omega^i_{\ j} v^j = 0$, $i = 1, 2, \cdots, n$. We shall abbreviate this to $\mu = dv + \omega v = 0$. Then

$$d\mu = d^2 v + d\omega\, v - \omega \wedge dv = d\omega\, v - \omega \wedge (dv + \omega v - \omega v)$$
$$= (d\omega + \omega \wedge \omega) v - \omega \wedge \mu$$
$$d\mu = \Theta v - \omega \wedge \mu.$$

Hence $d\mu \equiv \Theta v$ (recall that $\equiv$ means put $\mu = 0$). Certainly then $\Theta = 0 \Rightarrow d\mu \equiv 0$ and H is integrable. On the other hand suppose that H is integrable. Then from (25.2)

$$0 = d\mu^i \wedge \mu^1 \wedge \mu^2 \wedge \cdots \wedge \mu^n = (\Theta^i_{\ j} v^j - \omega^i_{\ j} \wedge \mu^j) \wedge \mu^1 \wedge \cdots \wedge \mu^n$$
$$= \Theta^i_{\ j} v^j \wedge \mu^1 \wedge \cdots \wedge \mu^n$$
$$= \Theta^i_{\ j} v^j \wedge (dv^1 + \omega^1_{\ k} v^k) \wedge \cdots \wedge (dv^n + \omega^n_{\ \ell} v^\ell)$$
$$= \Theta^i_{\ j} v^j \wedge dv^1 \wedge dv^2 \wedge \cdots \wedge dv^n + \text{terms where some } dv^j \text{ is missing.}$$

Hence $\Theta^i_{\ j} v^j = 0$ for $i = 1, \cdots, n$ and all v. This means $\Theta^i_{\ j} = 0$. □

(3) **Riemann's Theorem.** One can introduce local coordinates $y^1, \cdots y^n$ such that the given metric assumes the Endlidean (or "flat") form (40.4)

$$ds^2 = (dy^1)^2 + \cdots + (dy^n)^2$$

iff the Riemann curvature tensor vanishes

Proof. The only if part has already been discussed in § 1. Suppose now that the curvature vanishes, $\Theta = 0$. Then the horizontal distribution H

$$\mu^i = dv^i + \omega^i_{\ jk} v^k = dv^i + \Gamma^i_{jk} v^k dx^j = 0$$

is integrable. Since H is transverse to the fibers $\pi^{-1}(x)$, this

means (as in the system of Mayer-Lie, Lect. 26, §2) that we can solve

$$\begin{cases} \dfrac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k = 0 \\ v^i(x_0) = v^i_0 \quad \text{given} \end{cases} \tag{40.5}$$

In particular, given x_0 and given n linearly independent vectors $e_1^0, \dots, e_n^0$ at x_0, we may find n vector fields $e_1, \dots, e_n$ coinciding with $e_1^0, \dots, e_n^0$ at x_0 and such that their covariant differentials all vanish

$$\nabla_{\partial_j} e_\ell = 0.$$

But then $\nabla_X e_\ell = 0$ for all vectors X, i.e., the fields e are all "<u>covariant constant</u>". In particular, letting ω be the connection forms with respect to the frame e

$$\nabla e = e \otimes \omega = 0$$

and so $\omega = 0$. But then, for the dual frame σ, from (38.9),

$$d\sigma = -\omega \wedge \sigma = 0.$$

Thus $\sigma^i = dy^i$ locally, for some (coordinate) functions $y^1, \dots, y^n$.

Note, so far, we have shown that for any <u>symmetric</u> affine connection with curvature $\vartheta = 0$, one can find a frame of covariant constant vectors e

In our Riemannian case we may choose e^0 at x_0 to be o.n.; they then remain o.n. since, from (38.19),

$$d\langle e_i, e_j \rangle = \omega_{ij} + \omega_{ji} = 0.$$

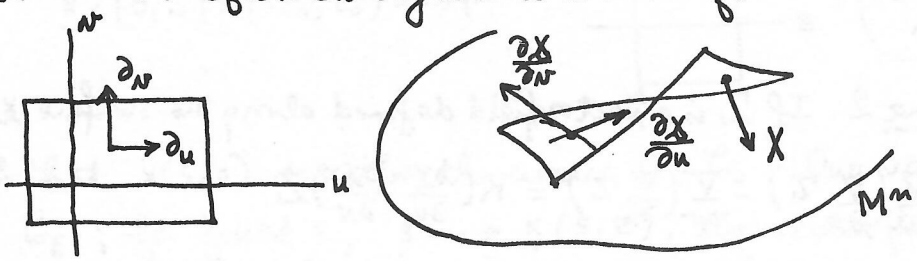
Since e is orthonormal and $e_i = \partial/\partial y^i$

$$ds^2 = (dy^1)^2 + \dots + (dy^n)^2. \qquad \square$$

Remark. e_i is obtained from e_i^0 by parallel translation, (see (40.5)) which is locally independent of path.

Lecture 41. Geodesics and Jacobi fields

(1) Vector fields along a surface in M^n. Consider a differentiable map $\underset{\sim}{x} : \mathbb{R}^2 \longrightarrow M^n$ of a rectangle into a manifold



We call this map a surface even though we make no requirements on the differential $\underset{\sim}{x}_*$; e.g. $\underset{\sim}{x}_* \partial/\partial u = \frac{\partial \underset{\sim}{x}}{\partial u}$ and $\underset{\sim}{x}_* \partial/\partial v = \frac{\partial \underset{\sim}{x}}{\partial v}$ need not be linearly independent. A map X that assigns to each (u,v) in the rectangle a tangent vector to M^n at $\underset{\sim}{x}(u,v)$ will be called a vector field along the surface $\underset{\sim}{x}$. For example $\partial \underset{\sim}{x}/\partial u$ and $\partial \underset{\sim}{x}/\partial v$ are vectors along $\underset{\sim}{x}$ that are "tangent to $\underset{\sim}{x}$".

Lemma 1. If the connection ∇ on M^n is symmetric, then

$$\frac{\nabla}{\partial u}\left(\frac{\partial \underset{\sim}{x}}{\partial v}\right) - \frac{\nabla}{\partial v}\left(\frac{\partial \underset{\sim}{x}}{\partial u}\right) = 0 \qquad (41.1)$$

Proof. Let $x^1, \dots, x^n$ be local coordinates for M^n. Then, e.g.

$$\frac{\partial \underset{\sim}{x}}{\partial v} = \frac{\partial x^i}{\partial v} \frac{\partial}{\partial x^i}.$$

Then, taking the covariant derivative of $\frac{\partial \underset{\sim}{x}}{\partial v}$ along the u curve, $x(u,v)$ for v fixed, gives

$$\frac{\nabla}{\partial u}\left(\frac{\partial x^i}{\partial v}\partial_i\right) = \frac{\partial^2 x^i}{\partial u\, \partial v}\partial_i + \frac{\partial x^i}{\partial v}\nabla_{\frac{\partial \underset{\sim}{x}}{\partial u}}(\partial_i).$$

But $\frac{\partial \underset{\sim}{x}}{\partial u} = \frac{\partial x^j}{\partial u}\partial_j$, and so

$$\nabla_{\frac{\partial \underset{\sim}{x}}{\partial u}}(\partial_i) = \frac{\partial x^j}{\partial u}\nabla_{\partial_j}\partial_i = \frac{\partial x^j}{\partial u}\Gamma^k_{ji}\partial_k, \text{ and so}$$

$$\frac{\nabla}{\partial u}\left(\frac{\partial x^i}{\partial v}\partial_i\right) = \frac{\partial^2 x^i}{\partial u\, \partial v}\partial_i + \frac{\partial x^i}{\partial v}\frac{\partial x^j}{\partial u}\Gamma^k_{ji}\partial_k$$

Subtracting $\frac{\nabla}{\partial v}\left(\frac{\partial x^j}{\partial u}\partial_j\right)$ gives 0, since $\Gamma^k_{ji} = \Gamma^k_{ij}$ □.

Note that Lemma 1 is a generalization of the trivial result $[\partial_i, \partial_j] = 0$, since $\partial x/\partial u$ and $\partial x/\partial v$ are not coordinate vector fields on M^n.

Lemma 2. If Z is a vector field defined along the surface x

$$\frac{\nabla}{\partial u}\left(\frac{\nabla}{\partial v} Z\right) - \frac{\nabla}{\partial v}\left(\frac{\nabla}{\partial u} Z\right) = R\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right) Z \tag{41.2}$$

Proof. $Z = Z^i \partial_i$ where $Z^i = Z^i(u, v)$. Then

$$\frac{\nabla}{\partial u}\frac{\nabla}{\partial v}(Z^i \partial_i) = \frac{\nabla}{\partial u}\left\{\frac{\partial Z^i}{\partial v}\partial_i + Z^i \frac{\nabla}{\partial v}\partial_i\right\}$$

$$= \frac{\partial^2 Z^i}{\partial u \partial v}\partial_i + \frac{\partial Z^i}{\partial v}\frac{\nabla}{\partial u}(\partial_i) + \frac{\partial Z^i}{\partial u}\frac{\nabla}{\partial v}\partial_i + Z^i \frac{\nabla}{\partial u}\frac{\nabla}{\partial v}\partial_i$$

and so

$$\frac{\nabla}{\partial u}\frac{\nabla}{\partial v} Z - \frac{\nabla}{\partial v}\frac{\nabla}{\partial u} Z = Z^i\left(\frac{\nabla}{\partial u}\frac{\nabla}{\partial v}\partial_i - \frac{\nabla}{\partial v}\frac{\nabla}{\partial u}\partial_i\right). \tag{41.3}$$

But $\frac{\nabla}{\partial v}\partial_i = \nabla_{\frac{\partial x^j}{\partial v}\partial_j}\partial_i = \frac{\partial x^j}{\partial v}\nabla_{\partial_j}\partial_i$, and so

$$\frac{\nabla}{\partial u}\left(\frac{\nabla}{\partial v}\partial_i\right) = \frac{\partial^2 x^j}{\partial u \partial v}\nabla_{\partial_j}\partial_i + \frac{\partial x^j}{\partial v}\frac{\partial x^k}{\partial u}\nabla_{\partial_k}\nabla_{\partial_j}(\partial_i)$$

thus

$$\frac{\nabla}{\partial u}\frac{\nabla}{\partial v}\partial_i - \frac{\nabla}{\partial v}\frac{\nabla}{\partial u}\partial_i = \left(\frac{\partial x^j}{\partial v}\frac{\partial x^k}{\partial u} - \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}\right)\nabla_{\partial_k}\nabla_{\partial_j}(\partial_i)$$

$$= \frac{\partial x^k}{\partial u}\frac{\partial x^j}{\partial v}\left(\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i\right)$$

$$= \frac{\partial x^k}{\partial u}\frac{\partial x^j}{\partial v} R(\partial_k, \partial_j)\partial_i = R\left(\frac{\partial x^k}{\partial u}\partial_k, \frac{\partial x^j}{\partial v}\partial_j\right)\partial_i$$

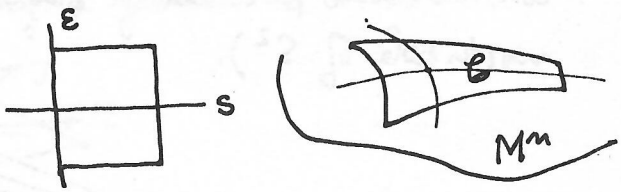
$$= R\left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right)\partial_j$$

Now plug into (41.3). □.

(2) Geodesics. We shall now return to our discussion of the variation of arc length of Lecture 35, but now we shall carry out the calculation intrinsically, i.e. without using an embedding of an M^2 into R^3 (and also doing the n dimensional version).

To "vary" a curve $\mathcal{C}$ on a manifold M^n is to consider a surface

$$\underset{\sim}{x} : [0, L] \times (-\varepsilon_0, \varepsilon_0) \longrightarrow M^n$$



such that $x(s,0)$ describes the curve $\mathcal{C} = \mathcal{C}_0$. Our varied curves are $\mathcal{C}_\varepsilon$; for fixed ε, $s \longmapsto \underset{\sim}{x}(s,\varepsilon)$. The vector field $\partial \underset{\sim}{x}/\partial \varepsilon$ defined along the surface $\underset{\sim}{x}$ is again the "variation vector field". As in Lecture 35

$$l(\varepsilon) = \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial s}, \frac{\partial \underset{\sim}{x}}{\partial s} \right\rangle^{1/2} ds.$$

We shall again require that the connection be Riemannian, i.e.

$$\frac{\partial}{\partial \varepsilon} \langle X, Y \rangle = \left\langle \frac{\nabla X}{\partial \varepsilon}, Y \right\rangle + \left\langle X, \frac{\nabla Y}{\partial \varepsilon} \right\rangle$$

for any pair of vector fields X, Y, defined along the surface of variation. Using this and Lemma (1) you should have no trouble mimicking the calculation of first variation of Lecture 35.

- Problem 72

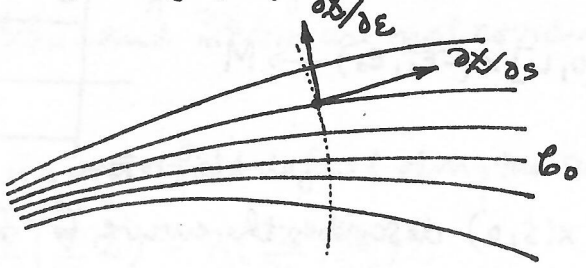
$$\frac{d}{d\varepsilon} \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial s}, \frac{\partial \underset{\sim}{x}}{\partial s} \right\rangle^{1/2} ds \Bigg|_{\varepsilon=0} = \left\langle \frac{\partial \underset{\sim}{x}}{\partial s}, \frac{\partial \underset{\sim}{x}}{\partial \varepsilon} \right\rangle \Bigg|_0^L - \int_0^L \left\langle \frac{\partial \underset{\sim}{x}}{\partial \varepsilon}, \frac{\nabla}{\partial s}\left(\frac{\partial \underset{\sim}{x}}{\partial s}\right) \right\rangle ds \qquad (41.4)$$

and so $\mathcal{C}$ is a geodesic again iff

$$\frac{\nabla}{\partial s}\left(\frac{\partial \underset{\sim}{x}}{\partial s}\right) = 0 \qquad (41.5)$$

(3) Jacobi fields. Let $\mathcal{C} = \mathcal{C}_0$ now be a geodesic on a Riemannian manifold and let us vary $\mathcal{C}_0$ by curves $\mathcal{C}_\varepsilon$ where each $\mathcal{C}_\varepsilon$ is itself a geodesic parameterized by arc length (for example

consider a 1-parameter family of great circles passing thru the north pole of S^2).



Then the variation vector field

$$J = \left.\frac{\partial x}{\partial \varepsilon}\right|_{\varepsilon=0}$$

satisfies an ordinary differential equation along $\mathcal{C}_0$ that we shall now derive.

Since each $\mathcal{C}_\varepsilon$ is a geodesic $\frac{\nabla}{\partial s}\left(\frac{\partial x}{\partial s}\right)=0$ for all ε. From (41.2) and (41.1)

$$0 = \frac{\nabla}{\partial \varepsilon}\left(\frac{\nabla}{\partial s}\left(\frac{\partial x}{\partial s}\right)\right) = \frac{\nabla}{\partial s}\left(\frac{\nabla}{\partial \varepsilon}\left(\frac{\partial x}{\partial s}\right)\right) + R\left(\frac{\partial x}{\partial \varepsilon}, \frac{\partial x}{\partial s}\right)\frac{\partial x}{\partial s}$$

$$= \frac{\nabla}{\partial s}\left(\frac{\nabla}{\partial s}\left(\frac{\partial x}{\partial \varepsilon}\right)\right) + R\left(\frac{\partial x}{\partial \varepsilon}, \frac{\partial x}{\partial s}\right)\frac{\partial x}{\partial s}.$$

Letting T be the unit tangent to $\mathcal{C} = \mathcal{C}_0$, and putting $\varepsilon = 0$

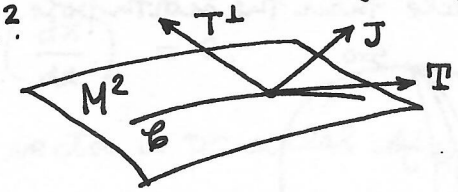
$$\boxed{\frac{\nabla^2}{\partial s^2} J + R(J,T)T = 0 \qquad \text{Jacobi's equation}} \tag{41.6}$$

Any vector field J defined along the geodesic $\mathcal{C}_0$ that satisfies Jacobi's equation will be called a <u>Jacobi field</u>. This is frequently written

$$\nabla_T \nabla_T J + R(J,T)T = 0 \tag{41.7}$$

even though J is only defined along $\mathcal{C}_0$. Note that (41.6) is a <u>linear</u> 2nd order differential equation for $J = J(s)$. This equation takes on an especially simple form when

$n=2$, ie when M^2 is itself a Riemannian surface. Consider then a Jacobi field J defined along a geodesic $\mathcal{C}$ on a Riemannian M^2.



Let T be the unit tangent along $\mathcal{C}$; $\frac{\nabla}{ds} T = 0$, i.e T is parallel displaced along $\mathcal{C}$. Let $T^\perp$ be a local tangent vector to M^2 that is a unit vector orthogonal to T; $T, T^\perp$ forms an o.n frame e_1, e_2, along $\mathcal{C}$. $T^\perp$ is also parallel displaced along $\mathcal{C}$ (why?); this is a special property for $n=2$. We expand

$$J(s) = x(s)\, T + y(s)\, T^\perp.$$

Then, from (41.6)

$$\frac{\nabla}{ds}\frac{\nabla}{ds} J(s) = \frac{d^2x}{ds^2} T + \frac{d^2y}{ds^2} T^\perp = -R(xT + yT^\perp, T)T$$

$$= -R(yT^\perp, T)T = -yR(T^\perp, T)T.$$

Then

$$\frac{d^2y}{ds^2} = -y\langle R(T^\perp, T)T, T^\perp\rangle = -y\langle R(e_2, e_1)e_1, e_2\rangle$$

Now

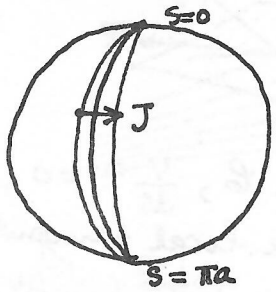
$$(R(X,Y)Z)^i = R^i{}_{jkl} X^k Y^l Z^j,$$

and from this one easily sees $\langle R(e_2,e_1)e_1, e_2\rangle = R_{2121} = R_{1212}$ (since v_{ij} is skew in i,j in an o.n frame). Hence, for the normal component y of a Jacobi field J on a surface Jacobi's equation becomes

$$\boxed{\begin{array}{c}\dfrac{d^2y}{ds^2} + Ky = 0 \\ \text{Jacobi, } n=2\end{array}} \qquad (41.8)$$

Thus, on a surface M^2 the <u>Gauss curvature</u> $K = R^{12}{}_{12}$ <u>controls the stability of a flow defined by a family of geodesics</u>. For example

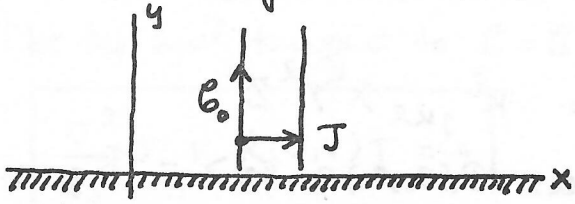
on the 2-sphere $M^2 = S^2_a$ of radius "a" in $\mathbb{R}^3$ we get a Jacobi field by rotating a great circle thru the north pole about the polar axis.



$\frac{d^2y}{ds^2} = -Ky = -\frac{1}{a^2}y$ and $y(0) = 0$ gives $y(s) = c \sin \frac{s}{a}$. This Jacobi field vanishes again at $s = \pi a$, ie at the south pole; this of course is a reflection of the fact that all the geodesics starting at the north pole pass thru the south pole.

Note that on a surface M^2 with $K \leq 0$, a Jacobi field that vanishes at $s=0$ will not vanish again unless the Jacobi field is identically the 0 field; in fact Jacobi fields in negative curvature continue to grow. For example

Problem 73. Consider again the Poincaré upper-half plane (see Prob. 71)



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}$$

Show that the vertical lines are geodesics, $\frac{\nabla e_2}{ds} = 0$ by using Cartan's moving frames, ie $\frac{\nabla e_2}{ds} = e_1 \omega^1_2(e_2)$. $J = \frac{\partial}{\partial x}$ is then a Jacobi field along such a e_0. Verify that Jacobi's equation (41.8) is indeed satisfied. Note that as $y \to 0$ $\|J\| \longrightarrow \infty$.

Lecture 42. Geodesics, Spiders and the Universe

(1) **Geodesics and Gaussian Coordinates.** We shall redefine a geodesic in a Riemannian manifold to be any curve $x = \gamma(t)$ such that

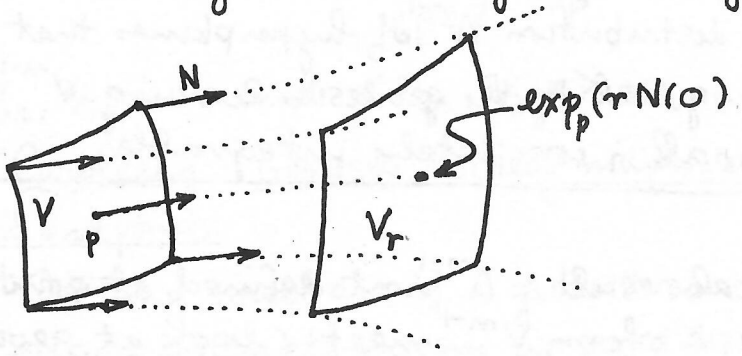
$$\frac{\nabla}{dt}\left(\frac{dx}{dt}\right) = 0, \tag{42.1}$$

ie its tangent vector is parallel displaced. Then $\frac{d}{dt}\langle \frac{dx}{dt}, \frac{dx}{dt}\rangle = 0$ shows that $\|\frac{dx}{dt}\|$ = constant along γ; this means that γ must be parameterized by a parameter t that is proportional to arc length s (where $\|\frac{dx}{dt}\| = 1$) except for an additive constant. There is a common, but strange, notation for geodesics. If v is a vector at $p \in M$ we let

$$\gamma(t) = \exp_p(tv)$$

denote the unique geodesic parameterized by t proportional to arc length such that $\gamma(0) = p$ and $d\gamma/dt\,(0) = v_p$. In particular, if v is a unit vector then t is arc length along γ. Since geodesics need not be defined for all t, $\exp_p(tv)$ might make sense for $|t| <$ some small number ε.

Given a point $p \in M^n$ and a hypersurface V^{n-1} passing thru p we can set up a system of coordinates for M^n near p as follows. Let $y^2, \ldots, y^n$ be an arbitary local coordinate system for V^{n-1} near p with p as origin. Let $N(y)$ be a unit normal field to V^{n-1} near p. If from each $y \in V$ we construct the geodesic thru y with tangent $N(y)$, and



if we travel along this geodesic for distance $|r|$, we shall get, if ε is small enough, a map

$$(-\varepsilon, \varepsilon) \times V^{n-1} \longrightarrow M^n$$
$$(r, y) \longrightarrow \exp_y(r\,N(y))$$

and it can be shown that this map is a diffeomorphism onto an open subset of M^n if V^{n-1} is small enough. (For a careful treatment of such matters see e.g. Milnors book "Morse Theory", Chapter II). This says in particular that any point $q \in M^n$ that is sufficiently close to p will lie on a unique geodesic of length $< \varepsilon$ that starts on V^{n-1} orthogonally to V^{n-1}. If $q = \exp_y(r\,N(y))$ we assign to q the "Gaussian coordinates"

$$(r, y^2, \cdots, y^n).$$

We can look at the hypersurface V^{n-1}_r of all points $\exp_y(r\,N(y))$ as y runs thru V^{n-1}, for fixed r; this is the "parallel hypersurface to V at distance r"

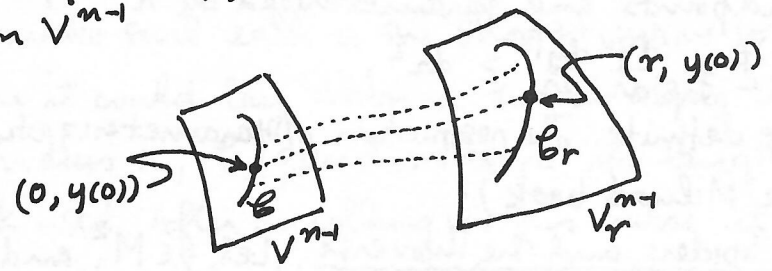
Gauss' Lemma The parallel hypersurface V^{n-1}_r to V is itself orthogonal to the geodesics leaving V orthogonal to V. (42.2)

Put another way

The distribution Δ^{n-1} of hyperplanes that are orthogonal to the geodesics leaving V^{n-1} orthogonally is completely integrable. (42.3)

This is a local result; Δ^{n-1} isn't defined at points where distinct geodesics from V^{n-1} meet (look at geodesics leaving the equator $V^1 \subset S^2$!).

Proof. Consider a curve $\mathcal{C}_r$; $y = y(\alpha)$ on V_r^{n-1} passing thru $r, y(0)$. This curve is the image of a curve $\mathcal{C}$ on V^{n-1}



Then the curves $\gamma_\alpha : s \longmapsto \exp_{0,y(\alpha)} (s\, N(0, y(\alpha))$ give a variation of the geodesic γ_0 by geodesics γ_α, and each geodesic has the same length r. From (41.4)

$$0 = l'(0) = \left\langle \frac{\partial x}{\partial s}, \frac{\partial x}{\partial \alpha} \right\rangle \Big|_{s=0}^{s=r}.$$

But $\left.\frac{\partial x}{\partial s}\right|_{s=0,\, y=y(0)} = N(0, y(0))$ is orthogonal to $\left.\frac{\partial x}{\partial \alpha}\right|_{s=0}$ which is tangent to V^{n-1}; hence $\frac{\partial x}{\partial \alpha}$ is orthogonal to the geodesic γ_0 at $s = r$. Thus the geodesic γ_0 is orthogonal to all curves on V_r^{n-1} passing thru $(r, y(0))$. □.

Corollary. In Gaussian coordinates $r, y^2, \dots y^n$ for M^n we have

$$ds^2 = dr^2 + \sum_{\alpha,\beta=2}^{n} g_{\alpha\beta}(r, y)\, dy^\alpha dy^\beta \qquad (42.4)$$

since $\langle \partial/\partial r, \partial/\partial r \rangle = 1$ and $\langle \partial/\partial r, \partial/\partial y^\alpha \rangle = 0$.

Corollary. Geodesics locally minimize arc length for fixed endpoints.

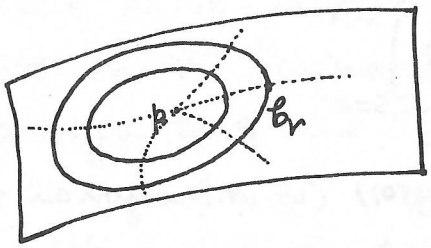
For any small arc of a geodesic can be embedded in a Gaussian coordinate system as one of the curves y = constant.

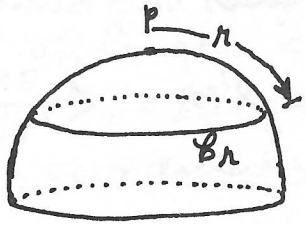
Then for any other curve lying in the Gaussian coordinate patch, joining the same endpoints and parameterized by r

$$ds^2 = dr^2 + \sum g_{\alpha\beta} \frac{dy^\alpha}{dr} \frac{dy^\beta}{dr} \geq dr^2,$$

since $(g_{\alpha\beta})$ is positive definite. The restriction of parameterization by r can be removed (see Milnor's book).

(2) Gaussian Curvature, Spiders, and the Universe. Let $p \in M^2$, and consider all geodesics on the surface starting at p. If on each geodesic we mark the point whose distance from p is r we obtain a "geodesic circle" $\mathcal{C}_r$.





One can show, by introducing coordinates analogous to "polar coordinates" in the plane, that $\mathcal{C}_r$ has a length given by

$$\ell(\mathcal{C}_r) = 2\pi r - 2\pi K(p) \frac{r^3}{3!} + \cdots$$

where $K(p)$ is the Gaussian curvature of M^2 at p. Consequently

$$\boxed{K(p) = \lim_{r \to 0} \frac{3}{\pi r^3} \left(2\pi r - \ell(\mathcal{C}_r)\right)} \qquad (42.5)$$

i.e, the Gauss curvature K at p is a measure, in the limit, of how the length of a geodesic circle of radius r differs from the length of a Euclidean circle of radius r. Here again we see Gauss' theorema egregium, since this formula exhibits K as an invariant under isometries.

A spider living on M^2 could mark off geodesic segments of length r by laying down its silk thread and experimenting to make sure each is the shortest curve joining its endpoints. Then it could lay down a thread along the "geodesic circle" of radius r, C_r, and measure its length by the amount of silk used. After enrolling in this course it could compute $K(p)$, approximately, by using (42.5). All this without any awareness of the enveloping $\mathbb{R}^3$.

What about us? We live in a 3 dimensional space, or a 4 dimensional space-time. To measure spatial distances we can use light rays, reflected by mirrors, measuring time with our atomic clocks. A similar construction also yields ds^2 for time-like intervals (see [Fr], p. 10). We seem to have an innate feeling for parallelism and parallel transport; in other words, <u>our world seems to be equipped with an affine connection</u>!

Where does our affine connection come from? It is a consequence of <u>the gravitational attraction of distant matter in the universe</u>; if we move so that our tangent vector is not parallel displaced we feel strange "forces" acting upon us. Since we seem to have a parallelism that is independent of path, our curvature tensor must be close to zero in some sense; our space is almost flat or Euclidean. But is it really zero? What is the curvature tensor of our space. Can we compute it by some simple experiment as the spider can on an M^2?

Einstein was the first to describe analytically <u>our affine connection as a physical field</u>, a "gauge field" as it is called today. He was the first to show that our Universe is <u>not flat</u>. We turn to these matters in our next lecture.

Lecture 43. Heuristics of Einstein's Gravitation

(1) The Metric Potentials. Consider space-time M^4; this should be some generalization of Minkowski space, with local coordinates $x^0 = t, x^1, x^2, x^3$. Einstein felt that a small test particle that moved only under the influence of gravitation should trace out a geodesic world line in M^4. This was a reflection of Einstein's belief that a body in free fall did not feel a "force of gravity" since gravitational forces were, in his eyes, fictional forces (like centrifugal force). The metric of M^4 should be some generalization of the Minkowski metric

$$ds^2 = g_{00}\,dt^2 + 2g_{0\beta}\,dt\,dx^\beta + g_{\alpha\beta}\,dx^\alpha dx^\beta \qquad (43.1)$$

where Greek indices now sum from 1 to 3, g_{00} must be negative, and where we are using units so that $c = 1$. Assume the simplest possible situation where the metric is time independent, ie g_{ij} are functions only of x^1, x^2, x^3. Such a metric is called "stationary". We shall further assume that the metric is static, ie

$$g_{0\beta} \equiv 0. \qquad (43.2)$$

(see [Fr], Chapter 3 for a discussion of these assumptions). This is supposed to correspond to a universe with no motion taking place

$$ds^2 = g_{00}(x)\,dt^2 + g_{\alpha\beta}(x)\,dx^\alpha dx^\beta \qquad (43.3)$$

We assume that our test particle is so small that it has a completely negligible effect on the "background metric" (43.3). Along its world line we again introduce "proper time" τ as parameter, the time kept by an atomic clock moving with the particle

$$d\tau^2 = -\,ds^2 \qquad (43.4)$$

$$\left(\frac{d\tau}{dt}\right)^2 = -g_{00} - g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}. \tag{43.5}$$

We shall assume that the particle is moving very slowly compared to light; consequently its unit velocity 4 vector is

$$T = \frac{dx}{d\tau} = \frac{dt}{d\tau}\begin{bmatrix} 1 \\ \frac{dx}{dt} \end{bmatrix} \sim \frac{dt}{d\tau}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim \frac{1}{\sqrt{-g_{00}}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We shall assume that the particle is moving in a very weak gravitational field so that M^4 is almost Minkowskian

$$g_{00} \sim -1.$$

(we shall not, however put derivatives of g_{00} equal to zero).

The particle is assumed to trace out a geodesic in M^4 then since $d\tau \sim dt$ (35.6) yields

$$\frac{d^2x^i}{dt^2} \sim \frac{d^2x^i}{d\tau^2} \sim -\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \sim -\Gamma^i_{00}.$$

In particular, for $\alpha = 1,2,3$, using $=$ instead of $\sim$

$$\frac{d^2x^\alpha}{dt^2} = -\Gamma^\alpha_{00} = -\frac{1}{2}g^{\alpha j}\left(\frac{\partial g_{0j}}{\partial x^0} + \frac{\partial g_{0j}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^j}\right)$$

$$= \frac{1}{2}g^{\alpha j}\frac{\partial g_{00}}{\partial x^j} \quad \left(\text{since we assume } \frac{\partial g_{ij}}{\partial t} = 0\right)$$

$$= \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} \quad (\text{since } g^{\alpha 0} = 0).$$

Consequently

$$\frac{dx^\alpha}{dt} = \left(\text{grad}\ \tfrac{1}{2}g_{00}\right)^\alpha. \tag{43.6}$$

If now we let φ be the <u>Newtonian</u> gravitational potential we must compare (43.6) with

$$\frac{dx^\alpha}{dt} = (\text{grad}\,\varphi)^\alpha, \tag{43.7}$$

yielding $\frac{1}{2}g_{00} \sim \varphi + \text{constant}$. If we assume that

$\varphi \longrightarrow 0$ "at infinity", in some sense, and if we assume that M^4 basically tends to the Minkowski metric "at infinity", ie

$$g_{00} \longrightarrow -1$$

Then we see that

$$g_{00} \sim 2\varphi - 1 \qquad (43.8)$$

Thus in this highly idealized case (which would correspond to a small planet, say, in the neighborhood of the sun) g_{00} plays a role similar to the Newtonian gravitational potential. But then what about the other metric coefficients? Surely they must play a role. We are led, with Einstein, to conclude that

(i) Newtonian gravitation is governed by a single potential φ, ie Newtonian gravitation is a scalar theory.

(ii) Electromagnetism is governed by a 4 vector potential

$$A = \varphi\, dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

(see Lecture 30, § 1). E&M is a vector theory.

(iii) Einsteins gravitation is governed by the 10 "metric potentials"

$$g_{ij}, \quad i,j = 0,1,2,3$$

i.e gravitation is a tensor theory (covariant, 2nd order, symmetric).

In (i) the potential φ satisfies a field equation

$$\nabla^2 \varphi = -4\pi K \rho \qquad 43.9$$

Poisson's equation.

Here ρ is the density of matter, K is Newton's gravitational constant.

In (ii) A satisfies a field equation of the form (see [Fr], p 127)

$$\Box A = 4\pi J.$$

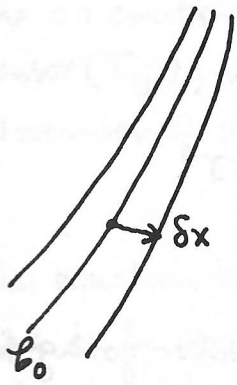
What are the field equations satisfied by the (g_{ij})?

(2) Einstein's Equations. Instead of considering a single particle, consider now a dust cloud of particles, having density ρ. By dust we mean that considered as a fluid the pressure vanishes identically, $p \equiv 0$; lack of a pressure gradient ensures us that the individual particles are falling freely under the influence of gravity alone. Each particle traces out a geodesic world line in M^4. We shall not restrict ourselves to static metrics

$$ds^2 = g_{00}\,dt^2 + 2\,g_{0\beta}\,dt\,dx^\beta + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

and the coefficients g_{ij} can depend on t also.

What is the Newtonian picture of the motion of the dust cloud. Follow the path of a particular particle and let



δx be the "variation vector" joining the base particle with a nearby one. From Newton's law (43.7) we get

$$\delta\,\frac{d^2x^\alpha}{dt^2} = \frac{d^2}{dt^2}\,\delta x$$

$$\delta\,\frac{\partial \varphi}{\partial x^\alpha} = \frac{\partial^2 \varphi}{\partial x^\beta \partial x^\alpha}\,\delta x^\beta$$

in Cartesian Coordinates. Thus the "equations of variation" are

$$\frac{d^2}{dt^2}(\delta x^\alpha) = \frac{\partial^2 \varphi}{\partial x^\beta \partial x^\alpha}(\delta x^\beta) \qquad (43.10)$$

a linear equation for δx along $\mathcal{C}_0$.

Now look at this same physical situation, but observed from our 4 dimensional space-time M^4. T is again the unit 4-velocity vector and J is the variational 4-vector. Since the

world lines are all geodesics, J is a Jacobi field

$$\frac{\nabla^2}{dt} J + R(J,T)T = 0 \quad , \text{ i.e}$$

$$\frac{\nabla^2 J}{dt^2} = - R(J,T)T \tag{43.11}$$

Again with our assumptions that the spatial velocities are small and the gravitational field is weak

$$T \sim \begin{bmatrix} 1 \\ \underset{\sim}{0} \end{bmatrix} , \quad g_{00} \sim -1$$

Jacobi's equation gives

$$\frac{\nabla^2 J^\alpha}{dt^2} \sim - R^\alpha{}_{ijk} J^j T^k T^i \sim - R^\alpha{}_{0j0} J^j$$

$$\sim - R^\alpha{}_{0\beta 0} J^\beta \tag{43.12}$$

since J has essentially no t component; J is a vector going from the base particle to a nearby particle "at the same time", in some relative sense. If we put $J^\beta = \delta x^\beta$ and compare (43.12) with (43.10) we see

$$- R^\alpha{}_{0\beta 0} \sim \frac{\partial^2 \varphi}{\partial x^\beta \partial x^\alpha} . \tag{43.13}$$

Consequently, since $R^j{}_{k00} = 0$

$$\nabla^2 \varphi = \sum_{\alpha=1}^{3} \frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\alpha} \sim \sum_{\alpha=1}^{3} - R^\alpha{}_{0\alpha 0} \sim - \sum_{i=0}^{3} R^i{}_{0i0} .$$

Now for any affinely connected manifold we define the <u>Ricci tensor</u>

$$R_{jk} \equiv R^i{}_{jik} \tag{43.14}$$

and so finally we have

$$\nabla^2 \varphi = -R_{00}.$$

Poisson's equation (43.9) then yields

$$\boxed{\nabla^2 \varphi = -4\pi\kappa\rho = -R_{00} \quad \text{dust in a weak field}} \tag{43.15}$$

We see from this simplest of all possible cases that

$$\boxed{\text{Space-time } M^4 \text{ must be curved in the presence of matter}} \tag{43.16}$$

It is known (see e.g. [Fr], Chap. 3) that the density ρ is essentially the 00 component of a physical tensor, the <u>stress-energy-momentum</u> tensor (T_{ij}); we are then tempted to write, as Einstein did in early November of 1915, as general field equations

$$R_{ij} = 4\pi\kappa T_{ij}.$$

This equation turns out to be incorrect; just as Ampere's equation had to be amended by Maxwell's displacement current $\partial D/\partial t$ in order to yield conservation of charge, so these equations were amended at the end of November 1915 to read

$$\boxed{R_{ij} - \tfrac{1}{2} g_{ij} R = 8\pi\kappa T_{ij} \quad \text{Einstein equations}} \tag{43.17}$$

Here

$$R \equiv R^i{}_i = g^{ik} R_{ki} \tag{43.18}$$

the trace of the Ricci tensor is the "<u>scalar curvature</u>". (43.17) is a system of second order partial differential equations for the g_{ij} in terms of T_{ij}.

Lecture 44. Tensor Analysis.

(1) **Covariant Differentiation of Tensors.** We have defined, in Lecture 37, §1 the covariant differential of the contravariant vector field $v = v^i \partial_i$ to be the mixed tensor with components

$$\nabla_j v^i = v^i{}_{|j} = \frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{jk} \tag{44.1}$$

while the covariant differential of the covector field $a_i\, dx^i$ is the covariant tensor with components

$$\nabla_j a_k = a_{k|j} = \frac{\partial a_k}{\partial x^j} - a_i \Gamma^i_{jk} \tag{44.2}$$

For a function f we have defined, for convenience

$$\nabla_j f = f_{|j} = \frac{\partial f}{\partial x^j} \tag{44.3}$$

Covariant differentiation will always increase the covariance of a tensor by 1.

In Lecture 37 we also defined Cartan's exterior covariant differential which took vector valued forms into vector valued forms using skew symmetrization; we shall not be concerned with exterior covariant differentiation in this lecture.

We generalize (44.1) and (44.2), defining the covariant differential of a general tensor T by

$$\begin{aligned} \nabla_j\, T^{i_1 \cdots i_p}{}_{k_1 \cdots k_r} &\equiv T^{i_1 \cdots i_p}{}_{k_1 \cdots k_r |j} \\ &\equiv \frac{\partial T^{i_1 \cdots i_p}{}_{k_1 \cdots k_r}}{\partial x^j} + T^{\ell i_2 \cdots i_p}{}_{k_1 \cdots k_r} \Gamma^{i_1}_{j\ell} + T^{i_1 \ell i_3 \cdots i_p}{}_{k_1 \cdots k_r} \Gamma^{i_2}_{j\ell} + \cdots \\ &\quad - T^{i_1 \cdots i_p}{}_{\ell k_2 \cdots k_r} \Gamma^{\ell}_{j k_1} - T^{i_1 \cdots i_p}{}_{k_1 \ell k_3 \cdots k_r} \Gamma^{\ell}_{j k_2} - \cdots \end{aligned} \tag{44.4}$$

We state two important facts about this definition of covariant differentiation. First recall that the tensor product of tensors $S^{a_1 a_2 \cdots}{}_{b_1 b_2 \cdots}$ and $T^{i_1 i_2 \cdots}{}_{k_1 k_2 \cdots}$ is the tensor with components

$$S^{a_1 a_2 \cdots}{}_{b_1 b_2 \cdots} T^{i_1 i_2 \cdots}{}_{k_1 k_2 \cdots}$$

Our first fact is that <u>covariant differentiation obeys a product rule</u>

$$(S^{\cdots}{}_{\cdots} T^{\cdots}{}_{\cdots})_{|j} = (S^{\cdots}{}_{\cdots|j} T^{\cdots}{}_{\cdots}) + (S^{\cdots}{}_{\cdots} T^{\cdots}{}_{\cdots|j}) \qquad (44.5)$$

Secondly <u>covariant differentiation commutes with contractions</u> For example, the covariant derivative of a mixed tensor $T^j{}_k$ is

$$T^j{}_{k|\ell} = \frac{\partial T^j{}_k}{\partial x^\ell} + T^m{}_k \Gamma_{\ell m}^{\ j} - T^j{}_m \Gamma_{\ell k}^{\ m}$$

which is a third rank tensor. Contract j, k and get a 1st rank tensor, ie a covector

$$T^j{}_{j|\ell} = \frac{\partial T^j{}_j}{\partial x^\ell} + T^m{}_j \Gamma_{\ell m}^{\ j} - T^j{}_m \Gamma_{\ell j}^{\ m}$$

$$= \frac{\partial T^j{}_j}{\partial x^\ell}$$

On the other hand we can first contract j, k in $T^j{}_k$ to get the 0^{th} rank tensor, ie <u>scalar</u> $T^j{}_j$. Its covariant derivative is

$$(T^j{}_j)_{|\ell} = \frac{\partial T^j{}_j}{\partial x^\ell}.$$

<u>Warning</u>. The covariant derivative of a tensor with constant components in some coordinate system need not be zero; there are Γ terms that won't vanish.

● <u>Problem 74</u>. Show that the identity tensor $\delta^i{}_j$ has $\delta^i{}_{j|k} = 0$.

- Problem 75. Show that the Codazzi equations (34.4) say

$$b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \qquad (44.6)$$

(2) Riemannian Connections & Bianchi Identities. The Riemannian or Levi-Civita connection satisfies 36.6

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$\frac{d}{d\varepsilon}\langle Y,Z\rangle$ i.e $= \langle \frac{\nabla}{d\varepsilon} Y, Z\rangle + \langle Y, \frac{\nabla}{d\varepsilon} Z\rangle$

$\nabla_X \langle Y,Z\rangle = d\langle Y,Z\rangle(X) = X(\langle Y,Z\rangle)$

$\nabla Y(X)$ = vector (36.8, 36 ∇Y = linear transformation)

$\nabla_X Y = \nabla Y(X)$

$$(g_{ij} Y^i Z^j)_{|k} X^k = g_{ij} Y^i_{|k} X^k Z^j + g_{ij} Y^i Z^j_{|k} X^k$$

But from the product rule (44.5) $X(\langle Y,Z\rangle) = X^i \frac{\partial}{\partial x^i}\langle Y,Z\rangle$

$$(g_{ij} Y^i Z^j)_{|k} = g_{ij|k} Y^i Z^j + g_{ij} Y^i_{|k} Z^j + g_{ij} Y^i Z^j_{|k}$$ $\frac{\nabla}{\partial x^i} = \nabla_i$

Comparing, since these are true for all X, Y, Z, we get

$$\boxed{g_{ij|k} = \nabla_k g_{ij} = 0} \; ; \qquad (44.7).$$

we say that the metric tensor is "covariant constant".

- Problem 76 Use $g^{ij} g_{jk} = \delta^i_k$ to show that

$$g^{ij}_{\ \ |k} = 0$$

Define the divergence of a symmetric contravariant tensor $T^{i_1 \cdots i_p}$ to be the symmetric $(p-1)$ tensor

$$(\mathrm{Div}\, T)^{j_1 \cdots j_{p-1}} \equiv T^{i\, j_1 \cdots j_{p-1}}_{\qquad\quad\ \ |i}$$

- Problem 77. Show that for a surface M^2 with mean curvature H

$$\mathrm{grad}\, H = \mathrm{Div}(b)$$

We now derive two important identities involving the Riemann tensor. We shall use (38.13) and (38.9); note that since we are dealing with a Riemannian connection, the connection is symmetric. ↳ 36.6 38.17

First

$$\begin{aligned}\vartheta\wedge\sigma &= (d\omega+\omega\wedge\omega)\wedge\sigma = d\omega\wedge\sigma+\omega\wedge(\omega\wedge\sigma)\\ &= d\omega\wedge\sigma-\omega\wedge d\sigma = d(\omega\wedge\sigma)\\ &= -d(d\sigma) = 0.\end{aligned}$$

Consequently $\vartheta^i{}_j\wedge\sigma^j=0$, all i, i.e.

$$\tfrac{1}{2}R^i{}_{jkl}\,\sigma^k\wedge\sigma^l\wedge\sigma^j=0.$$

But this means that $R^i{}_{jkl}$ when skew symmetrized with respect to jkl must be 0. Since $R^i{}_{jkl}$ is already skew in kl we conclude

$$\boxed{R^i{}_{jkl}+R^i{}_{ljk}+R^i{}_{klj}=0} \qquad (44.8)$$

1st Bianchi identity, ∇ symmetric

$\vartheta\wedge\sigma=0$

Next,

$$\begin{aligned}d\vartheta &= d(d\omega+\omega\wedge\omega) = d(\omega\wedge\omega)\\ &= d\omega\wedge\omega-\omega\wedge d\omega\\ &= (\vartheta-\omega\wedge\omega)\wedge\omega-\omega\wedge(\vartheta-\omega\wedge\omega)\\ &= \vartheta\wedge\omega-\omega\wedge\vartheta\end{aligned}$$

or $$d\vartheta+\omega\wedge\vartheta-\vartheta\wedge\omega=0$$

thus $$d\vartheta^i{}_j+\omega^i{}_r\wedge\vartheta^r{}_j-\vartheta^i{}_r\wedge\omega^r{}_j=0$$

this becomes

$$\left(\frac{\partial R^i{}_{jkl}}{\partial x^m}+\Gamma^i_{mr}R^r{}_{jkl}-\Gamma^r_{mj}R^i{}_{rkl}\right)dx^m\wedge dx^k\wedge dx^l=0$$

Γ symmetric

But $(R^i_{j\,rl}\,\Gamma^r_{mk} + R^i_{j\,kr}\,\Gamma^r_{ml})\,dx^m \wedge dx^k \wedge dx^l = 0$ (why?)
and so we may subtract this from our above expression to get

$$R^i_{jkl|m}\,dx^m \wedge dx^k \wedge dx^l = 0$$

As before we conclude

$$\boxed{\begin{array}{c} R^i_{jkl|m} + R^i_{jmk|l} + R^i_{jlm|k} = 0 \\ 2^{nd}\ \text{Bianchi identity} \\ \mathcal{D}\vartheta \equiv d\vartheta + \omega \wedge \vartheta - \vartheta \wedge \omega = 0 \end{array}} \qquad (44.9)$$

(3) Second Derivatives; the Ricci Identities. The covariant derivative of a vector field Z is the mixed tensor with components $Z^i_{|j}$. The covariant derivative of this mixed tensor is the third rank tensor with components $Z^i_{|j|k}$, which is traditionally written

$$Z^i_{|jk} \equiv Z^i_{|j|k}.$$

We wish to evaluate $Z^i_{|jk} - Z^i_{|kj}$. We prefer to do this as follows. Let X and Y be vectors at a given point. Extend them also to be vector fields. We have $Z = Z^i \partial_i$, etc. Then

$$\nabla_X(\nabla_Y Z) = \nabla_X(Z^i_{|j} Y^j \partial_i) = (Z^i_{|j} Y^j)_{|k} X^k \partial_i$$

$$= (Z^i_{|jk} Y^j + Z^i_{|j} Y^j_{|k}) X^k \partial_i$$

and $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \left[(Z^i_{|jk} - Z^i_{|kj}) Y^j X^k + Z^i_{|j}(X^k Y^j_{|k} - Y^k X^j_{|k})\right]\partial_i$.

$$= (Z^i_{|jk} - Z^i_{|kj}) Y^j X^k \partial_i + \nabla_{[X,Y]} Z$$

But $[\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z = R(X,Y)Z = R^i_{l\,kj} X^k Y^j Z^l \partial_i$

We conclude

$$\boxed{Z^i_{\ljk} - Z^i_{\lkj} = Z^l R^i_{\ l kj}}$$

or

$$\boxed{\nabla_k \nabla_j Z^i - \nabla_j \nabla_k Z^i = Z^l R^i_{\ l kj}} \qquad (44.10)$$

Ricci's identity

- Problem 78. Use (44.9) to show (using several contractions) that

$$\boxed{\frac{\partial R}{\partial x^l} = 2 R^j_{\ l|j}} \qquad (44.11)$$

where, recall, $R = R^j_{\ j}$ is the scalar curvature.

Note then that (44.11) says $(\text{Div Ricci})_l = \frac{1}{2} \partial R / \partial x^l$. Einstein's equations (43.17) can be written

$$R^i_{\ j} - \tfrac{1}{2} \delta^i_{\ j} R = 8\pi\kappa T^i_{\ j}$$

and so, taking Div of both sides

$$R^i_{\ j|i} - \tfrac{1}{2} \delta^i_{\ j} R_{|i} = 8\pi\kappa T^i_{\ j|i}.$$

But (44.11) gives

$$\left(R^i_{\ j} - \tfrac{1}{2} \delta^i_{\ j} R\right)_{|i} = 0$$

and then $T^i_{\ j|i} = 0$, i.e $\text{Div } T = 0$. This is the physical reason for subtracting the term $\frac{1}{2} g_{ij} R$ from R_{ij} in Einstein's field equations.

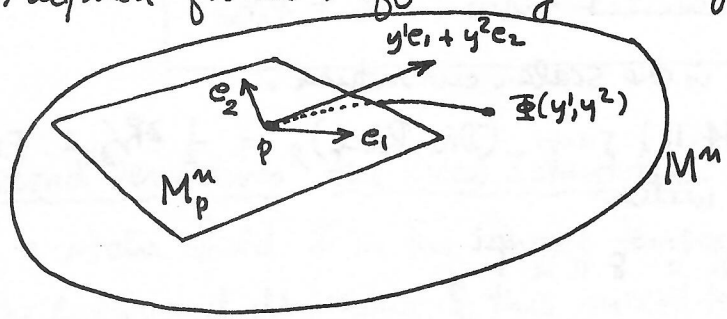
Lecture 45. Hilbert's Action Principle

(1) Normal Coordinates, Divergence & Laplacian. Let $p \in M^n$ be a given point in a (pseudo-) Riemannian manifold. Let $e_1, \dots, e_n$ be an o.n frame at p (if M is pseudo-Riemannian with indefinite metric of type $(-1, +1, \dots, +1)$ we mean by o.n that $\langle e_i, e_j \rangle = \eta_{ij} =$ diagonal $(-1, +1, \dots, +1)$.). The map

$$(y^1, y^2, \dots, y^n) \xrightarrow{\Phi} \exp_p (y^1 e_1 + \cdots + y^n e_n) \tag{45.1}$$

$$\mathbb{R}^n = M^n_p \longrightarrow M^n$$

is defined for all sufficiently small y, $(y^1)^2 + \cdots + (y^n)^2 < \varepsilon$.



See Lect 42, § 1 for notation $\exp_p$.

For differential Φ_* at $(0, \dots, 0)$

$$\Phi_* \left(\frac{\partial}{\partial y^\alpha} \right) \Big|_{y=0} = \frac{\partial}{\partial y^\alpha} \exp_p (y^\alpha e_\alpha \text{ no sum}) \Big|_{y=0} = e_\alpha$$

Since the geodesic generated by e_α has tangent vector e_α at p. Thus Φ_* is an isomorphism and so Φ must be a diffeomorphism of some open ball $\subset M^n_p$ onto a nbhd of p in M^n. This means that we may use $y^1, \dots, y^n$ as a local coordinate system in some nbhd of p (i.e, any point q near p lies on a unique small geodesic from p, its "distance" from p being

$$\left(\sum_{\alpha=1}^n (y^\alpha)^2 \right)^{1/2}.$$

The coordinates $y^1, \dots, y^n$ are "geodesic normal coordinates" with "pole" $p \in M^n$. By construction

$$g_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle \Big|_p = \langle e_i, e_j \rangle.$$

that is, the y coordinate system is o.n at the pole p. It should be clear that the geodesic thru p with tangent vector

$$\lambda^1 e_1 + \cdots + \lambda^n e_n$$

there (i.e. $\lambda^1, \ldots, \lambda^n$ are the "direction cosines" of the unit tangent) has equations given by the linear equations

$$y^i(s) = \lambda^i s \qquad i = 1, \ldots, n.$$

But the geodesics satisfy $\frac{d^2 y^i}{ds^2} + \Gamma^i_{jk}(y(s)) \frac{dy^j}{ds} \frac{dy^k}{ds} = 0$, and so

$$\Gamma^i_{jk}(\lambda s)\lambda^j \lambda^k = 0 \quad .$$

In particular, this is true at p, ie at $s=0$. Furthermore, at p this is true for all choices of $\lambda^1, \ldots, \lambda^n$

$$\Gamma^i_{jk}(p)\lambda^j \lambda^k = 0 \quad , \text{ all } \lambda.$$

Since Γ^i_{jk} is symmetric in j, k, we conclude that at the pole p of geodesic normal coordinates y we have

$$\Gamma^i_{jk}(p) = 0 \quad \text{all } i, j, k. \tag{45.2}$$

We also have at the pole p

$$0 = g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - g_{rj}\Gamma^r_{ki} - g_{ir}\Gamma^r_{kj} = \frac{\partial g_{ij}}{\partial x^k}$$

and so all first partial derivatives of the metric tensor g_{ij} vanish at the pole of geodesic normal coordinates

$$\frac{\partial g_{ij}}{\partial x^k}(p) = 0 \tag{45.3}$$

As an application of the use of these coordinates, consider the divergence of a vector field v. We have defined div v in (18.12) to be

$$\text{div } v = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i}\left(\sqrt{g}\, v^i\right).$$

Now at the pole p of geodesic normal coordinates we clearly

have

$$\frac{\partial}{\partial y^i}\sqrt{g}\,\Big|_p = 0$$

and so at the pole p

$$\operatorname{div} v = \frac{\partial}{\partial y^i} v^i$$

But consider now the scalar $v^i{}_{/i}$. At the pole

$$v^i{}_{/i} = \frac{\partial v^i}{\partial y^i} + v^j \Gamma^i_{ij} \qquad (45.4)$$
$$= \frac{\partial v^i}{\partial y^i}.$$

We conclude that at the pole $\operatorname{div} v = v^i{}_{/i}$. But $\operatorname{div} v$ and $v^i{}_{/i}$ are well defined scalars, <u>independent of coordinates</u>; we conclude that in <u>any</u> coordinate system

$$\boxed{\operatorname{div} v = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, v^i) = v^i{}_{/i}} \qquad (45.5)$$

This in turn means

$$\frac{\partial v^i}{\partial x^i} + v^i \frac{\partial}{\partial x^i} \log\sqrt{g} = \frac{\partial v^i}{\partial x^i} + v^i \Gamma^k_{ki}$$

i.e

$$\frac{\partial}{\partial x^i} \log\sqrt{g} = \Gamma^k_{ki} \qquad (45.6)$$

a <u>very</u> useful expression!

We have already defined the gradient of a function f to be the contravariant vector

$$(\operatorname{grad} f)^i = g^{ij} \frac{\partial f}{\partial x^j} = g^{ij} f_{/j} \qquad (45.7)$$

and the <u>Laplacian</u> of f is

$$\Delta f \equiv \nabla^2 f \equiv \operatorname{div}\operatorname{grad} f = (g^{ij} f_{/j})_{/i}$$

$$\boxed{\nabla^2 f = g^{ij} f_{/ji}} \qquad (45.8)$$

Thus

$$\nabla^2 f = g^{ij}\left[\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \Gamma_{ij}^{k}\frac{\partial f}{\partial x^k}\right]$$

$$= g^{ij}\left[\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^{k}\frac{\partial f}{\partial x^k}\right] \qquad (45.9)$$

• Problem 79. Consider a surface $M^2 \subset R^3$. R^3 has cartesian coordinates x^1, x^2, x^3 while M^2 has local coordinates u^1, u^2. For each fixed i, x^i is a function on M^2. Show that the surface Laplacian of x^i (using the induced Riemannian metric $g_{\alpha\beta}(u)$ for M^2) is given by

$$\nabla^2 x^i = H N^i \qquad (45.10)$$

where N^i is the i^{th} component of the unit normal and H is the mean curvature.

> In particular, M^2 is a minimal surface iff each coordinate function x^1, x^2, x^3 is a "harmonic" function on M^2
>
> $$\nabla_M^2 x^i = 0 \qquad i = 1, 2, 3$$
>
> where the Laplace operator is the surface Laplacian

(2) Hilbert's Variational Approach to General Relativity.

Let R be the scalar curvature of a (pseudo-) Riemannian M^n

$$R = g^{ik} R_{ik} = g^{ik} R^{l}{}_{ilk}. \qquad (45.11)$$

While our approach will hold for any n, we shall write the computation in terms of space-time M^4. The volume element for space-time is of the form

$$\sqrt{g}\, d^4x \equiv \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

(recall that $\det g_{ij} = g < 0$ for a pseudo metric). Consider R is a function of the metric. We shall (with Hilbert) take the first variation of the functional

$$\delta \int_{M^4} R \sqrt{-g}\, d^4x \qquad (45.12)$$

for a 1-parameter variation of the metric g_{ij}. For our purposes it is more convenient to vary the inverse of the metric

$$\begin{cases} g_\varepsilon^{ij} = g_0^{ij} + \varepsilon \eta^{ij} \\ \dot{g}^{ij} = \eta^{ij} \end{cases} \tag{45.13}$$

We are concerned with

$$\left(\frac{d}{d\varepsilon}\int_{M^4} R\sqrt{-g}\, d^4x\right)_{\varepsilon=0} = \int_{M^4} (R\sqrt{-g})^{\cdot}\, d^4x \tag{45.14}$$

Now

$$(R\sqrt{-g})^{\cdot} = (g^{ik}R_{ik}\sqrt{-g})^{\cdot} = (g^{ik}R_{ik})^{\cdot}\sqrt{-g} + R(\sqrt{-g})^{\cdot} \tag{45.15}$$

$$(g^{ik}R_{ik})^{\cdot} = \dot{g}^{ik}R_{ik} + g^{ik}\dot{R}_{ik}. \quad \text{We need } \dot{R}_{ik}!$$

Now from (34.3)

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma\cdot\Gamma - \Gamma\cdot\Gamma$$

and so, since ε is independent of x

$$\dot{R}_{ik} = \frac{\partial \dot{\Gamma}^l_{ik}}{\partial x^l} - \frac{\partial \dot{\Gamma}^l_{il}}{\partial x^k} + \dot{\Gamma}\Gamma + \Gamma\dot{\Gamma} - \dot{\Gamma}\Gamma - \Gamma\dot{\Gamma}$$

We shall work at the pole p of a geodesic normal coordinate system for the metric g_{ij}. Then $\Gamma = 0$ at the pole and so

$$\dot{R}_{ik} = \frac{\partial \dot{\Gamma}^l_{ik}}{\partial x^l} - \frac{\partial \dot{\Gamma}^l_{il}}{\partial x^k} \qquad \text{at } p. \tag{45.16}$$

Now while Γ^l_{ik} is <u>not</u> a tensor, $\dot{\Gamma}^l_{ik}$ <u>is</u>. To see this, recall that a connection transforms as (see (38.27))

$$\omega' = P^{-1}\omega P + P^{-1}dP$$

where P is the matrix for a change of frame. Since P is independent of ε

$$\dot{\omega}' = P^{-1}\dot{\omega}P$$

shows that $\dot{\omega}$ is a tensorial form, and it follows easily that

$$\dot{\Gamma}^{\dot{l}}_{ik}$$

transforms as a mixed tensor (i.e, "the difference of two affine connections is a tensor"). Thus we can write at our pole

$$\dot{R}_{ik} = \dot{\Gamma}^{\dot{l}}_{ik/l} - \dot{\Gamma}^{\dot{l}}_{il/k} \tag{45.17}$$

and since this is a tensor equation it holds everywhere, in any coordinate system. In (45.17), the covariant derivatives are computed using the <u>original metric</u> g°_{ij}. Thus, when $\varepsilon = 0$ (ie $g_{ij} = g^{\circ}_{ij}$)

$$\left.\begin{aligned} g^{ik}\dot{R}_{ik} &= \left(g^{ik}\dot{\Gamma}^{\dot{l}}_{ik}\right)_{/l} - \left(g^{ik}\dot{\Gamma}^{\dot{l}}_{il}\right)_{/k} \\ &\equiv W^{r}_{/r} = \operatorname{div} W \end{aligned}\right\} \tag{45.18}$$

Now look at the second term in (45.15), $R(\sqrt{-g})^{\bullet}$. But to differentiate the determinant we need

$$\frac{\partial g}{\partial g_{ik}} = \underset{\text{cofactor}}{G^{ik}} = g^{ik} g$$

Likewise

$$\frac{\partial g^{-1}}{\partial g^{ik}} = g_{ik}\, g^{-1}, \text{ i.e } \quad \frac{\partial g}{\partial g^{ik}} = -\, g_{ik}\, g$$

Thus

$$\frac{\partial \sqrt{-g}}{\partial g^{ik}} = -\frac{1}{2} g_{ik} \sqrt{-g} \tag{45.19}$$

and so

$$(\sqrt{-g})^{\bullet} = -\frac{1}{2} g_{ik} \sqrt{-g}\, \dot{g}^{ik} \tag{45.20}$$

We finally have

$$\left(\frac{d}{d\varepsilon}\int_M R\sqrt{-g}\,d^4x\right)_{\varepsilon=0} = \int_M \left(R_{ik} - \tfrac{1}{2}g_{ik}R\right)\dot{g}^{ik}\sqrt{-g}\,d^4x + \int_M \operatorname{div} W\,\sqrt{-g}\,d^4x \Bigg\} \quad (45.21).$$

Suppose now that we consider variations that vanish outside some compact subset, ie $\dot{g}^{ik}=0$ outside a compact set M_0 of M. Letting M_{00} be even larger, so that M_0 lies inside M_{00}

$$\int_M \operatorname{div} W\,\sqrt{-g}\,d^4x = \int_{M_{00}} \operatorname{div} W\sqrt{-g}\,d^4x \quad = 0$$

by applying the divergence theorem ($W=0$ on ∂M_{00}). We get

$$\boxed{\left(\frac{d}{d\varepsilon}\int_M R\sqrt{-g}\,d^4x\right)_{\varepsilon=0} = \int_M \left(R_{ik} - \tfrac{1}{2}g_{ik}R\right)\dot{g}^{ik}\sqrt{-g}\,d^4x} \quad (45.22)$$

Hilbert, Nov. 1915

Define now the (Hilbert) <u>action</u> for the gravitational field

$$S_{grav} = \int_M \frac{R\sqrt{-g}}{8\pi K}\,d^4x$$

considered as a (highly non-linear) fctnal of the metric tensor g^{ik}. As usual we define the <u>variational</u> or fctnal derivative $\frac{\delta S}{\delta g^{ik}}$ by

$$\left(\frac{d}{d\varepsilon}\int_M \frac{R\sqrt{-g}}{8\pi K}d^4x\right)_{\varepsilon=0} = \int_M \frac{\delta S_{grav}}{\delta g^{ik}}\dot{g}^{ik}\,d^4x \quad (45.23)$$

(M is assumed that the variation $\dot{g}^{ik}$ is compactly supported)

and so

$$\frac{\delta S_{grav}}{\delta g^{ik}} = (8\pi k)^{-1}\left(R_{ik} - \tfrac{1}{2} g_{ik} R\right)\sqrt{-g} \qquad (45.24)$$

Let $S_{nongrav}$ be the action for the non gravitational fields present (e.g. electromagnetic); it is given via some Lagrange density

$$S_{nongrav} = \int_M \mathcal{L}\, d^4x = \int_M L\sqrt{-g}\, d^4x. \qquad (45.25)$$

The "stress energy momentum tensor" T for the non gravitational fields is defined by

$$\frac{\delta S_{nongrav}}{\delta g^{ik}} = -T_{ik}\sqrt{-g} \qquad (45.26)$$

The total action, gravitational + non gravitational, is

$$S = S_{grav} + S_{nongrav}.$$

and we see

$$\frac{\delta S}{\delta g^{ik}} = (8\pi k)^{-1}\left[R_{ik} - \tfrac{1}{2} g_{ik} R\right]\cdot\sqrt{-g} - T_{ik}\sqrt{-g} \qquad 45.27.$$

and Einstein's equations follow from Hilbert's action principle

$$\frac{\delta S}{\delta g^{ik}} = 0 \qquad 8\pi k\, T_{ik} = R_{ik} - \tfrac{1}{2} g_{ik} R$$

Classically one wrote

$$\delta \int \left[(8\pi k)^{-1} R + L\right]\sqrt{-g}\, d^4x = 0 \qquad 45.28.$$

We shall discuss action principles (including L for the electromagnetic field) next quarter.

In any case, it is natural to call

$$(8\pi k)^{-1} R\sqrt{-g}$$

the Lagrange density for the gravitational field.

(79 1/2) Lagrange density for E&M is (up to a multiple)

$$\mathcal{L} = F_{ik} F^{ik} \sqrt{|g|}$$

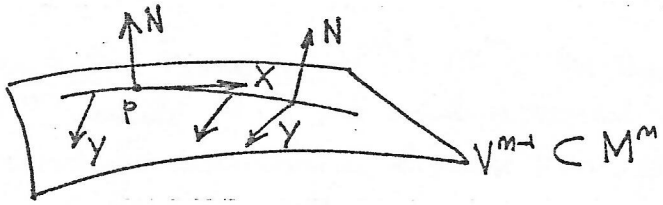
Compute the energy momentum tensor T_{ik}.

$$\frac{\delta \mathcal{L}}{\partial g^{ik}} = -T_{ik} \sqrt{|g|}$$

Lecture 46. Sectional Curvature and Gauss Equations.

(1) The Induced Metric and Connection for $V^{n-1} \subset M^n$. Let M^n be a Riemannian manifold and let V^{n-1} be a submanifold. If we restrict the scalar product $\langle , \rangle$ for M^n to vectors X, Y tangent to V we shall then have a Riemannian metric for V^{n-1}, the "induced metric". We use the same notation $\langle , \rangle$ for this induced metric.

Let ∇ be the Riemannian connection for M^n. Define a new connection for V^{n-1} as follows. Let X be a tangent vector to V at p and let Y be a tangent vector field to V near p. Let $x^1, \dots, x^n$ be local coordinates for M^n, $u^1, \dots, u^{n-1}$ local coordinates for V^{n-1}.



Then

$$\nabla_X Y = \nabla_{X^\alpha \frac{\partial}{\partial u^\alpha}} Y = X^\alpha \nabla_{\frac{\partial}{\partial u^\alpha}} Y = X^\alpha \frac{\nabla}{\partial u^\alpha} Y$$

makes sense since Y is a vector field along the submanifold V^{n-1} (we talked about such things for a $V^2 \subset M^n$ in Lecture 44, § 1). Let N be a unit vector field along V^{n-1} that is normal to V^{n-1}. Then

$$\bar{\nabla}_X Y \equiv \text{projection of } \nabla_X Y \text{ into the tangent space } V^{n-1}_p$$

$$\bar{\nabla}_X Y = \nabla_X Y - \langle \nabla_X Y, N \rangle N \tag{46.1}$$

is a well defined tangent vector to V^{n-1} at p. We claim that $\bar{\nabla}$ defines an affine connection, i.e. a covariant differentiation for V. While we have defined an affine connection in a slightly different fashion in Lecture 36 it is equivalent to say the following

> An affine connection $\bar{\nabla}$ on a manifold V assigns to each vector field Y and to each vector X at p a vector $\bar{\nabla}_X Y$ at p such that
>
> $$Y \longmapsto \bar{\nabla}_X Y$$
>
> is linear in Y, and if f is a function
>
> $$\bar{\nabla}_X (fY) = X(f)\, Y + f \bar{\nabla}_X Y$$

(see (36.3)).

One checks immediately that (46.1) does define a connection $\bar{\nabla}$ on V^{n-1}

We claim more; $\bar{\nabla}$ is the Riemannian connection for the induced metric on V^{n-1}.

Problem 80. Let X, Y, Z, be tangent vector fields to V^{n-1}. Extend them in any way you wish to vector fields on M^n.

(i) Show that

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X \text{ is the Lie Bracket } [X,Y] \text{ on } V^{n-1}$$

and so the connection $\bar{\nabla}$ is symmetric (see the discussion after (38.6)).

(ii) Show that

$$X\langle Y,Z\rangle = \langle \bar{\nabla}_X Y, Z\rangle + \langle Y, \bar{\nabla}_X Z\rangle$$

and hence $\bar{\nabla}$ is the Levi-Civita connection for V^{n-1}.

(2) The Second Fundamental Form for $V^{n-1} \subset M^n$. In the following $X, Y, Z, \ldots$ are tangent to V^{n-1}, while N is the unit normal field to V^{n-1} (locally). The second fundamental linear transformation

$$b : V^{n-1}_p \longrightarrow V^{n-1}_p$$

is defined by

$$b(X) = -\nabla_X N. \tag{46.2}$$

As usual, define the second fundamental form $B(X,Y)$ by

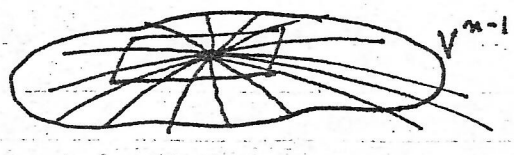
$$B(X,Y) \equiv \langle X, b(Y)\rangle = \langle X, -\nabla_Y N\rangle$$

$$= Y(\langle X, -N\rangle) - \langle \nabla_Y X, -N\rangle$$

$$= \langle \nabla_Y X, N\rangle = \langle \nabla_X Y, N\rangle \text{ (why)} = B(Y,X)$$

where again we take extensions of the fields X, Y. b is then a self adjoint linear transformation, it has $(n-1)$ real eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ called principal normal curvatures; the eigendirections are called principal directions and they can always be chosen to be orthogonal.

V^{n-1} is said to be "geodesic" at p provided that near p V^{n-1} is made up of all the geodesics of M^n that are tangent to V^{n-1} at p



If V^{n-1} is geodesic at p, then at p

$$B(X,X) = \langle \nabla_X X, N \rangle = 0$$

since X can be chosen to be the tangent field to an M^n geodesic. Thus all the eigenvalues of b vanish at p and so

$$B(X,Y) \equiv 0 \qquad (46.3)$$

at p; <u>the second fundamental form vanishes at any point at which V is "geodesic"</u>!

As for $V^2 \subset R^3$ we define the <u>mean curvature</u> H of V^{n-1} by

$$H = tr(b) = \kappa_1 + \kappa_2 + \cdots + \kappa_{n-1}$$

and this is again significant for variation of vol^{n-1} of V^{n-1}! The other invariants $\sum_{\alpha<\beta} \kappa_\alpha \kappa_\beta$, $\sum_{\alpha<\beta<\gamma} \kappa_\alpha \kappa_\beta \kappa_\gamma$, $\cdots$, $\kappa_1 \kappa_2 \cdots \kappa_{n-1}$ are also useful, though not to the same extent as K is for $V^2 \subset R^3$.

(3) <u>Equations of Gauss and Codazzi</u>. M^n has connection ∇ and curvature tensor R; V^{n-1} has connection $\bar{\nabla} = \nabla_V$ and curvature $\bar{R} = R_V$. How are their curvatures related. In other words, if X, Y, Z are tangent to V^{n-1}, how are the vectors

$$R(X,Y)Z \quad \text{and} \quad \bar{R}(X,Y)Z$$

related? The calculation starts out as follows. Again let $x^1 \cdots x^n$ be coordinates for M, $u^1, \ldots, u^{n-1}$ coordinates for V^{n-1}. Denote the basis coordinate vectors for V by

$$\partial_\alpha \equiv \frac{\partial}{\partial u^\alpha}, \qquad \alpha = 1, 2, \ldots, n-1$$

From (41.2)

$$(\nabla_{\partial_\alpha} \nabla_{\partial_\beta} - \nabla_{\partial_\beta} \nabla_{\partial_\alpha}) \partial_\gamma = R(\partial_\alpha, \partial_\beta) \partial_\gamma$$

where R is the curvature tensor for M^n. On the other hand

$$(\bar{\nabla}_{\partial_\alpha} \bar{\nabla}_{\partial_\beta} - \bar{\nabla}_{\partial_\beta} \bar{\nabla}_{\partial_\alpha}) \partial_\gamma = \bar{R}(\partial_\alpha, \partial_\beta) \partial_\gamma.$$

If now we use, systematically,

$$\nabla_{\partial_\alpha} \partial_\gamma = \bar{\nabla}_{\partial_\alpha} \partial_\gamma + \langle \nabla_{\partial_\alpha} \partial_\gamma, N \rangle N$$

and take second derivatives (using $\nabla_{\partial_\beta} N = -b(\partial_\beta), \ldots$)

then by a calculation entirely similar in spirit to Gauss' in Prob. 62 one finally gets

$$\left.\begin{aligned}(\nabla_{\partial_\alpha}\nabla_{\partial_\beta} - \nabla_{\partial_\beta}\nabla_{\partial_\alpha})\,\partial_\gamma &= (\bar\nabla_{\partial_\alpha}\bar\nabla_{\partial_\beta} - \bar\nabla_{\partial_\beta}\bar\nabla_{\partial_\alpha})\,\partial_\gamma \\ &+ B(\partial_\alpha,\partial_\gamma)\,b(\partial_\beta) - B(\partial_\beta,\partial_\gamma)\,b(\partial_\alpha) \\ &+ \left[\frac{\partial}{\partial u^\alpha} B(\partial_\beta,\partial_\gamma) - B(\partial_\beta, \bar\nabla_{\partial_\alpha}\partial_\gamma)\right. \\ &\left. - \frac{\partial}{\partial u^\beta} B(\partial_\alpha,\partial_\gamma) + B(\partial_\alpha, \bar\nabla_{\partial_\beta}\partial_\gamma)\right] N\end{aligned}\right\} \qquad (46.4)$$

you should do it yourself!

The last two lines can be simplified. It is <u>important</u> (and not difficult to see) that our prescription for taking the covariant derivative of a tensor (44.4) says the following in the case of the covariant tensor B on V^{n+1}. Differentiate the function $B(Y,Z)$ along the vector X

$$X\big(B(Y,Z)\big) = (\nabla_X B)(Y,Z) + B(\nabla_X Y, Z) + B(Y, \nabla_X Z) \qquad (46.5)$$

(there are similar formulas for <u>any</u> tensor). In our case

$$\frac{\partial}{\partial u^\alpha} B(\partial_\beta,\partial_\gamma) = (\bar\nabla_{\partial_\alpha} B)(\partial_\beta,\partial_\gamma) + B(\bar\nabla_{\partial_\alpha}\partial_\beta, \partial_\gamma) + B(\partial_\beta, \bar\nabla_{\partial_\alpha}\partial_\gamma).$$

Using (41.1), i.e $\bar\nabla_{\partial_\alpha}\partial_\beta - \bar\nabla_{\partial_\beta}\partial_\alpha = 0$, we see that the last two lines of (46.4) just yield

$$\left[(\bar\nabla_{\partial_\alpha} B)(\partial_\beta,\partial_\gamma) - (\bar\nabla_{\partial_\beta} B)(\partial_\alpha,\partial_\gamma)\right] N = \left[B_{\beta\gamma\|\alpha} - B_{\alpha\gamma\|\beta}\right] N \qquad (46.6)$$

where we use double slash for covariant differentiation using $\bar\nabla$. (This should be no surprise after (44.6)!).

(46.4) and (46.6) then yield the following

$$R(\partial_\alpha,\partial_\beta)\partial_\gamma = \bar{R}(\partial_\alpha,\partial_\beta)\partial_\gamma + B(\partial_\alpha,\partial_\gamma)b(\partial_\beta) - B(\partial_\beta,\partial_\gamma)b(\partial_\alpha) + \left[\bar{\nabla}^{(B)}_{\partial_\alpha}(\partial_\beta,\partial_\gamma) - \bar{\nabla}^{(B)}_{\partial_\beta}(\partial_\alpha,\partial_\gamma)\right] N \tag{46.7}$$

Finally, multiplying by $X^\alpha Y^\beta Z^\gamma$ and summing over α,β,γ gives

$$R(X,Y)Z = \bar{R}(X,Y)Z + B(X,Z)b(Y) - B(Y,Z)b(X) + \left[(\bar{\nabla}_X B)(Y,Z) - (\bar{\nabla}_Y B)(X,Z)\right] N \tag{46.8}$$

normal are Codazzi tensors or Gauss

Compare to 34.4 with $R=0$

In particular, we shall get two consequences. First

$$\begin{aligned}\langle R(X,Y)Y,X\rangle &= \langle \bar{R}(X,Y)Y,X\rangle + B(X,Y)\langle b(Y),X\rangle - B(Y,Y)\langle b(X),X\rangle \\ &= \langle \bar{R}(X,Y)Y,X\rangle + B^2(X,Y) - B(X,X)B(Y,Y).\end{aligned} \tag{46.9}$$

finding $R'_{212} = K$

Now it is easy to see that if we make a substitution $X \mapsto X' = aX + bY$, $Y \mapsto Y' = cX + dY$, then

should be $\bar{R}$ with $R=0$

$$\langle R(X',Y')Y',X'\rangle = \begin{vmatrix} a & b \\ c & d\end{vmatrix}^2 \langle R(X,Y)Y,X\rangle$$

On the other hand, if we denote the area of the parallelogram spanned by X and Y by $\|X\wedge Y\|$, so

$$\|X\wedge Y\|^2 \equiv \|X\|^2\|Y\|^2 \sin^2\measuredangle X,Y = \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2$$

then we also have, under the same substitution

$$\|X'\wedge Y'\|^2 = \begin{vmatrix} a & b \\ c & d\end{vmatrix}^2 \|X\wedge Y\|^2.$$

Consequently, if X and Y are not dependent, then

$$K(X\wedge Y) \equiv \frac{\langle R(X,Y)Y,X\rangle}{\|X\wedge Y\|^2} \tag{46.10}$$

<u>depends only on the plane "$X\wedge Y$" spanned by X and Y, not the basis X,Y itself!</u>

$K(X \wedge Y)$ is called the (Riemann) "sectional curvature" for the 2 plane spanned by X and Y. If, in (46.9) we take X and Y to be orthonormal, we get

$$\boxed{K_M(X \wedge Y) = K_V(X \wedge Y) + B(X,Y)B(X,Y) - B(X,X)B(Y,Y)} \quad (46.11)$$

Gauss' eqn for a hypersurface $V^{n-1} \subset M^n$

We also have, from (46.8)

$$\boxed{\langle R(X,Y)Z, N \rangle_M = (\bar{\nabla}_X B)(Y,Z) - (\bar{\nabla}_Y B)(X,Z)} \quad (46.12)$$

Codazzi eqns for $V^{n-1} \subset M^n$

(4) Interpretation of the Sectional Curvature. Suppose now we consider, instead of a hypersurface (of dimension $n-1$) a submanifold

$$V^r \subset M^n$$

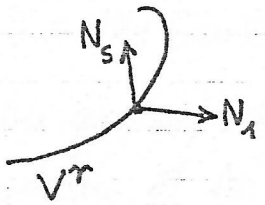
M^n being again Riemannian. We can again define an induced connection $\bar{\nabla}$ by

$\bar{\nabla}_X Y$ = projection of $\nabla_X Y$ onto the tangent space to V^r.

The normal space to V^r at p

$$(V^r_p)^\perp \subset M^n_p$$

now has dimension $(n-r)$; let N_A, $A = 1, 2, \ldots, (n-r) = s$ be orthonormal normal vector fields to V^r, locally.



Then

$$\bar{\nabla}_X Y = \nabla_X Y - \sum_{A=1}^{s} \langle \nabla_X Y, N_A \rangle N_A$$

For each normal N_A we shall have a second fundamental linear transformation

$$b_A : V^r_p \longrightarrow V^r_p$$

defined by

$$b_A(X) = -\bar{\nabla}_X(N_A) \tag{46.13}$$

where we allow $\bar{\nabla}_X$ to operate on any vector field U (not necessarily tangent to V^n) again by

$$\bar{\nabla}_X U = \text{projection of } \nabla_X U \text{ into } V^n_p.$$

This is required since $\nabla_X N_A$ is orthogonal to N_A but not necessarily tangent to V^n! A calculation very similar to (46.11) then yields

$$K_M(X\wedge Y) = K_V(X\wedge Y) + \sum_{A=1}^{s}\left[B_A(X,Y)B_A(X,Y) - B_A(X,X)\,B_A(Y,Y)\right] \tag{46.14}$$

Now let X, Y be any pair of orthonormal vectors at a point $p\in M^n$. Consider the 2 dimensional surface $V^2 \subset M^n$ generated by all geodesics of M^n that are tangent to $X\wedge Y$ at p. For this V^2 we have all $b_A \equiv 0$ at p, just as in (46.3). Thus

$$K_M(X\wedge Y) = K_V(X\wedge Y)$$

Putting $X = e_1$, $Y = e_2$, we see

$$K_{V^2}(e_1\wedge e_2) = \langle R_{V^2}(e_1,e_2)e_2, e_1\rangle = R_{V^2 1212} = K$$

the Gaussian curvature of the surface V^2. Thus

$$\boxed{\begin{array}{l} K_M(X\wedge Y) = \text{the Gaussian curvature } R_{1212} \text{ at } p \\ \quad \text{of the geodesic disc } V^2 \text{ generated by the} \\ \quad \text{geodesics of } M^n \text{ tangent to the plane} \\ \quad \text{spanned by } X \text{ and } Y. \end{array}} \tag{46.15}$$

Lecture 47. The Geometry of Einstein's Equations

(1) Equations of Einstein and Gauss. When dealing with the space-time manifold M^4 of general relativity the equations of Gauss and Codazzi discussed in our last lecture must be amended because of the indefinite metric. Many of those questions are discussed in Chapter 4 of [Fr] which the student could now read rather easily.

A vector is null if its norm vanishes, $\|Z\|^2 = \langle Z,Z\rangle = 0$. If N is not a null vector, define its indicator ε_N by

$$\varepsilon_N \equiv \operatorname{sign} \langle N, N\rangle.$$

Consider a spatial hypersurface $V^3 \subset M^4$, ie a hypersurface whose unit normal N is time-like, $\varepsilon_N = -1$. There exist local coordinates $t = x^0, x^1, x^2, x^3$ such that V^3 is defined by

$$V^3 \;;\quad t = \text{constant}.$$

It is a "spatial slice" of M^4. The Gauss equations for V^3 differ from (46.11) only in a single minus sign coming from ε_N

$$K_{M^4}(X\wedge Y) = K_{V^3}(X\wedge Y) + B(X,X)B(Y,Y) - B^2(X,Y) \qquad (47.1).$$

Define the "Einstein tensor", the symmetric covariant tensor $G = (G_{ij})$ for M^4 by

$$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R. \qquad (47.2)$$

The Einstein equations imply, for our normal N to V^3

$$\left.\begin{aligned} G(N,N) = G_{ij}N^iN^j &= R_{ij}N^iN^j + \tfrac{1}{2}R \\ &= 8\pi K\, T_{ij}N^iN^j = 8\pi K\, T(N,N)\end{aligned}\right\}$$

Let $N = e_0, e_1, e_2, e_3$ be an on. frame (and thus e_1, e_2, e_3 are tangent to V^3); then in [Fr] it is shown that

$$8\pi K\, T(e_0,e_0) = K_M(e_1\wedge e_2) + K_M(e_1\wedge e_3) + K_M(e_2\wedge e_3). \qquad (47.3)$$

If we choose e_1, e_2, e_3 to be principal directions for the second fundamental b of V^3, this yields from (47.1)

$$\left.\begin{aligned} 8\pi k\, T(e_0,e_0) &= K_V(e_1\wedge e_2) + K_V(e_1\wedge e_3) + K_V(e_2\wedge e_3) \\ &\quad + k_1k_2 + k_1k_3 + k_2k_3 \end{aligned}\right\} \qquad (47.4)$$

(note that since V^3 is spatial, the induced metric on V^3 is Riemannian, ie positive definite).

Now in a Riemannian V^n one can show the following (see [Fr]). Let $e_1, \dots, e_n$ be an o.n frame. Then the Ricci quadratic form (or "Ricci curvature")

$$Ric_V(X,X) \equiv Ric_V(X) \equiv R_{\alpha\beta}X^\alpha X^\beta$$

is a sum of sectional curvatures; for example

$$Ric_V(e_1,e_1) = \sum_{\alpha=2}^{n} K(e_1\wedge e_\alpha)$$

(negative signs come in for pseudo-Riemannian V^n). Furthermore, the scalar curvature R is a sum of Ricci curvatures

$$R_V = R^\alpha{}_\alpha = \sum_{\substack{\beta,\gamma \\ \beta\neq\gamma}} K_V(e_\beta\wedge e_\gamma).$$

In particular, for a Riemannian V^2 (ie surface)

$$Ric_V(e_1,e_1) = K_V(e_1\wedge e_2) = K, \qquad (47.5)$$

the Gauss curvature, while

$$R_V = 2K. \qquad (47.6)$$

Then (47.4) can be written (again $e_0 = N$)

$$\left.\begin{aligned} G(e_0,e_0) &= 8\pi k\, T(e_0,e_0) \\ &= \tfrac{1}{2}R_{V^3} + (k_1k_2 + k_1k_3 + k_2k_3) \end{aligned}\right\} \qquad (47.7)$$

The stress energy momentum contribution $T(e_0,e_0)$ is defined to be the energy density ρ

$$\boxed{8\pi k T(e_0,e_0) = 8\pi k \rho = \tfrac{1}{2} R_{V3} + (k_1k_2 + k_1k_3 + k_2k_3)} \qquad (47.8)$$

We may as well call this the Einstein-Gauss equation, although Einstein never wrote such equations down.

(47.8) has a remarkably simple interpretation. Recall that Gauss' theorema egregium for $V^2 \subset R^3$ states, in our new notation

$$0 = \tfrac{1}{2} R_{V2} - k_1k_2 .$$

For a Riemannian $V^3 \subset$ Minkowski space M^4_0 we would have from (47.7) (and putting $G(e_0,e_0)=0$ since M^4_0 is flat)

$$0 = \tfrac{1}{2} R_{V3} + (k_1k_2 + k_1k_3 + k_2k_3) .$$

This is simply the straightforward generalization of Gauss' theorema egregium, the sign difference coming from the indefinite metric of Minkowski space. Einstein's equation (47.8) says that

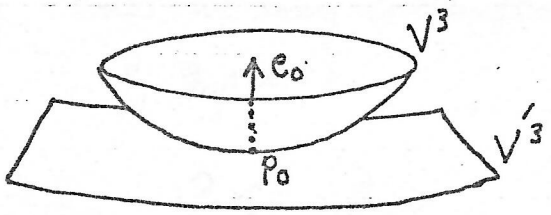
$$\tfrac{1}{2} R_{V3} + (k_1k_2 + k_1k_3 + k_2k_3)$$

is <u>not</u> zero in the physical space time M^4 in which we live and, in fact, <u>this curvature expression is essentially a measure of the density of energy</u> (<u>due to matter, electromagnetism, stresses...</u>).

This interpretation has been emphasized many times by J. A. Wheeler. The student is now ready to read <u>Gravitation</u>, by Misner, Thorne and Wheeler.

I like to interpret (47.8) as follows. Let e_0 be a time-like unit vector at a given point p_0 of our space-time M^4. Let V^3 be <u>any</u> hypersurface of M^4 that is normal to e_0 at p_0. Since V^3 is a Riemannian manifold (in the induced metric from M^4) it has a scalar curvature R_{V3} at p_0. Of course

changing V^3 will change R_{V3} at p_0. As a submanifold



of M^4, V^3 will also have a 2nd fundamental & and principal curvatures $\kappa_1, \kappa_2, \kappa_3$ at p_0. Of course if V^3 is changed, these curvatures will also change. However, Gauss' theorema egregium (47.7) is telling us that the particular combination

$$\frac{1}{2}R_{V3} + (\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)$$

will not change! It is a property of the manifold M^4, the point p, and the direction e_0. Einstein's equations are telling us that this combination is a measure of the energy density.

Recall that R_{V3} is an intrinsic curvature, a bending invariant determined by the metric for V^3. It can be determined by measurements carried out entirely in V^3 in a manner similar to (42.5). On the other hand, $\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3$ is a measure of extrinsic curvature, ie how V^3 is curving and bent in M^4. Wheeler likes to state (47.8) as

energy density = intrinsic curvature + extrinsic curvature	(47.8

(2) Interpretation of the 2nd Fundamental Form. Consider in space time M^4 a coordinate system in which the metric assumes the form

$$ds^2 = g_{00}\,dt^2 + h_{\alpha\beta}\,dx^\alpha dx^\beta$$

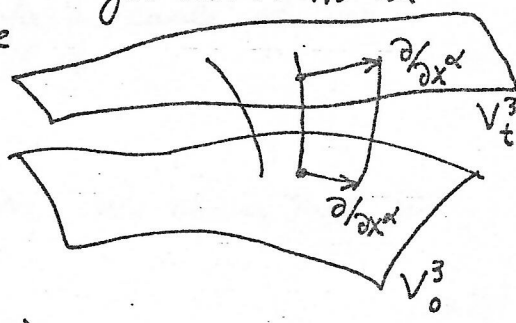
i.e $g_{0\beta} = 0$, and $g_{\alpha\beta} = h_{\alpha\beta}$ is the induced metric tensor on the spatial slices $t = \text{constant}$. Such coordinate systems always exist; in fact we could introduce "Gaussian coordinates" based on the hypersurface

$$V_0^3 \;;\; t = 0$$

in which case $g_{00} = -1$. We shall not require this; consequently

the t-lines ($x^\alpha = \text{const.}$, $\alpha = 1,2,3$) not need be geodesics. As we proceed along t lines we may contemplate

$$\frac{\partial}{\partial t} g_{\alpha\beta}(x,t).$$



Now

$$\frac{\partial}{\partial t} g_{\alpha\beta} = \frac{\partial}{\partial t} h_{\alpha\beta} = \frac{\partial}{\partial t} \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle$$

$$= \left\langle \frac{\nabla}{\partial t} \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle + \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\nabla}{\partial t} \frac{\partial}{\partial x^\beta} \right\rangle$$

$$= \left\langle \frac{\nabla}{\partial x^\alpha} \frac{\partial}{\partial t}, \frac{\partial}{\partial x^\beta} \right\rangle + \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\nabla}{\partial x^\beta} \frac{\partial}{\partial t} \right\rangle$$

If we define

$$\varphi \equiv \sqrt{-g_{00}} \tag{47.10}$$

then $u = \varphi^{-1} \partial/\partial t$ is the unit normal to each each V_t^3

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \left\langle \frac{\nabla}{\partial x^\alpha} \varphi u, \frac{\partial}{\partial x^\beta} \right\rangle + \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\nabla}{\partial x^\beta} \varphi u \right\rangle$$

$$= \varphi \left\langle \frac{\nabla}{\partial x^\alpha} u, \frac{\partial}{\partial x^\beta} \right\rangle + \varphi \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\nabla}{\partial x^\beta} u \right\rangle$$

$$= -\varphi (b_{\alpha\beta} + b_{\beta\alpha})$$

$$\boxed{\frac{\partial}{\partial t} g_{\alpha\beta} = -2\varphi b_{\alpha\beta}} \tag{47.11}$$

This shows that $(b_{\alpha\beta})$ essentially is a measure of the rate of change of the spatial metric $g_{\alpha\beta}$ as one moves along the normal to V^3 (i.e. as one moves in time).

Problem 81 Consider in a Riemannian M^n a coordinate system for which

$$ds^2 = \varphi^2 dt^2 + \sum_{\alpha,\beta=}^{n-1} h_{\alpha\beta} dx^\alpha dx^\beta.$$

Then $\partial/\partial t$ is a variation vector along the hypersurfaces

$$V_t^{n-1} : t = \text{constant}$$

Show that

$$\frac{\partial}{\partial t} \sqrt{\det h_{\alpha\beta}} = -\varphi H \qquad H = \text{mean curvature of } V_t^{n-1} \tag{47.12}$$

Hint: Compute $\dfrac{\partial \sqrt{\det h_{\alpha\beta}}}{\partial h_{\mu\nu}}$ (see lect. 45).

(3) The Einstein Codazzi Equations. First we shall require a famous identity that the Riemann tensor satisfies for the case of the Riemannian connection. We already know that

$$R^{h}_{ijk}$$

is skew symmetric in j,k, for any connection. We claim that in a (pseudo-) Riemannian manifold

$$\boxed{R_{hijk} = -R_{ihjk}} \qquad (47.13)$$

is also skew in h,i.

We have seen in §1 of Lecture 39 that $(\vartheta^h{}_i)$ is skew in h,i in an o.n. frame

$$\vartheta_{hi} = -\vartheta_{ih}.$$

This is an o.n. frame.

$$R_{hijk}\, X^j Y^k Z^h W^i = \langle R(X,Y)W, Z\rangle$$

is skew in W,Z. But R is a tensor, hence this property is, in fact, independent of frame!

Problem 82. Using the 1st Bianchi identity show that we also have

$$\boxed{R_{hijk} = R_{jkhi}} \qquad (47.14)$$

Back to relativity! Assume coordinates again for which

$$ds^2 = g_{00}\,dt^2 + h_{\alpha\beta}\,dx^\alpha dx^\beta$$

The Codazzi equations (46.12) are unchanged in the pseudo-Riemannian case when N is time like. We are looking at V^3_t, t = constant.

Problem 83 (i) Show that (46.12) has the coordinate expression

$$R_{0\gamma\alpha\beta} = b_{\gamma\beta\|\alpha} - b_{\gamma\alpha\|\beta}$$

(ii) use $g_{ij} = \begin{pmatrix} g_{00} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}$, $g^{ij} = \begin{pmatrix} g_{00}^{-1} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}$ to show

$$-h^{\mu\gamma} R_{\gamma 0\alpha\beta} = b^{\mu}_{\beta\|\alpha} - b^{\mu}_{\alpha\|\beta}$$

$$\therefore -R^{\mu}_{0\alpha\beta} = -g^{\mu i} R_{i0\alpha\beta} = b^{\mu}_{\beta\|\alpha} - b^{\mu}_{\alpha\|\beta}$$

(iii) Show $R_{\alpha\beta} = R^{i}_{0i\beta} = R^{\alpha}_{0\alpha\beta}$.

Problem 83 then shows that

$$-R_{0\beta} = b^{\alpha}{}_{\beta\|\alpha} - b^{\alpha}{}_{\alpha\|\beta}$$

and so Einstein's equations yield

$$\boxed{8\pi K T_{0\beta} = \frac{\partial H}{\partial x^{\beta}} - b^{\alpha}{}_{\beta\|\alpha}}$$

$$8\pi K T_{0\beta} = R_{0\beta} - \tfrac{1}{2} g_{0\beta} \cancel{R}^{\,0}$$
$$= b^{\alpha}{}_{\alpha\|\beta} - b^{\alpha}{}_{\beta\|\alpha}$$
$$= H_{\|\beta} - b^{\alpha}{}_{\beta\|\alpha}$$
$$= \frac{\partial H}{\partial x^{\beta}} - b^{\alpha}{}_{\beta\|\alpha} \qquad (47.15)$$

the "Einstein Codazzi" equations. The components $T_{0\beta}$, $\beta = 1, 2, 3$ are the linear momentum densities!

Suppose that one attempts to consider Einstein's equations as an initial value problem. Suppose that one assigns the metric tensor

$$g_{ij}(0, x)$$

on the initial hypersurface V_0^3; $t = 0$, and also one assigns

$$\frac{\partial g_{ij}}{\partial t}(0, x).$$

One then hopes to find $g_{ij}(t, x)$ that will yield a solution to Einsteins equations. Note however that we do not have complete freedom in prescribing the metric and its time derivative. For example, in a metric of the form

$$ds^2 = g_{00}\, dt^2 + g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$$

note that prescribing $g_{\alpha\beta}(0, x)$ will prescribe R_{V^3}, while prescribing $\dot{g}_{\alpha\beta}(0, x)$ will, from (47.11), prescribe $b_{\alpha\beta}$. Then (47.8) and (47.15) will give initial value constraints that must be satisfied! For a nice treatment of the initial value problem I recommend the book General Relativity, by Robt. Wald.

Lecture 48. The Hodge Operators

(1) The * operator. On a (pseudo-) Riemannian M^n we introduce a pointwise scalar product between p-forms by

$$\left.\begin{aligned} \langle \alpha^p, \beta^p \rangle &= \underset{\underset{I}{\rightarrow}}{\alpha} \beta^I = \underset{\underset{i_1 \dots i_p}{\rightarrow}}{\alpha_{i_1 \dots i_p}} \beta^{i_1 \dots i_p} \\ &= \sum_{i_1 < \dots < i_p} \alpha_{i_1 \dots i_p} \beta^{i_1 \dots i_p} \end{aligned}\right\} \tag{48.1}$$

It is not difficult to check that if $e_1, \dots, e_n$ is an o.n basis for tangent vectors at a point, then $\sigma^1, \dots, \sigma^n$ is an o.n basis for 1-forms there and also

$$\sigma^{\underset{I}{\rightarrow}} = \sigma^{i_1} \wedge \sigma^{i_2} \wedge \dots \wedge \sigma^{i_p}$$

gives an o.n basis for p-forms there.

We now introduce a <u>global</u> or <u>Hilbert space</u> scalar product by

$$(\alpha^p, \beta^p) \equiv \int_M \langle \alpha^p, \beta^p \rangle \, \mathrm{vol} \tag{48.2}$$

whenever this makes sense (always, if M is compact, also if α or β has compact support)

Define now, for each p-form α^p an $(n-p)$ form $*\alpha$ by

$$\left.\begin{aligned} *\alpha^p &= \alpha^*_{\underset{J}{\rightarrow}} \, dx^J \\ \alpha^*_{\underset{J}{\rightarrow}} &= \sqrt{|g|} \, \alpha^K \, \varepsilon_{\underset{K}{\rightarrow} \underset{J}{\rightarrow}} \end{aligned}\right\} \tag{48.3}$$

i.e

$$\alpha^*_{j_1 \dots j_{n-p}} = \sqrt{|g|} \, \alpha^{k_1 \dots k_p} \, \varepsilon_{\underset{k_1 \dots k_p}{\rightarrow} j_1 \dots j_{n-p}}$$

Note that for a given $j_1 < \dots < j_{n-p}$ there is at most <u>one</u> non-vanishing term in the sum on the right, namely when $k_1 < \dots < k_p$ is a complementary multi-index to $j_1 \dots j_{n-p}$.

Then

$$\alpha^p \wedge *\beta^p = (\alpha \wedge *\beta)_{12\cdots n}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

and

$$\begin{aligned}(\alpha^p \wedge *\beta^p)_{12\cdots n} &= \delta^{JK}_{12\cdots n}\, \alpha_{\underrightarrow{J}} (*\beta)_{\underrightarrow{K}} \\ &= \varepsilon^{JK}\, \alpha_{\underrightarrow{J}} \sqrt{|g|}\, \beta^L\, \varepsilon_{\underrightarrow{L}\,\underrightarrow{K}} \\ &= \sqrt{|g|}\, \alpha_{\underrightarrow{J}} \beta^J \qquad \text{(why?)} \\ &= \sqrt{|g|}\, \langle \alpha, \beta \rangle\end{aligned}$$

and so

$$\boxed{\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, vol^n} \tag{48.4}$$

In particular this shows us that

$$* : p\text{-forms} \longrightarrow \text{pseudo } (n-p) \text{ forms}$$

and conversely.

For 0-forms, i.e. functions, we define

$$*f = f \cdot vol^n$$

and in particular, for the constant function 1

$$\boxed{*1 = vol^n}$$

If $e_1, \ldots, e_n$ is an o.n basis (allowing $\|e_1\|^2 = -1$ if pseudo-Riemannian), then $\sigma^1, \ldots, \sigma^n$ is o.n basis, and one easily sees that, since

$$\sigma^1 \wedge \sigma^2 \wedge \cdots \wedge \sigma^n = \pm\, vol^n$$

we have

$$*(\sigma^{i_1} \wedge \cdots \wedge \sigma^{i_p}) = \pm (\sigma^{j_1} \wedge \cdots \wedge \sigma^{j_{n-p}}) \tag{48.5}$$

where J is complementary to I. By looking at $\alpha = \sigma^{\underrightarrow{I}}$ you should have no trouble in showing

Problem 84

$$*(*\alpha^p) = \begin{cases} (-1)^{p(n-p)}\alpha & \text{if } M \text{ Riemannian} \\ -(-1)^{p(n-p)}\alpha & \text{if } M \text{ pseudo-Riemannian} \end{cases}$$

Problem 85 Show that in Minkowski space M^4_0

$$*\mathcal{F} = *(\mathcal{E}^1 \wedge dt + \mathcal{B}^2) = -\mathcal{H} \wedge dt + \mathcal{D}^1$$

(this explains the notation used in §4 of Lecture 30)
(Hint: for a computation like this it is simplest just to use (48.4) and (48.5)).

Next note that if X is a vector and $\mathcal{X}$ its associated 1-form then

$$i_X \text{vol}^m = i_X \left(\sqrt{|g|}\, \varepsilon_{\underrightarrow{i_1 \cdots i_m}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_m} \right)$$

$$= \sqrt{|g|}\, X^j \varepsilon_{j\, \underrightarrow{i_2 \cdots i_m}}\, dx^{i_2} \wedge \cdots \wedge dx^{i_m}$$

$$= (*\mathcal{X}^1)_{\underrightarrow{i_2 \cdots i_m}}\, dx^{i_2} \wedge \cdots \wedge dx^{i_m}$$

i.e

$$\boxed{i_X \text{vol}^m = *\mathcal{X}^1} \qquad (48.6)$$

(2) The Co-differential Operator $\delta = d^*$. We shall define the Hilbert space adjoint d^* to the operator $d : \Lambda^p(M^n) \longrightarrow \Lambda^{p+1}(M^n)$. We want

$$(d\alpha^{p-1}, \beta^p) = (\alpha^{p-1}, d^*\beta^p) \qquad (48.7)$$

at least if M is compact, or if α or β has compact support. Consider the Riemannian case first. Now from (48.4)

$$\langle d\alpha, \beta \rangle * 1 = d\alpha \wedge *\beta.$$

But

$$d(\alpha^{p-1} \wedge *\beta^p) = d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1} \alpha \wedge d*\beta^p$$

$$= d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1} (-1)^{(n-p+1)(p-1)} \alpha \wedge **d*\beta$$

$$= d\alpha \wedge *\beta + (-1)^{(p-1)(n-p)} \alpha \wedge **d*\beta$$

$$= d\alpha \wedge *\beta + (-1)^{n(p+1)} \alpha \wedge *(*d*\beta)$$

and so

$$d\alpha^{p-1} \wedge *\beta^p = (-1)^{n(p+1)+1} \alpha \wedge *(*d*\beta^p) + d(\alpha^{p-1} \wedge *\beta^p) \quad (48.8)$$

Then if α or β has compact support (for example if M^m is a closed manifold)

$$(d\alpha^{p-1}, \beta^p) = (\alpha^{p-1}, (-1)^{n(p+1)+1} * d * \beta^p) + \int_M d(\alpha \wedge *\beta) \quad (48.9)$$

$$= (\alpha^{p-1}, d^*\beta^p)$$

where

$$d^*\beta^p = \begin{cases} (-1)^{n(p+1)+1} * d * \beta, & \text{Riemannian} \\ (-1)^{n(p+1)} * d * \beta, & \text{pseudo-Riemannian} \end{cases} \quad (48.10)$$

(why?)

We shall need to have a coordinate expression for d^*

$$\boxed{(d^*\beta^p)_K = -\nabla_j \beta^j{}_K = -\beta^j{}_{K|j}} \quad (48.11)$$

ie

$$(d^*\beta)_{k_2 \cdots k_p} = -\beta^j{}_{k_2 \cdots k_p | j}$$

(note that

$$d^* : p\text{-forms} \longrightarrow (p-1)\text{ forms}$$

not $(p-1)$ pseudo forms).

We shall call the right hand side of (48.11) the negative of the divergence of the form β

$$(\mathrm{Div}\, \beta^p)_K \equiv -\nabla_j \beta^j{}_K \quad (48.12)$$

We shall verify (48.11) by showing that

$$(\alpha^{p-1},\, d^*\beta^p) = (\alpha^{p-1},\, -\mathrm{Div}\,\beta^p)$$

holds for all compactly supported $(p-1)$ forms α; surely if

$$\int_M \alpha_{\underset{J}{\rightarrow}}\, \gamma^J \mathrm{vol} = \int_M \alpha_{\underset{J}{\rightarrow}}\, \gamma'^J \mathrm{vol}$$

for all forms α then $\gamma = \gamma'$!

Now $(\alpha^{p-1},\, d^*\beta^p) = \int_M \langle d\alpha^{p-1},\, \beta^p\rangle * 1$ and

$$\langle d\alpha^{p-1},\, \beta^p\rangle = (d\alpha)_{\underset{I}{\rightarrow}}\, \beta^I = (\delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \partial_j \alpha_{\underset{K}{\rightarrow}})\beta^I$$

We have mentioned a number of times that in $(d\alpha)_I$ we may use covariant derivatives

$$\boxed{(d\alpha)_{\underset{I}{\rightarrow}} = \delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \nabla_j \alpha_{\underset{K}{\rightarrow}}}\,; \qquad (48.13)$$

Since the connection is symmetric and the α_K's are skew symmetric the terms involving the Christoffel symbols will disappear! Then

$$\langle d\alpha^{p-1},\, \beta^p\rangle = (\delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \nabla_j \alpha_{\underset{K}{\rightarrow}})\beta^I$$

$$= \nabla_j(\delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \alpha_{\underset{K}{\rightarrow}}\beta^I) - \alpha_{\underset{K}{\rightarrow}}\, \nabla_j(\delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \beta^I)$$

But

$$\delta^{j\underset{K}{\rightarrow}}_{\underset{I}{\rightarrow}}\, \beta^I = \beta^{j\underset{K}{\rightarrow}} \quad \text{(why?)}$$

and so

$$\langle d\alpha^{p-1},\, \beta^p\rangle = \nabla_j(\alpha_{\underset{K}{\rightarrow}}\, \beta^{jK}) - \alpha_{\underset{K}{\rightarrow}}\, (\nabla_j \beta^{jK})$$

Thus

$$\langle d\alpha^{p-1}, \beta^p \rangle = \nabla_j X^j - \alpha_{\underset{K}{\rightarrow}} (\mathrm{Div}\,\beta)^K \tag{48.14}$$

where

$$X^j = \alpha_{\underset{K}{\rightarrow}} \beta^{j\underset{K}{\rightarrow}} = \frac{1}{(p-1)!}\, \alpha_K \beta^{jK}$$

are the components of a contravariant vector X. But if α has compact support, so does X and

$$\int_M \nabla_j X^j * 1 = \int_M \mathrm{div}\, X \, \mathrm{vol}^m = 0$$

by the divergence theorem. We conclude from (48.14) that

$$(\alpha^{p-1}, d^*\beta^p) = (d\alpha^{p-1}, \beta^p) = \int_M \langle d\alpha^{p-1}, \beta^p \rangle * 1$$

$$= -\int_M \alpha_{\underset{K}{\rightarrow}} (\mathrm{Div}\beta)^K * 1 = (\alpha^{p-1}, -\mathrm{Div}\beta^p) \qquad \square$$

(3) Manifold with Boundary. Note that if M^m is a compact manifold with boundary, then (48.9) gives, via Stokes' theorem

$$(d\alpha^{p-1}, \beta^p) - (\alpha^{p-1}, d^*\beta^p) + \int_{\partial M} \alpha^{p-1} \wedge *\beta^p \tag{48.15}$$

This, in turn, is a complicated way of writing

$$\int_{M^m} d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1} \int_{M^m} \alpha^{p-1} \wedge d*\beta^p = \int_{\partial M} \alpha^{p-1} \wedge *\beta^p \tag{48.16}$$

which is an immediate consequence of

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d*\beta$$

Note that if α and β are true forms, then $\alpha \wedge *\beta$ is a pseudo form, and so (48.15) and (48.16) are true even if M^m is not orientable!

Lecture 49. Harmonic Forms on Closed Riemannian Manifolds

(1) The Laplace operator Δ on Forms. Since $d^* = \pm * d *$ and $** = \pm 1$, we see that

$$d^* d^* = 0$$

and $(d + d^*)^2 = (d+d^*)(d+d^*) = dd^* + d^*d$. We define this to be the Laplace operator on forms

$$\Delta \equiv dd^* + d^*d$$

$$\Delta : \Lambda^p(M^m) \longrightarrow \Lambda^p(M^m)$$

- Problem 86. In $\underline{R}^3$, show the following

(i) $d^* f^0 = 0$

(ii) $d^* \alpha^1 \longleftrightarrow -\operatorname{div} \underline{A}$

(iii) $d^* \beta^2 = d^*(i_{\underset{\sim}{B}} vol^3) = *(i_{curl \underset{\sim}{B}} vol^3)$ is the 1-form version of $curl \underset{\sim}{B}$.

(iv) $d^* \gamma^3 = d^*(*f) = -i_{grad f} vol^3$ is the 2-form version of $-grad f$.

(v) $\Delta f^0 = -\nabla^2 f$

(vi) $\Delta \alpha^1 \longleftrightarrow curl\, curl \underset{\sim}{A} - grad\, div \underset{\sim}{A}$ (usually called $-\nabla^2_{\underset{\sim}{A}}$).

(vii) $\Delta \beta^2 \longleftrightarrow curl\, curl \underset{\sim}{B} - grad\, div \underset{\sim}{B}$.

(viii) $\Delta(*f) = -\nabla^2 f * 1$.

- Problem 87. Let F^p be a p-form on a (pseudo-) Riemannian M^m. Show

$$(Div\, F^p)^K = F^{jK}{}_{|j} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j}(\sqrt{|g|}\, F^{jK})$$

(Hint; see (45.6)).

(2) <u>Δ for a 1-form on a Riemannian M^n</u>. Let $\alpha = a_j\, dx^j$ be a 1-form. Consider $\Delta\alpha = d\,d^*\alpha + d^*\,d\alpha$.

$$d\alpha = \sum_{i<j} \left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}\right) dx^i \wedge dx^j$$

$$= \sum_{i<j} (a_{j|i} - a_{i|j})\, dx^i \wedge dx^j \equiv \sum_{i<j} c_{ij}\, dx^i \wedge dx^j$$

$$(d^*c)_j = -\, c^i{}_{j|i} = -\, a_j{}^{|i}{}_{|i} + a^i{}_{|ji}$$

where we put

$$a_j{}^{|i} \equiv \nabla^i a_j \equiv g^{ik} a_{j|k}.$$

Thus

$$(d^*d\,\alpha)_j = -\, a_j{}^{|i}{}_{|i} + a^i{}_{|ji} \qquad (4$$

Also

$$d^*\alpha = -\, a^l{}_{|l}$$

and so

$$d(d^*\alpha) = -\, a^l{}_{|lj}\, dx^j, \text{ i.e}$$

$$(d\,d^*\alpha)_j = -\, a^l{}_{|lj} \qquad (49.2$$

Consequently

$$(\Delta\alpha)_j = -\, a_j{}^{|i}{}_{|i} + a^i{}_{|ji} - a^i{}_{|ij} \qquad (49.3$$

But Ricci's identity (44.10) says (see (43.14))

$$a^i{}_{|ji} - a^i{}_{|ij} = a^k R^i{}_{kij} = a^k R_{kj} \qquad (49.4$$

where R_{kj} is the Ricci tensor. We conclude

$$\left.\begin{aligned}\Delta\alpha^1 &= (-a_j{}^{lk}{}_{lk} + a^k R_{kj})\,dx^j \\ &= (-g^{lk} a_{j/lk} + a^k R_{kj})\,dx^j\end{aligned}\right\} \qquad (49.5)$$

The first term

$$-g^{lk} a_{j/lk} = -\nabla^2(a_j)$$

is simply the negative of the ordinary Laplacian of the function a_j. The second term is a curvature contribution

$$\boxed{(\Delta\alpha^1)_j = -\nabla^2 a_j + a_k R^k{}_j} \qquad (49.6)$$

A form α^p on M^n is said to be a harmonic form if

$$\Delta\alpha = d d^*\alpha + d^* d\alpha = 0$$

In particular, in R^n with the usual metric, α^1 is harmonic iff each component a_j is a harmonic function on R^n.

(3) Δ on a Closed, Riemannian M^n. Let $(\,,\,)$ again denote the Hilbert space scalar product for p-forms

$$(\alpha^p, \beta^p) = \int_{M^n} \alpha^p \wedge *\beta^p = \int_{M^n} \langle \alpha^p, \beta^p \rangle *1$$

$$= \int_{M^n} a_{\underset{\rightarrow}{I}}\, b^I \, vol^n.$$

When M^n is Riemannian (i.e. not pseudo) $(\alpha^p, \alpha^p) \geq 0$ and $= 0$ iff $\alpha^p \equiv 0$ (for, at the pole of geodesic coordinates

$$a_{\underline{I}}\, a^{I} = \sum_{i_1 < \cdots < i_p} (a_{i_1 \cdots i_p})^2 \quad).$$

When M^n is closed, ie a compact manifold without boundary, d^* is the adjoint of d, and so

$$\begin{aligned}(\Delta\alpha, \alpha) &= (dd^*\alpha + d^*d\alpha, \alpha) \\ &= (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha) \\ &= \| d^*\alpha \|^2 + \| d\alpha \|^2 \geq 0\end{aligned}$$

and so we conclude

<u>On a closed Riemannian M^n, α^p is harmonic</u>

$$\Delta\alpha^p = 0$$

<u>iff</u> α <u>is closed and "co-closed"</u>

$$d\alpha = 0 \quad \& \quad d^*\alpha = 0$$

(49)

A form satisfying $d\alpha = 0$ and $d^*\alpha = 0$ on any manifold is sometimes called a harmonic <u>field</u>. This condition is stronger than being a harmonic <u>form</u>, $\Delta\alpha = 0$ in non compact M^n. For example, the only harmonic 0-form <u>fields</u> in R^n are constants, since

$$df^0 = 0 \text{ says } f = \text{constant}.$$

In Euclidean space R^n with the usual metric the Laplace operator Δ is an <u>elliptic</u> operator; it is simply the negative of the usual ∇^2 operating on each component. Δ is elliptic on any <u>Riemannian</u> M^n. However, in Minkowski space, Δf, say, means

$$\Delta f = \frac{\partial^2 f}{\partial t^2} - \nabla^2 f$$

ie Δ is the d'Alembertian operator, which is hyperbolic. To apply elliptic theory we need to consider only Riemannian M^n! The proof of the following is too long and difficult to be given here.

The Fundamental Theorem (Hodge's Theorem)
Let M^n be a closed Riemannian manifold. Then the vector space of Harmonic p-forms

$$\mathcal{H}^p(M^n) = \{h \in \Lambda^p(M^n) \mid dh = d^*h = 0\}$$

is finite dimensional and the equation

$$\Delta \alpha^p = \rho^p \tag{49.8}$$

has a solution α^p iff ρ^p is orthogonal (in the Hilbert space sense) to the space of harmonic p-forms

$$(\rho^p, h^p) = 0 \text{ , all } h^p \in \mathcal{H}^p$$

For example, if f^0 is a harmonic function on a closed M^n, then $df = 0$ and f is a constant. Thus

$$\mathcal{H}^0(M^n) = \text{space of constant fctns}$$

is a 1 dimensional vector space with basis given by the constant $h^0 = 1$. Hodge's theorem states that if we are given a function ρ^0 such that

$$0 = (\rho^0, 1) = \int_{M^n} \rho \, \text{vol}^n = \int_{M^n} \rho * 1$$

then one can always solve

$$\nabla^2 f = \rho$$

for a function f; ie Poisson's equation is always soluble if the integral of the "density" ρ over M^n vanishes. It is clear that the condition

$$\int_{M^n} \rho * 1 = 0$$

is necessary, for if $\Delta f = \rho$

$$(\rho, 1) = (\Delta f, 1) = (d^*df, 1) = (df, d1) = 0 \, !$$

(this is simply a fancy way of using the divergence theorem)

Suppose now that ρ^p is an arbitrary p-form on the closed M^m. Let $h^p_{(1)}, \dots, h^p_{(r)}$ be a basis for the harmonic p-forms. Then

$$\rho^p - \sum_i (\rho^p, h^p_{(i)})\, h^p_{(i)}$$

is clearly orthogonal to $\mathcal{H}^p$ and so we can solve

$$\Delta \alpha^p = \rho^p - h^p$$

for α^p, where $h^p = \sum_i (\rho^p, h^p_{(i)}) h^{(p)}_i$ is the harmonic part of ρ^p. Thus

$$d(d^* \alpha^p) + d^*(d\alpha^p) = \rho^p - h^p$$

can be solved for α. In other words, for any ρ^p on M^m we can write

$$\boxed{\rho^p = d(d^*\alpha^p) + d^*(d\alpha^p) + h^p} \tag{49}$$

for some p-form α. This is of the form

$$\rho^p = d\mu^{p-1} + d^*\nu^{p+1} + h^p$$

{ This generalizes the Helmholtz decomposition of a vector field in R^3

Note that the three parts are mutually orthogonal

$$(d\mu, d^*\nu) = (\mu, d^*d^*\nu) = 0$$
$$(d\mu, h) = (\mu, d^*h) = 0$$
$$(d^*\nu, h) = (\nu, dh) = 0.$$

We have derived the Hodge decomposition theorem

We have, on the closed M^m an orthogonal decomposition

$$\Lambda^p = d\Lambda^{p-1} \oplus d^*\Lambda^{p+1} \oplus \mathcal{H}^p.$$

Thus we have a unique decomposition

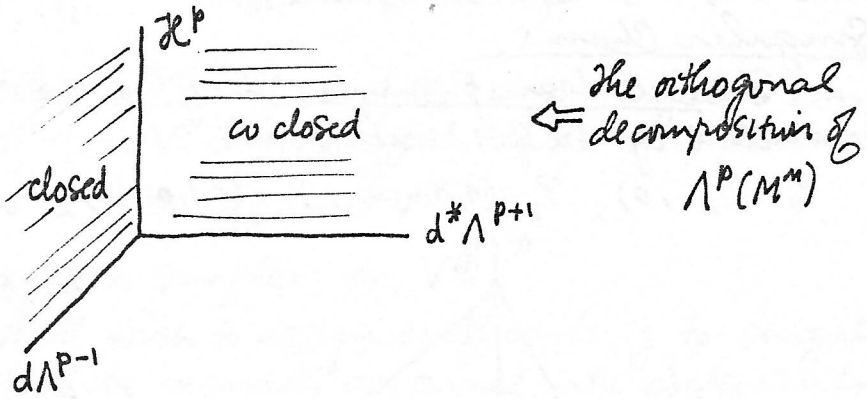
$$\rho^p = d\mu^{p-1} + d^*\nu^{p+1} + h \tag{49}$$

Uniqueness is immediate

$$0 = d(\mu - \mu') + d^*(\nu - \nu^*) + (h - h')$$

gives, using orthogonality

$$\|h - h'\|^2 = (h - h', h - h') = 0, \quad \text{etc. etc.}$$



⇐ The orthogonal decomposition of $\Lambda^p(M^n)$

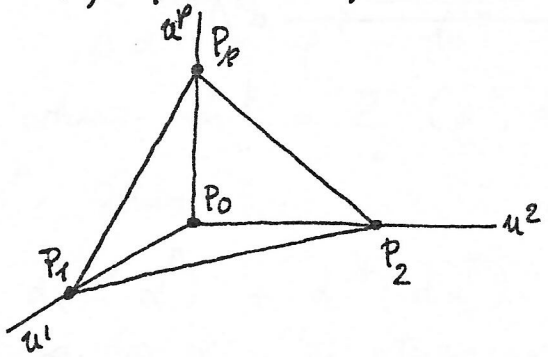
Lecture 50. Singular Chains and Their Boundaries

We shall be concerned with global properties of fields, and also with certain boundary value problems. For both subjects we shall need to know more about "algebraic topology" than was discussed in Lecture 22.

(1) Singular Chains

The Euclidean p-simplex in $\mathbb{R}^p$ is the convex set $\Delta_p \subset \mathbb{R}^p$ generated by the $(p+1)$ unit points

$$P_0 = (0, \cdots, 0),\ P_1 = (1, 0 \cdots, 0),\ P_2 = (0, 1, 0 \cdots, 0),\ \cdots,\ P_p\ (0, 0, \cdots, 1)$$

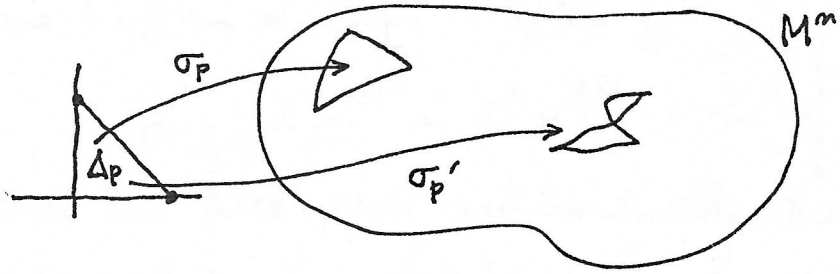


We shall write

$$\Delta_p = (P_0, P_1, \cdots, P_p)$$

A singular p-simplex in a manifold M^n is a differentiable map

$$\sigma_p : \Delta_p \longrightarrow M^n$$



Note that a singular simplex is a special case of a "parameterizable subset" of M^n (see § 2 of Lecture 13). These are natural objects over which one integrates forms on M^n via the pull back σ_p^*.

Note that the k^{th} face of Δ_p

$$\Delta_{p-1}^{(k)} = (P_0, \cdots, \hat{P}_k, \cdots P_p) \tag{50.1}$$

is not a standard Euclidean simplex because it is sitting in $\mathbb{R}^p$ instead of $\mathbb{R}^{p-1}$. However, we shall consider it as a singular $(p-1)$ simplex of $\mathbb{R}^p$ under the map

$$f_k : \Delta_{p-1} \longrightarrow \Delta_p \subset \mathbb{R}^p$$

where f_k is the unique <u>affine</u> map that assigns

$$\begin{array}{ccccccccc} P_0 & & P_1 & \cdots & P_{k-1} & & P_k & \cdots & P_{p-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_0 & & P_1 & \cdots & P_{k-1} & & P_{k+1} & & P_p \end{array}$$

Note that if $\sigma : \Delta_p \longrightarrow M^m$ is a singular simplex of M^m and if $\varphi : M^m \longrightarrow V^r$ is a differentiable map, then the composition

$$\varphi_\# \sigma \equiv \varphi \circ \sigma : \Delta_p \longrightarrow V^r$$

defines a singular simplex on V^r.

We shall wish to assign coefficients g to singular simplexes. While we shall be mainly concerned with coefficients g that are real numbers, other coefficient "groups" will arise. Thus let G be an arbitrary <u>abelian</u> (= commutative) <u>group</u>. The main groups of interest to us are

$$G = \begin{cases} \mathbb{Z} \text{ ; the group of integers} \cdot n \\ \mathbb{R} \text{ ; the additive group of real numbers } r \\ \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \text{ the group of integers mod 2} \end{cases}$$

the notation $\mathbb{Z}/2\mathbb{Z}$ meaning that in the group $\mathbb{Z}$ of integers we <u>shall identify any two integers that differ by an even integer</u>. Thus $\mathbb{Z}_2$ consists of merely two elements

$$\mathbb{Z}_2 = \{\breve{0}, \breve{1}\} \qquad \begin{array}{l} \breve{0} \text{ is the equivalence class of } 0, \pm 2, \pm 4, \pm 6, \cdots \\ \breve{1} \text{ is } \quad " \quad \quad " \quad \quad " \quad " \quad \pm 1, \pm 3, \pm 5, \cdots \end{array}$$

with addition law

$$\breve{0} + \breve{0} = \breve{0}, \quad \breve{0} + \breve{1} = \breve{1}, \quad \breve{1} + \breve{1} = \breve{0}$$

and we usually write $\{0, 1\}$, leaving off the hats $\breve{}$. One can likewise consider

$$\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$$

the group of integers mod p.

A singular p-chain on M^n with coefficient group G is a finite formal sum

$$c_p = \sum_\ell g_\ell \, \sigma_p^\ell$$

where each σ_p^ℓ, $\ell = 1, \dots, R$, is a singular simplex and each $g_\alpha \in G$. The collection

$$C_p(M^n; G) = \text{all singular p-chains on } M^n$$

is in an obvious way itself an abelian group (we all know how to add formal sums; for example the zero element of $C_p(M^n; G)$ is the singular p-chain on M^n that assigns the group element $0 \in G$ to each singular p-simplex σ).

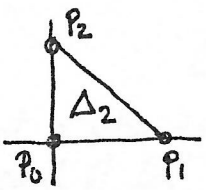
$$C_p(M^n; G) \equiv \text{the singular p-chain group on } M^n \text{ with coefficient group } G$$

Singular p-chains with real coefficients, $G = \mathbb{R}$ arise because they are domains of integration for p-forms on M^n

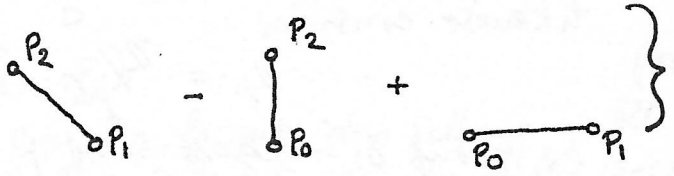
$$\int_{\sum_\ell r_\ell \sigma_p^\ell} \alpha^p \equiv \sum_\ell r_\ell \int_{\sigma_p^\ell} \alpha^p \equiv \sum_\ell r_\ell \int_{\Delta_p} (\sigma_p^\ell)^* \alpha^p$$

Back to the Euclidean p-simplex $\Delta_p = (P_0, \dots, P_p)$. We define its boundary $\partial \Delta_p$ to be the singular $(p-1)$ chain, integer coefficients on $\Delta_p \subset \mathbb{R}^p$ defined by

$$\partial \Delta_p = \sum_{k=0}^{p} (-1)^k (P_0, \dots, \hat{P}_k, \dots, P_p) = \sum_{k=0}^{p} (-1)^k \Delta_{p-1}^{(k)}$$



$$\partial \Delta_2 = \text{“}\left\{ \right.$$



$$\left. \right\}$$

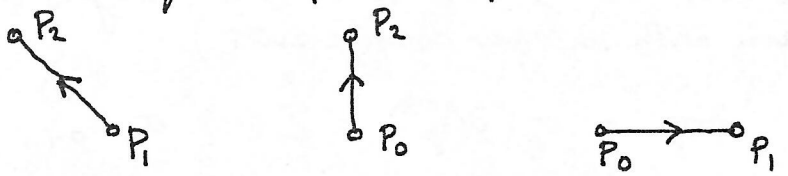
By definition $\partial \Delta_0 = 0$, as the boundary of a vertex is 0.

Note that if α^{p-1} is a $p-1$ form on Δ^p, then

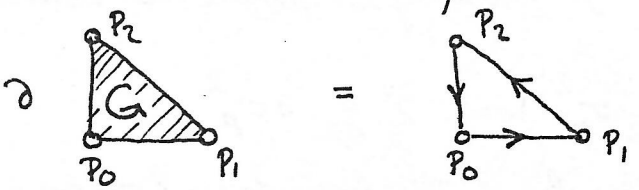
$$\int_{\partial\Delta_p} \alpha^{p-1} = \sum_{k=0}^{p} (-1)^k \int_{\Delta_{p-1}^{(k)}} \alpha^{p-1}$$

is well defined because each $\Delta_{p-1}^{(k)}$ is a specific map of the oriented Euclidean $(p-1)$ simplex Δ_{p-1}

Each of the faces is "ordered" by the order of their vertices; since each face is a diffeomorphic copy of the basic oriented $(p-1)$ simplex Δ_{p-1}, we see that the faces are also oriented objects. For example in $\partial\Delta_2$

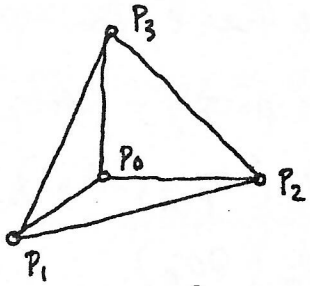


Then the coefficients ± 1 in $\partial\Delta_2$ effectively preserve or reverse those orientations. Pictorially

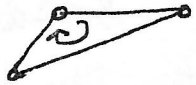


In the three dimensional case

$$\partial(P_0,P_1,P_2,P_3) = (P_1,P_2,P_3) - (P_0P_2,P_3) + (P_0P_1P_3) - (P_0,P_1,P_2)$$



the orientation of the bottom face (P_0,P_1,P_2) inherited from the face map $f_3 : \Delta_2 \to \Delta_3$ is

Then $-(P_0,P_1,P_2)$ is interpreted as oriented as 

, which is the correct orientation as given by the outward pointing normal to Δ_3 (See Lecture 15, § 1), using right hand orientation of $\mathbb{R}^3$, (the

right-hand orientation is that of $\vec{P_0P_1}, \vec{P_0P_2}, \vec{P_0P_3}$).

If $\varphi : M^n \longrightarrow V^r$, then we have an induced <u>homomorphism</u> of groups

$$\varphi_\# : C_p(M;G) \longrightarrow C_p(V;G)$$

defined by

$$\varphi_\#\left(\sum_\ell g_\ell \sigma_p^\ell\right) = \sum_\ell g_\ell \varphi_\# \sigma_p^\ell \qquad (5\text{-}4)$$
$$= \sum_\ell g_\ell \varphi\circ\sigma_p^\ell$$

If $\sigma_p : \Delta_p \to M^n$ is a singular p-simplex on M^n we define its boundary $\partial\sigma_p$ to be the integer $(p-1)$ chain (i.e., chain with integer coefficients)

$$\partial\sigma_p = \sigma_\#(\partial\Delta_p) = \sum_{k=0}^{p} (-1)^k \sigma_\# \Delta_{p-1}^{(k)}$$

(ie, crudely, "the boundary of the image of Δ_p is the image of the boundary of Δ_p"). Finally we define the boundary of any p-chain by

$$\partial\left(\sum_\ell g_\ell \sigma_p^\ell\right) \equiv \sum_\ell g_\ell \partial\sigma_p^\ell \qquad (5\text{a}.5)$$

By construction we then have the <u>boundary homomorphism</u>

$$\boxed{\partial : C_p(M^n;G) \longrightarrow C_{p-1}(M^n;G)} \qquad (5\text{b}.6)$$

By construction, we also have, for $\varphi : M^n \longrightarrow V^r$

$$\partial(\varphi_\#\sigma_p) = \partial(\varphi\circ\sigma_p) = (\varphi\circ\sigma_p)_\#(\partial\Delta_p)$$
$$= \varphi_\#(\sigma_{p\#}(\partial\Delta_p))$$
$$= \varphi_\#(\partial\sigma_p)$$

ie.

$$\boxed{\partial\,\varphi_\# = \varphi_\#\,\partial} \qquad (5\text{c}.7)$$

("the boundary of a map is the map of the boundary"). We can then say that we have a "commutative diagram"

$$\begin{array}{ccc} C_p(M;G) & \xrightarrow{\varphi_\#} & C_p(V;G) \\ \partial \downarrow & & \downarrow \partial \\ C_{p-1}(M,G) & \xrightarrow{\varphi_\#} & C_{p-1}(V;G) \end{array}$$

i.e., for each $c_p \in C_p(M;G)$, $\partial \circ \varphi_\#(c_p) = \varphi_\# \circ \partial(c_p)$.

Proposition. $\partial^2 = \partial \circ \partial = 0$ (508)

Proof. Consider first a Euclidean simplex Δ_p. Then

$$\partial \Delta_p = \partial(P_0, \cdots, P_p) = \sum_{k=0}^{p} (-1)^k (P_0, \cdots, \hat{P}_k, \cdots P_p)$$

$$\partial(\partial \Delta_p) = \sum_{k=0}^{p} (-1)^k \partial(P_0, \cdots, \hat{P}_k, \cdots P_p)$$

$$= \sum_{k=0}^{p} (-1)^k \sum_{j<k} (-1)^j (P_0, \cdots, \hat{P}_j, \cdots \hat{P}_k, \cdots P_p)$$

$$+ \sum_{k=0}^{p} (-1)^k \sum_{j>k} (-1)^{j-1} (P_0, \cdots, \hat{P}_k, \cdots \hat{P}_j, \cdots P_p)$$

$$= 0 \quad \text{(cancellation in pairs)}$$

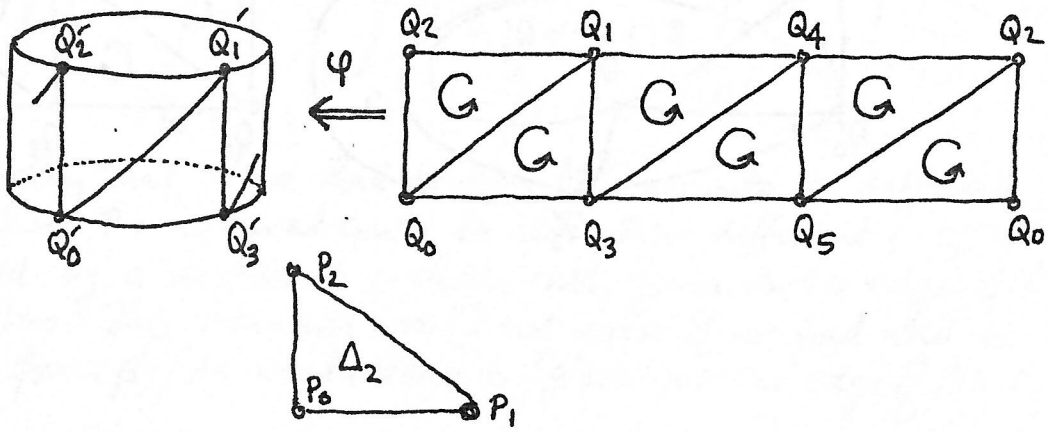
Next, for a singular simplex

$$\partial(\partial \sigma_p) = \partial(\sigma_\#(\partial \Delta_p)) = (\text{from (50.7)})$$

$$= \sigma_\# \partial(\partial \Delta_p) = \sigma_\# 0 = 0 \qquad \square$$

(2) Some Examples

A cylinder E^2 is a rectangular band with two vertical edges sewn together



The singular simplex (Q_0', Q_1', Q_2') for example results from the affine map

$$(P_0, P_1, P_2) \longrightarrow (Q_0, Q_1, Q_2)$$

followed by the slight bending into (Q_0', Q_1', Q_2') etc. We exhibit an integer 2-chain on $\mathcal{C}^2$. It is the image under the the bending $\varphi_\#$ of the following chain on the flat band

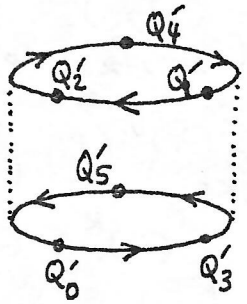
$$c_2 = (Q_0, Q_1, Q_2) - (Q_0\, Q_1\, Q_3) + (Q_1 Q_3\, Q_4) - (Q_3\, Q_4\, Q_5)$$
$$+ (Q_2\, Q_4\, Q_5) + (Q_0, Q_2, Q_5)$$

Compute that

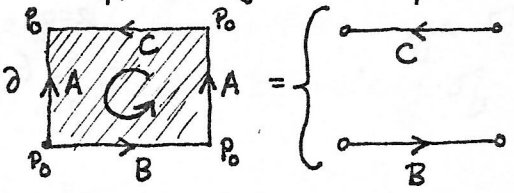
$$\partial c_2 = (Q_0 Q_3) + (Q_3 Q_5) - (Q_0 Q_5)$$
$$+ (Q_2 Q_4) - (Q_1 Q_4) + (Q_1 Q_2)$$

i.e

$$\varphi_\# \partial c_2 =$$

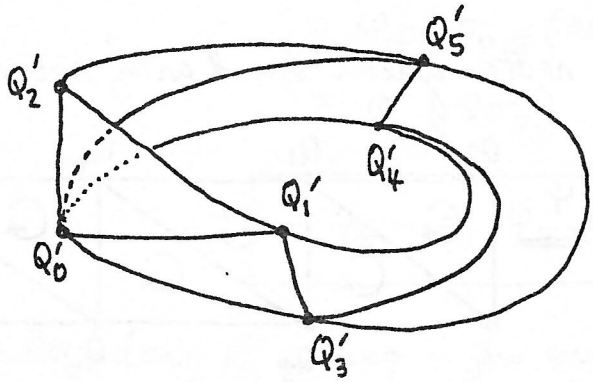


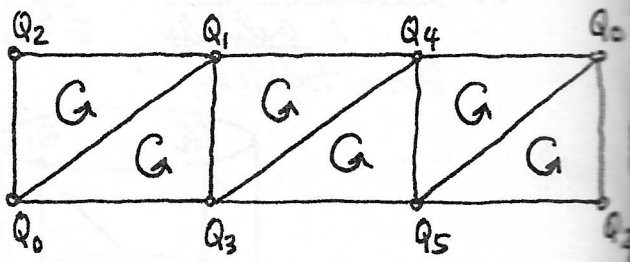
Briefly, omitting the triangles



which coincides with the usual notion of the boundary of the cylinder (with correct orientations). The fact that only the edges of $\mathcal{C}^2$ are left in ∂c_2 is a reflection of the fact that the cylinder $\mathcal{C}^2$ is orientable.

For a second example we consider a Möbius band $M\ddot{o}^2$

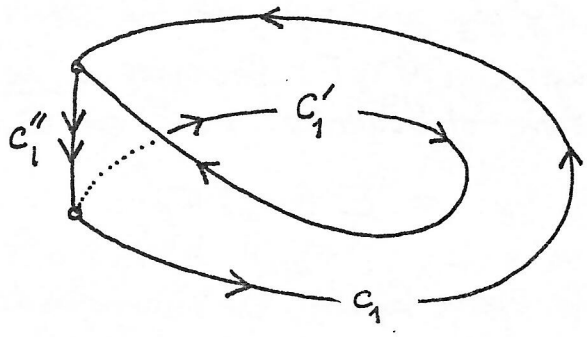




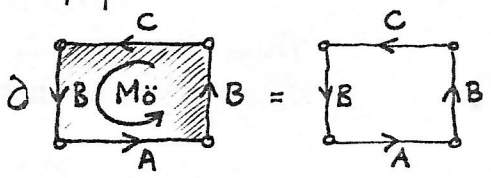
$$c_2 = (Q_0 Q_1 Q_2) - (Q_0 Q_1 Q_3) + (Q_1 Q_3 Q_4) - (Q_3 Q_4 Q_5)$$
$$+ (Q_0 Q_4 Q_5) - (Q_0 Q_2 Q_5)$$

Compute

$$\partial c_2 = (Q_0 Q_3) + (Q_3 Q_5) - (Q_2 Q_5)$$
$$+ (Q_0 Q_4) - (Q_1 Q_4) + (Q_1 Q_2)$$
$$- 2(Q_0 Q_2)$$



Briefly



$$\partial[M^2_{\ddot{o}}] = A+B+C+B = A+C+2B$$

Thus the boundary of c_2 consists of half of the single edge traversed in one direction, the other half traversed in the opposite direction, together with twice a 1-simplex $Q_0 Q_2$ which isn't even on the edge of Mö !! This is a reflection of the fact that Mö is not orientable.

$$\partial(\varphi_\# c_2) = c_1 + c_1' + 2c_1''.$$

If α' is any 1-form on $M^2_{\ddot{o}}$ (a true 1-form, not a pseudo form), then Stokes' theorem gives

$$\int_{\varphi_\# c_2} d\alpha' = \int_{\partial \varphi_\# c_2} \alpha' = \int_{c_1} \alpha' + \int_{c_1'} \alpha' + 2\int_{c_1''} \alpha'' \quad !$$

Note especially that if we had broken $M^2_{\ddot{o}}$ up into a different 2-chain then the integral could be altogether different; c_1'' would be replaced by a different 1-chain not lying on the edge of $M^2_{\ddot{o}}$. This is useless! This situation would not arise if we had used a pseudo 1-form β'; $\int d\beta'$ would reduce to $\int \beta'$ over just the edge of Mö !

Lecture 51. Singular Homology Groups

(1) **Cycles, Boundaries, and Homology.** We have defined the (singular) p-chain group for M^n with coefficients in the abelian group G, $C_p(M^n; G)$ and the boundary homomorphism

$$\partial : C_p(M^n; G) \longrightarrow C_{p-1}(M^n; G)$$

with $\partial^2 = \partial \circ \partial = 0$.

We remark that if the coefficient group G is actually a field, i.e., if besides the additive structure G also possesses a multiplicative structure for which each $0 \neq g \in G$ has a multiplicative inverse, for example $G = \mathbb{R}$, or $G = \mathbb{Z}_2$ with multiplication

$$\check{0} \cdot \check{0} = \check{0}, \quad \check{0} \cdot \check{1} = 0, \quad \check{1} \cdot \check{1} = \check{1},$$

then the chain group $C_p(M^n; F)$ becomes a vector space. We have an obvious action of "scalars" $r \in F$ on $C_p(M^n; F)$ by

$$r \cdot \sum_\ell a_\ell \sigma_p^\ell = \sum_\ell (r a_\ell) \cdot \sigma_p^\ell \qquad a_\ell \in F$$

In this case the vector space $C_p(M^n; F)$ is infinite dimensional (there are "many" singular simplexes), and the boundary homomorphism ∂ is a linear transformation.

We define the (singular) p-cycles on M^n to be those p-chains with 0 boundary; the p-cycle group is the kernel of ∂

$$Z_p(M^n; G) = \ker \partial : C_p(M^n; G) \longrightarrow C_{p-1}(M^n; G)$$
$$\equiv \left\{ p \in C_p(M^n; G) \;\middle|\; \partial p = 0 \right\}.$$

When $G = F$ is a field, Z_p is again a vector subspace of C_p and is the "null space" of the linear transformation ∂. We shall consider some examples in but a moment.

We define the (singular) p-boundaries on M^n to be those p-chains that are boundaries of $(p+1)$ chains; the p-boundary group is the image of C_{p+1} under ∂

$$B_p(M^n; G) = \mathrm{Im}(\partial) : C_{p+1}(M^n; G) \xrightarrow{\partial} C_p(M^n; G)$$
$$\equiv \left\{ \beta_p \in C_p(M^n; G) \;\middle|\; \beta_p = \partial c_{p+1}, \text{ some } c_{p+1} \in C_{p+1}(M^n; G) \right\}$$

Since $\partial\beta_p = \partial(\partial c_{p+1}) = 0$ we see that B_p is a subgroup of Z_p

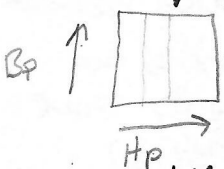
$$B_p(M^n; G) \subset Z_p(M^n; G)$$

(if G = field F, B_p is a vector subspace of Z_p). Every boundary is a cycle.

Finally, as in Lecture 22, we shall wish to identify two p-cycles z_p and z_p' if they differ by a p-boundary

$$z_p' \sim z_p \quad \text{iff} \quad z_p' - z_p = \partial c_{p+1}$$

for some $(p+1)$ chain c_{p+1}. Algebraically this is accomplished by forming the "quotient group" (it is really the "difference group" since we are dealing with the additive structure)



$$H_p(M^n; G) = \frac{Z_p(M^n; G)}{B_p(M^n; G)} \qquad \text{the } p^{\text{th}} \text{ homology group} \qquad (51.1)$$

When $G = F$ is a field $H_p(M^n; F)$ is a vector space. When $F = \mathbb{R}$ the dimension of this vector space is called the p^{th} Betti number written $b_p(M^n)$

$$b_p(M^n) = \dim H_p(M^n; \mathbb{R}). \qquad (51.2)$$

b_p can be infinite.

It is rather obvious from our definitions that the groups $H_p(M^n; G)$ are topological invariants, i.e. if $\varphi: M^n \to V^n$ is a diffeomorphism then

$$H_p(M^n; G) \approx H_p(V^n; G)$$

i.e their homology groups are isomorphic! This is because

$$\varphi_\# : C_p(M^n; G) \longrightarrow C_p(V^n; G)$$

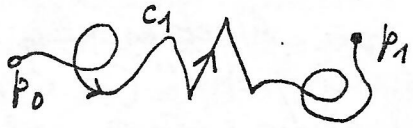
will be 1:1 and onto (why), and so will $Z_p(M^n; G) \xrightarrow{\varphi_\#} Z_p(V^n; G)$ ($\varphi_\#$ will take cycles into cycles), $B_p(M^n; G) \xrightarrow{\varphi_\#} B_p(V^n; G)$ ($\varphi_\#$ will take boundaries into boundaries). Thus

$$z_p' \sim z_p \quad \text{iff} \quad \varphi_\#(z_p') \sim \varphi_\#(z_p).$$

(2) Some Basic Tools for Computing First some important general tools.

By definition, any point $p \in M^n$ is a 0-cycle, since $\partial(p) = 0$.

Let us say that a manifold is (pathwise) connected if any pair of points can be joined by a piecewise smooth path (i.e., a 1-chain). Thus given p_0 and p_1, let c_1 be a path joining them, going from p_0 to p_1.



Then $\partial c_1 = (p_1) - (p_0)$ shows that $(p_1) \sim (p_0)$. It is also rather clear that p_0 is homologically non-trivial, ie. there is no 1-chain c_1 such that $\partial c_1 = (p_0)$. Likewise one can see that if g and g' are distinct coefficients then $g(p_0) \neq g'(p_0)$. Thus

> When M^n is connected, $H_0(M^n; G)$ is the group of all multiples $g(p_0)$ of a given point p_0. This group is isomorphic to G and is written Gp_0
>
> $$H_0(M^n; G) \approx Gp_0.$$

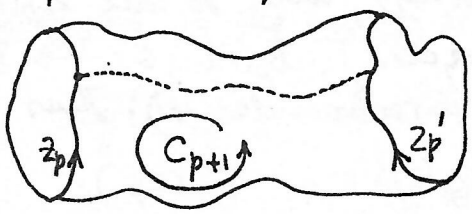
(51.3)

Next, we shall require a

> Deformation lemma. If a cycle z_p can be "deformed" into a cycle z_p', then z_p and z_p' are homologous, $z_p' \sim z_p$.

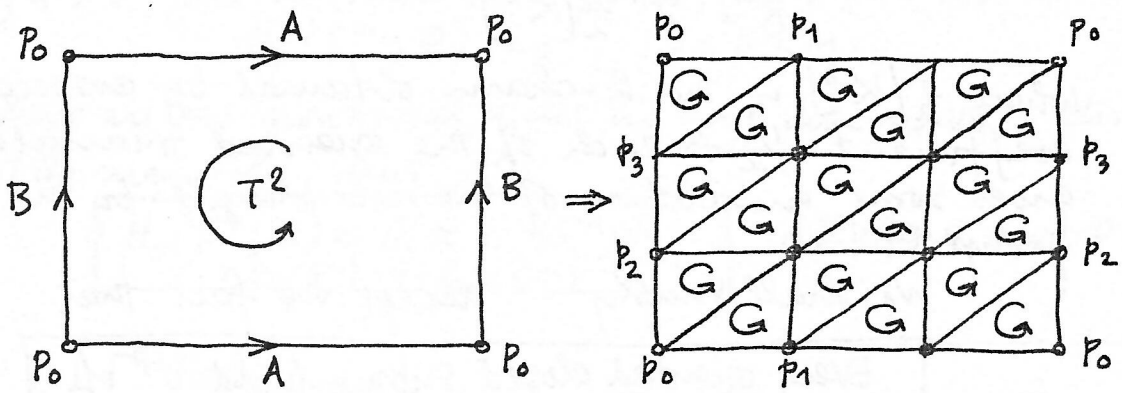
(51.4)

for it can be shown that in the process of deformation a "deformation" $(p+1)$ chain c_{p+1} is swept out



$$z_p' - z_p = \partial c_{p+1}$$

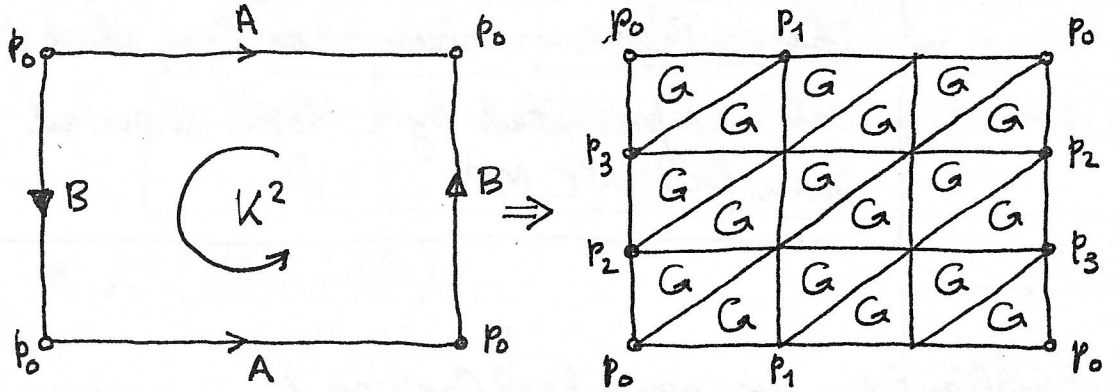
Next consider a p dimensional, oriented, compact manifold V^p without boundary. By triangulating V^p we can show that V^p always defines a p cycle. For example consider the torus



If we associate the integer 1 to each of the triangles of the triangulation of T^2 we shall have a singular 2-chain $[T^2]$ whose boundary will clearly be

$$\partial [T^2] = A + B - A - B = 0$$

and this will work for any orientable closed manifold V^p. Consider on the other hand a non-orientable closed manifold, the Klein bottle. It can be represented by (note arrows)



Orient each of the triangles as indicated and give to each such triangle the coefficient 1, yielding a singular 2-chain $[K^2]$

$$\partial [K^2] = A + B - A + B = 2B \neq 0.$$

$[K^2]$ is not a cycle, and this is a reflection of the fact that we have not been able to orient the triangles coherently; K^2 is not orientable. We note also the surprising fact that

B is a 1-cycle, B does not bound (using integer coefficients) but $2B$ does bound, $2B = \partial[K^2]$. Also note that if we had allowed ourselves to use real coefficients then B itself would bound

$$B = \partial\ \tfrac{1}{2}[K^2]$$

where $\frac{1}{2}[K^2]$ is the 2-chain obtained by associating the coefficient $\frac{1}{2}$ to each of the oriented triangles. This gives some indication of the role played by the coefficient group G!

We shall simply accept the fact that

> Every oriented closed submanifold V^p of an M^n defines a p-cycle by associating the same coefficient $g \in G$ to each p-simplex of a triangulation. We call this p-cycle $g[V^p]$. (51.5)

There is a deep converse to this result in the case of real coefficients

> Thom's Theorem: Every homology class $\in H_p(M^n; \mathbb{R})$ can be represented by a closed oriented submanifold $V^p \subset M^n$ (51.

(3) Familiar Examples, using Real Coefficients.

(a) S^n; The n-sphere. $H_0(S^n; \mathbb{R}) \approx \mathbb{R}$ since S^n is connected. $H_n(S^n; \mathbb{R}) \approx \mathbb{R}$, with $[S^n]$ a generator, since S^n is orientable. We could write

$$H_n(S^n; \mathbb{R}) = \mathbb{R}[S^n].$$

If z_p, $0<p<n$ is a cycle of S^n it is homologous (by Thom's Theorem) to an oriented p dimensional closed submanifold V^p. Let $Q \in S^n$ be a point that does not lie on V^p. By pushing everything in $S^n - Q$ to the antipode of Q, V^p gets deformed to a point, ie $V^p \sim 0$. Thus

$$H_p(S^n; \mathbb{R}) = 0 \quad , \quad 0<p<n.$$

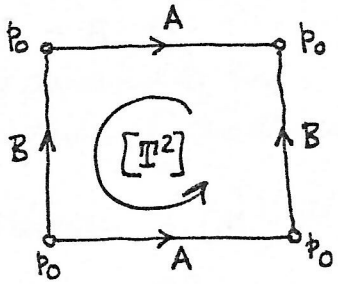
There are no nontrivial cycles in any dimension > dimension of the manifold. Thus

$$\boxed{H_0(S^n;\mathbb{R}) \approx \mathbb{R} \approx H_n(S^n;\mathbb{R}) \; .} \qquad \text{All others are } 0. \qquad (51.7)$$
$$b_0(S^n) = 1 = b_n(S^n)$$

(b) T^2 (Torus)

$$H_0(T^2;\mathbb{R}) = \mathbb{R} \text{ (Connected)}$$

Orient the triangles as indicated (see p.137) Then (assigning $1 \in \mathbb{R}$ to each triangle)



$$\partial[T^2] = A + B - A - B = 0$$

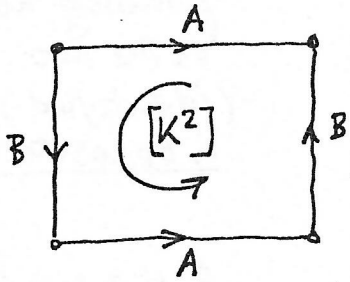
$[T^2]$ is a 2-cycle (showing orientability!) $H_2(T^2;\mathbb{R}) = \mathbb{R}[T^2]$.

Any 1-cycle can be pushed out to the edge. It can be shown that any 1-cycle is homologous to $aA + bB$ for some $a \in \mathbb{R}$, $b \in \mathbb{R}$, (assign real number a to some simplex say P_0P_1 on p 137, and see what this implies for the other simplexes in order to have a 1-cycle!)

$$\boxed{\begin{array}{ccc} H_0(T^2;\mathbb{R}) = \mathbb{R}, & H_1(T^2;\mathbb{R}) = \mathbb{R}A + \mathbb{R}B, & H_2(T^2;\mathbb{R}) = \mathbb{R}[T^2] \\ b_0 = 1 & b_1 = 2 & b_2 = 1 \end{array}} \qquad (51.8)$$

(c) K^2 (Klein bottle)

$$H_0(K^2;\mathbb{R}) = \mathbb{R}.$$



$$\partial[K^2] = A + B - A + B = 2B \neq 0$$

$[K^2]$ is not a cycle, $H_2(K^2;\mathbb{R}) = 0$.

A and B are 1-cycles. Now however there is a relation $\partial[K^2] = 2B$ thus $B = \partial\left(\frac{1}{2}[K^2]\right)$, so $B \sim 0$. $\therefore H_1(K^2;\mathbb{R}) = \mathbb{R}A$

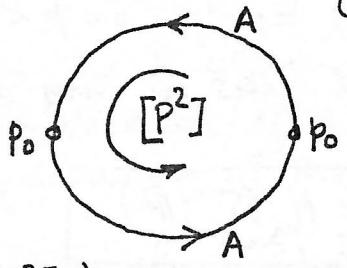
$$\boxed{\begin{array}{ccc} H_0(K^2;\mathbb{R}) = \mathbb{R} & H_1(K^2;\mathbb{R}) = \mathbb{R}A & H_2(K^2;\mathbb{R}) = 0 \\ b_0 = 1 & b_1 = 1 & b_2 = 0 \end{array}} \qquad (51.9$$

$H_2(K^2, \mathbb{Z}_2)$ $\begin{cases} H_0 = \mathbb{Z}_2 P_0 \\ H_1 = \mathbb{Z}_2 A + \mathbb{Z}_2 B \\ H_2 = \mathbb{Z}_2 K^2 \end{cases}$

Note: if we had used $\mathbb{Z}$ as coefficients we would have, since $2B \sim 0$

$H_0(K^2; \mathbb{Z}) = \mathbb{Z}$ $H_1(K^2; \mathbb{Z}) = \mathbb{Z}A + \mathbb{Z}_2 B$, $H_2(K^2; \mathbb{Z}) = 0$.

(d) $P^2(\mathbb{R})$ (the real projective plane). Model is the 2-disc with antipodal points on boundary circle identified.



$H_0(P^2(\mathbb{R}); \mathbb{R}) = \mathbb{R}$

A is a cycle, $\partial A = (P_0) - (P_0) = 0$

However $\partial [P^2] = A + A = 2A \neq 0$

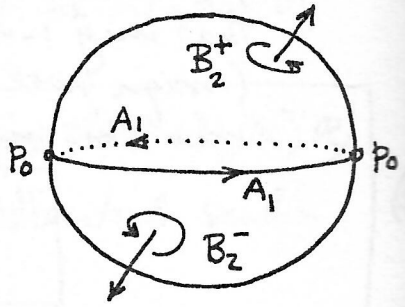
ie $A = \partial \left(\frac{1}{2}[P^2]\right)$, $A \sim 0$. Conclude

$H_0(P^2(\mathbb{R}); \mathbb{R}) = \mathbb{R}$	$H_1(P^2(\mathbb{R}); \mathbb{R}) = 0$	$H_2(P^2(\mathbb{R}); \mathbb{R}) = 0$
$b_0 = 1$	$b_1 = 0$	$b_2 = 0$

If we had used $\mathbb{Z}$ for coefficients

$H_0(P^2(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}$ $H_1(P^2(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}_2 A$ $H_2(P^2(\mathbb{R}); \mathbb{Z}) = 0$.

(e) $P^3(\mathbb{R})$ (the real projective 3-space).



The model is the solid ball with antipodal points on the boundary 2-sphere identified. Orient the ball (and all 3-simplexes in a triangulation by using the right hand. Then by means of the outward pointing normal we get induced orientations on the two 2 dimensional hemispheres. If we call these (identified) hemispheres B_2^+ note that the orientation of B^+ is opposite to that of B^-!

$H_0(P^3(\mathbb{R}); \mathbb{R}) = \mathbb{R}$

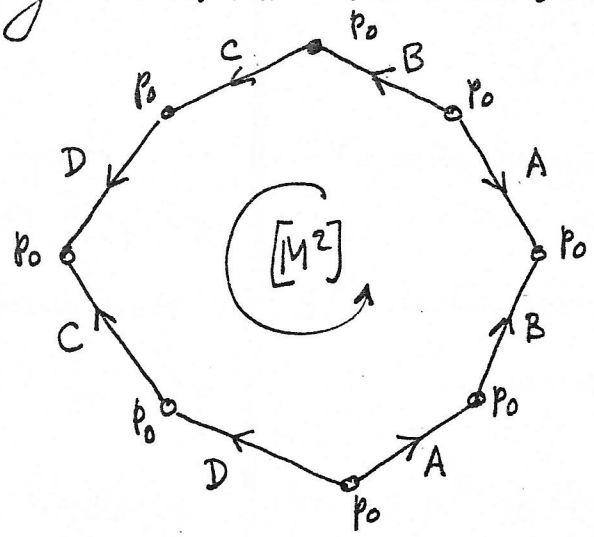
A_1 is a 1-cycle but $\partial B_2^+ = 2A_1$ $\therefore A_1 = \partial \frac{1}{2} B_2^+$, $A_1 \sim 0$, and so $H_1(P^3(\mathbb{R}); \mathbb{R}) = 0$. Neither B_2^+ nor B_2^- is a 2-cycle (they are two identified copies of a projective plane $P^2(\mathbb{R})$ which is not orientable) $\partial B_2^+ = 2A_1$, $\partial B_2^- = -2A_1$. $H_2(\mathbb{P}^3(\mathbb{R}); \mathbb{R}) = 0$. However

$\partial [P^3] = B_2^+ + B_2^- = 0$

Since $B_2^+ = B_2^-$ but they have been given opposite orientations! We conclude

$$\boxed{\begin{array}{c} H_0(P^3;\mathbb{R}) = \mathbb{R}, \quad H_1(P^3;\mathbb{R}) = 0, \quad H_2(P^3;\mathbb{R}) = 0, \quad H_3(P^3;\mathbb{R}) = \mathbb{R}P^3 \\ b_0 = 1 \quad , \quad b_1 = 0 = b_2 \quad , \quad b_3 = 1 \end{array}} \qquad (51.11)$$

- Problem 88. Compute using $\mathbb{Z}$ coefficients $H_i(P^3(\mathbb{R});\mathbb{Z})$.
- Problem 89 A certain closed surface M^2 has as model an octagon with the indicated identifications on the boundary.



Note carefully the direction of the arrows!

Write down $H_i(M^2;\mathbb{R})$ and $H_i(M^2;\mathbb{Z})$ for all i. What are the Betti numbers b_i for M^2? Is M^2 orientable?

Lecture 52. de Rham's Theorem

(1) de Rham's Theorem. In this lecture we shall concern ourselves only with homology with real coefficients, $G = \mathbb{R}$. As we know, the chain, cycle, boundary and homology groups become vector spaces. Let us compare these vector spaces with vector spaces associated with differential forms on M^n. As M^n is fixed and as $G = \mathbb{R}$, M^n and $\mathbb{R}$ will be omitted in the notation (thus $C_p(M^n; \mathbb{R})$ will be written simply as C_p).

C_p = p-chains on M^n	A^p = vector space of p-forms on M^n
$\partial : C_p \longrightarrow C_{p-1}$, $\partial^2 = 0$	$d : A^p \longrightarrow A^{p+1}$, $d^2 = 0$.
Z_p = p-cycles = $\ker \partial : C_p \to C_{p-1}$	F^p = vector subspace of closed p-forms = $\ker d : A^p \longrightarrow A^{p+1}$
B_p = p-boundaries = $\operatorname{Im} \partial : C_{p+1} \longrightarrow C_p$	$\mathcal{E}^p$ = vector subspace of exact p-forms = $\operatorname{Im} d : A^{p-1} \longrightarrow A^p$
H_p = p^{th} homology vector space = Z_p / B_p = equivalence classes of cycles, where $z_p' \sim z_p$ iff $z_p' - z_p = \partial c_{p+1}$ for some $c_{p+1} \in C_{p+1}$	$de\,R^p$ = p^{th} de Rham vector space $\equiv F^p / \mathcal{E}^p$ = equivalence classes of closed p-forms β where $\beta'^p \sim \beta^p$ iff $\beta'^p - \beta^p = d\alpha^{p-1}$ for some $(p-1)$ form $\alpha^{p-1} \in A^{p-1}$.

We shall be concerned with the observation that integration allows us to consider differential forms α^p as linear functionals on p-chains; to α^p we associate the linear functional $I\alpha^p$ on C_p by

$$I\alpha^p(c_p) \equiv \int_{c_p} \alpha^p.$$

We shall be especially concerned with thinking of <u>closed</u> p-forms as being linear functionals on p-<u>cycles</u>

$$I : \mathcal{F}^p \longrightarrow Z_p^*$$

$$I\beta^p(z_p) = \int_{z_p} \beta^p$$

We claim that $I\beta^p$ can also be considered as a linear functional on the <u>homology</u> vector space H_p

$$I : \mathcal{F}^p \longrightarrow H_p^*.$$

What does this mean? An element of H_p is an equivalence class of p-cycles. Let h_p be such an equivalence class, $h_p \in H_p$. We can <u>represent</u> this class by some specific p-cycle z_p, ie h_p is the equivalence class of some cycle z_p. As before we put

$$I\beta^p(z_p) = \int_{z_p} \beta^p.$$

But the choice of z_p as representative of h_p is not unique. If z_p represents h_p, then so does $z_p + \partial c_{p+1}$ for any $c_{p+1} \in C_{p+1}$. But, since $d\beta^p = 0$

$$I\beta^p(z_p + \partial c_{p+1}) = \int_{z_p + \partial c_{p+1}} \beta^p = \int_{z_p} \beta^p + \int_{\partial c_{p+1}} \beta^p$$

$$= \int_{z_p} \beta^p + \int_{c_{p+1}} d\beta^p = I\beta^p(z_p)$$

shows that

$$I\beta^p(h_p) \equiv I\beta^p(z_p)$$

<u>is independent of the representative z_p chosen for h_p</u>!

$$I : \mathcal{F}^p \longrightarrow H_p^*$$

<u>is</u> well defined!

We claim even more. β^p is but a single representative of a de Rham class; the most general representative of the same de Rham class is of the form

$$\beta^p + d\alpha^{p-1}$$

where α^{p-1} can be any $(p-1)$ form $\alpha^{p-1} \in \mathcal{A}^{p-1}$. But note that

$$I(\beta^p + d\alpha^{p-1})\ (z_p) = \int_{Z_p} \beta^p + \int_{Z_p} d\alpha^{p-1}$$

$$= \int_{Z_p} \beta^p + \int_{\partial Z_p = 0} \alpha^{p-1} \quad = I\beta^p(z_p)$$

shows that $I\beta^p(z_p)$ depends only on the de Rham class of the closed form β^p, not the form β^p itself. In other words I really defines a linear transformation

$$I : deR^p \longrightarrow H_p^*$$

of the de Rham vector space deR^p into the vector space of all linear functals on H_p. de Rham goes on to show that I is an isomorphism, ie I is 1:1 and onto!

Theorem (de Rham): $I : deR^p \longrightarrow H_p^*$ is an <u>isomorphism</u>. First of all it is <u>onto</u>; this means that <u>any</u> linear functional on homology classes, ie on H_p, is of the form $I\beta^p$ for some <u>closed</u> p-form β^p. In particular, if

$$z_p^{(1)}, \ldots, z_p^{(b_p)} \qquad b_p = p^{th} \text{ Betti number}$$

is a basis for $H_p(M^n; \mathbb{R})$, and if $\pi_1, \ldots, \pi_{b_p}$ are arbitrary real numbers then there exists a <u>closed</u> p-form β^p such that

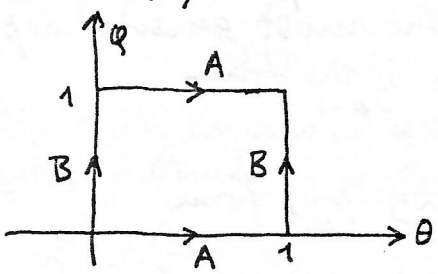
$$\int_{Z_p^{(i)}} \beta^p = \pi_i \qquad i = 1, 2, \ldots, b_p.$$

Second, I is 1:1; this means that if $I\beta^p(z_p) = \int_{Z_p} \beta^p = 0$ for <u>all</u> cycles z_p, then β^p is <u>exact</u>

$$\beta^p = d\alpha^{p-1} \quad \text{some } \alpha^{p-1} \in \mathcal{A}^{p-1}$$

We remark that the dual space $H^*_p(M^n; \mathbb{R})$ is called the p^{th} cohomology vector space, and is written $H^p(M^n; \mathbb{R})$

(2) An Example. First, as a warm-up, consider the 2-torus T^2



$d\mathbb{R}^0$ has basis the 0-form 1 (the constant function)

$d\mathbb{R}^1$ has basis given by the closed 1-forms "$d\theta$" and "$d\varphi$" (recall, since θ and φ are multiple valued, that these forms are not exact). We have

$$I("d\theta")(A) = \int_A d\theta = 1 \qquad I("d\theta")(B) = \int_B d\theta = 0$$

$$I("d\varphi")(A) = \int_A d\varphi = 0 \qquad I("d\varphi")(B) = \int_B d\varphi = 1$$

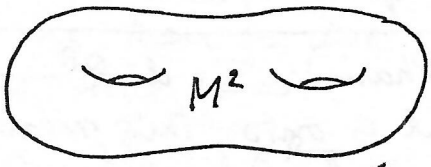
This shows that "$d\theta$", "$d\varphi$" gives the dual basis to A, B.

For $d\mathbb{R}^2$ basis we can take the "area form" $d\theta \wedge d\varphi$

$$\int_{T^2} d\theta \wedge d\varphi = 1.$$

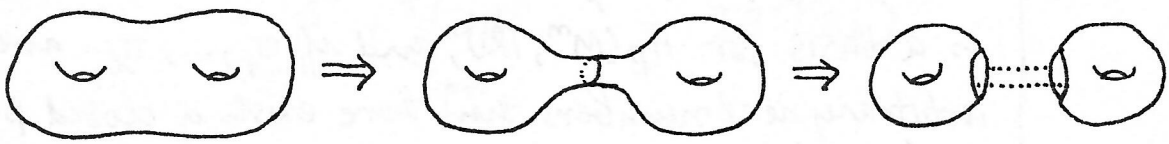
This was too easy because of the global "coordinates" θ & φ.

Consider next a closed surface of genus 2

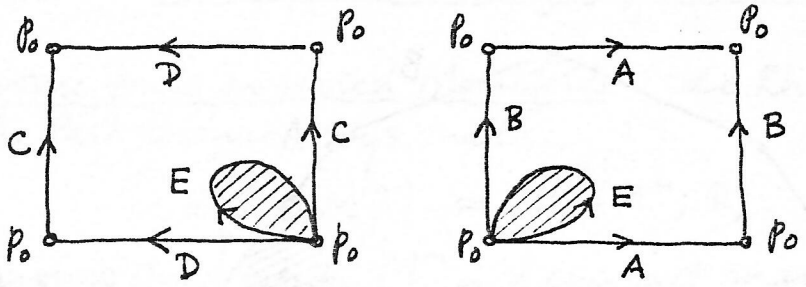


$d\mathbb{R}^0$: $f = 1$.
$d\mathbb{R}^2$: area form.

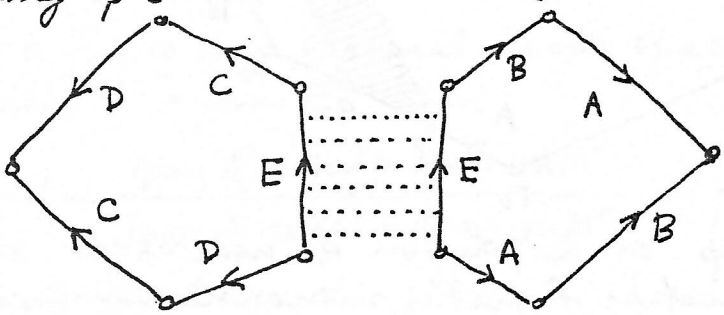
We need only worry about $d\mathbb{R}^1$. Let's write M^2 as an <u>octagon with sides identified</u>. We proceed as follows



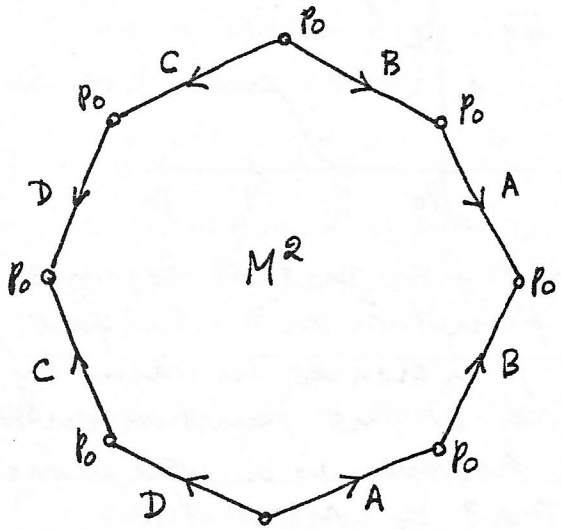
where the last figure represents two tori, each with a disc removed, and where the edges of the two holes are to be identified. But this last figure can be represented by



The two edges of the two holes are to be identified; they form a single closed curve E. We finally get the desired polygon by opening up E



or

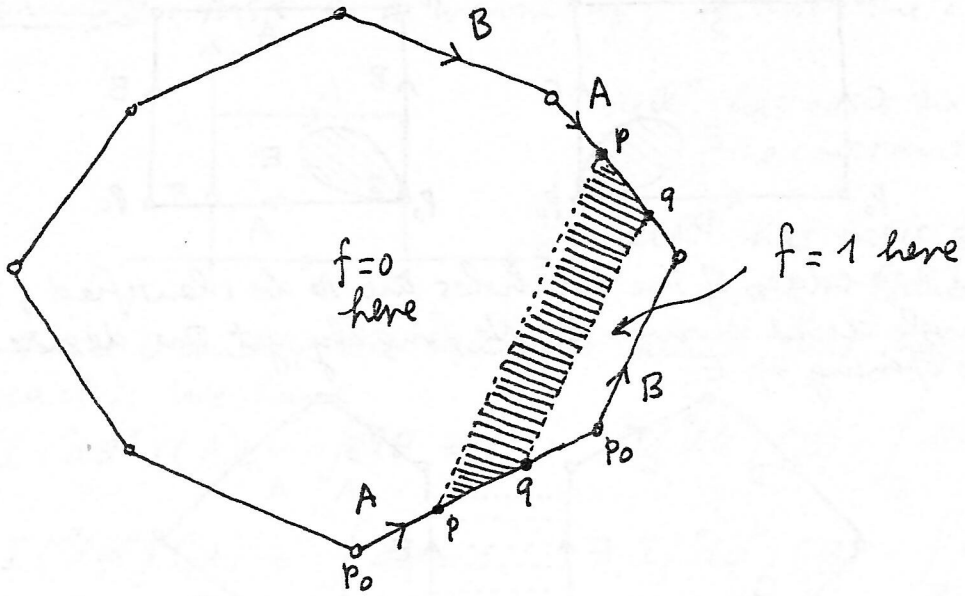


the seam E no longer being required. The surface of genus 2 is topologically an octagon with the eight sides identified pairwise with the above prescription. (The reader should compare this with the figure of Prob. 89) Here clearly $b_1(M) = 4$ with generators for $H_2(M^2; \mathbb{R})$ given by

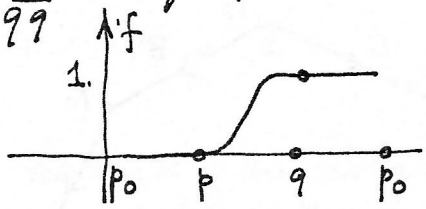
$$A, \quad B, \quad C, \quad D.$$

Suppose we wish to exhibit a closed 1-form β with "periods"

$$\int_A \beta^2 = \pi_A = 1, \quad \int_B \beta^2 = \int_C \beta^2 = \int_D \beta^2 = 0, \text{ say.}$$



Take a "strip" on the polygon as indicated. In this strip take a smooth function f that is monotonically increasing, starts with value $f=0$ on the edge $\overline{pp}$ and ends with the value $f=1$ on the edge $\overline{qq}$



We can then extend f so that it becomes a smooth multiple valued function on the surface; everytime one crosses the strip f increases in value by 1. We have shown one branch of this function on the polygon. Its differential, df, however is a well defined smooth 1-form and it is clear that by construction

$$\int_A \text{"}df\text{"} = 1, \quad \int_{B \text{ or } C \text{ or } D} \text{"}df\text{"} = 0,$$

as desired.

The proof of de Rham's theorem is too difficult and long to be given here.

Note that the statement in de Rham's theorem that I is 1:1, ie. a closed 1-form with zero periods is exact, is what we had called "de Rham's first theorem" in Lecture 22.

Lecture 53 Some Applications of Hodge's Theorem

(1) **Harmonic Forms on Closed Manifolds.** de Rham's theorem (see the last lecture) says that

$$\text{de}R^p(M^n) \approx H_p^*(M^n; \mathbb{R}).$$

It is known that when M^n is a compact manifold that $H_p(M^n; \mathbb{R})$ is a finite dimensional vector space

$$b_p(M^n) < \infty \quad \text{if } M^n \text{ compact.}$$

But for a finite dimensional vector space E we know (Lect. 5) that $\dim E = \dim E^*$. Thus

$$\dim \frac{\text{Space of closed } p\text{-forms on } M^n}{\text{Space of exact } p\text{-forms on } M^n} = b_p(M^n)$$

if M^n is compact.

Let β^p be a closed p-form on M^n, $d\beta^p = 0$. From the Hodge decomposition (49.10)

$$\beta^p = d\alpha^{p-1} + d^*\gamma^{p+1} + h.$$

But this is an orthogonal decomposition, and get

$$(\beta^p, d^*\gamma^{p+1}) = (d\beta^p, \gamma^{p+1}) = 0$$

and so

$$\boxed{d\beta^p = 0 \implies \beta^p = d\alpha^{p-1} + h^p} \tag{53.1}$$

and so $\beta^p \sim h^p$

$$\boxed{\begin{array}{l}\text{In each de Rham class of closed}\\ p\text{-forms on a compact } M^n \text{ there}\\ \text{is exactly one harmonic } p\text{-form}\end{array}} \tag{53.2}$$

In particular, again letting $\mathcal{H}^p$ be the vector space of harmonic p-forms on M^n

$$\dim \mathcal{H}^p(M^n) = b_p \tag{53.3}$$

We may further say

Let $z_p^{(1)}, \dots, z_p^{(b_p)}$ be a basis for $H_p(M^n; \mathbb{R})$ (the p-cycles on M^n). Let $\pi_{(1)}, \dots, \pi_{(b_p)}$ be p arbitrary real numbers. Then if M^n is compact there is a <u>unique</u> harmonic p-form h^p with these periods (53.4)

$$\int_{z_p^{(i)}} h^p = \pi_{(i)} .$$

Finally note that on a compact M^n a harmonic h^p form is exact iff it is identically $h^p = 0$ (orthogonal decomposition). Suppose now that M^n is also <u>orientable</u>. If α^p is a (true) p-form, we know that $*\alpha$ is a pseudo $(n-p)$ form, but in an orientable manifold, by picking an orientation we may identify a pseudo form with a true form, hence $*\alpha$ may be considered a true form. But note that if h^p is harmonic, $dh = 0 = d^*h = \pm * d * h$ then so is $*h$

$$d(*h) = \pm * d^* h = 0$$
$$d^*(*h) = \pm * d * * h = \pm * dh = 0$$

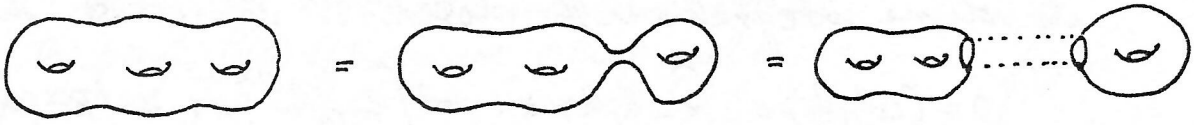
We conclude that

If M^n is compact and orientable and $h^p_{(1)}, \dots, h^p_{(b_p)}$ is a basis for the harmonic p-forms, then $*h^p_{(1)}, \dots, *h^p_{(b_p)}$ is a basis for the harmonic $(n-p)$ forms. In particular (53.5)

$$b_{n-p} = b_p$$

The last equality is a special case of "<u>Poincaré duality</u>" Orientability is important; look for example at (51.9) and (51.10)

(2) Bochner's Theorem. The usual metric on the 2-sphere S has positive curvature (in fact $K=1$). The Betti numbers of S^2 are (see § 3, Lect. 51) $b_0=1$, $b_1=0$, $b_2=1$. The closed surface of genus g, M_g^2 has $b_0=1$, $b_1=2g$, $b_2=1$. (The case $g=2$ was described in the last lecture; for a surface of genus $g=3$ we may consider it as a surface of genus 2 with a "handle" attached



i.e. we take an $M^2_{g=2}$ and a torus $T^2 = M^2_{g=1}$, cut a hole in each and identify the boundary circles. By the same process as in § 2 of the last lecture one gets that $M^2_{g=3}$ can be considered as a polygon with $8+4=12$ sides, with sides identified in pairs (do it!). The six pairs will yield a 1-cycles A, B, C, D and F, G, giving $b_1=6$. One can continue; $M^2_{g=4}$ is $M^2_{g=3}$ with a handle attached, etc, etc. In any case, for $g>0$, $b_1(M^2_g) > 0$. But note that in the "standard picture" for these surfaces, i.e. common embeddings $K<0$ at some points. It is a fact that no metric on M^2_g for $g>0$ can have strictly positive curvature $K>0$. This is a consequence of the following general theorem relating curvature (a local concept) with Betti number (a global "topological" concept. It is due to S. Bochner.

> Theorem. A closed M^n with positive Ricci curvature
> $$Ric(X,X) > 0$$
> has first Betti number $b_1(M^n) = 0$

Proof. We shall show that the only harmonic 1-form h^1 is $h^1=0$. From (53.3) we shall have $b_1 = \mathcal{H}^1 = 0$. Let h^1 be harmonic, $\Delta h^1 = 0$. From (45.5)

$$(\Delta h^1)_j = -g^{lk} h_{j|lk} + h^k R_{kj}$$

and so

$$\langle \Delta h, h\rangle = (\Delta h)_j h^j = -g^{lk} h_{j|lk} h^j + h^R R_{kj} h^k$$
$$= -g^{lk}(h_{j|l} h^j)_{|k} + g^{lk} h_{j|l} h^j{}_{|k} + \mathrm{Ric}(h,h)$$
$$= -(g^{lk} h_{j|l} h^j)_{|k} + h_{j|l} h^{j|l} + \mathrm{Ric}(h,h)$$

Now $(g^{lk} h_{j|l} h^j)_{|k}$ is the divergence of the vector $g^{lk} h_{j|l} h^j$ & its volume integral over the closed M^n vanishes. Thus

$$0 = (\Delta h, h) = \int_{M^n} \langle \Delta h, h\rangle * 1 = \int_{M^n} h_{j|l} h^{j|l} * 1 + \int \mathrm{Ric}(h,h) * 1.$$

But $h_{j|l} h^{j|l}$ = twice the norm squared of the covariant tensor $(h_{j|l})$

and so is ≥ 0. By assumption $\mathrm{Ric}(h,h) > 0$ if $h \neq 0$. We conclude then that $\mathrm{Ric}(h,h) = 0$ and so $h = 0$. □

Note that only integrals of pseudo n-forms arise above; thus this result holds even if M^n is not orientable.

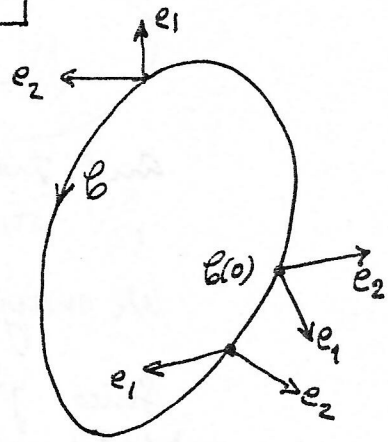
A slightly stronger result than Bochner's was proven somewhat earlier by S.B. Myers, by altogether different means. However Bochner's theorem was the model for a vast generalization by Kodaira (the "vanishing theorems" of complex manifold theory).

(3) Maintaining Currents in a Closed M^3. In §3 of Lecture 23 we showed that one cannot maintain a current in a loop of a closed M^3 if the loop C does not bound a real 2-chain (i.e, if C is a nontrivial cycle in $H_1(M^3; \mathbb{R})$). We stated, but did not prove, that a current in C can be maintained if C bounds, $C \sim 0$. We gave a sketch of this result here. For simplicity we shall consider only the case of an orientable M^3. We shall prove the following

> Given $C \sim 0$, we shall construct a closed 2-form j^2 whose support lies in an ε-tubular neighborhood of C (a "wire" of radius ε). We shall exhibit a 1-form $\mathcal{H}'$ such that
> $$d\mathcal{H}' = 4\pi j^2 \quad \text{and} \quad d * \mathcal{H}' = d\mathcal{B}^2 = 0$$

We can then interpret (since j^2 will be time independent) H as the magnetic field 1-form due to the current j. The charge density and electric field vanish identically.

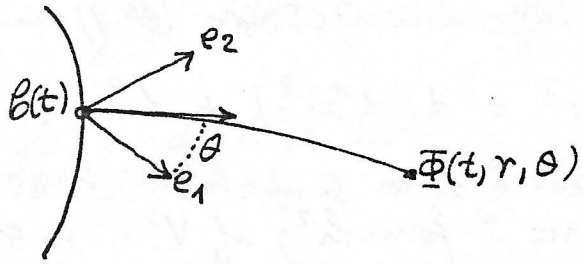
"maintained"



Proof. Let e_1, e_2 be orthonormal vectors at the point $\mathcal{C}(0)$ of the curve, orthogonal to the curve. Transport them smoothly (in any way you wish) around $\mathcal{C}$, keeping them o.n and orthogonal to $\mathcal{C}$. When you've almost returned to the starting point $\mathcal{C}(0)$ you may have to rotate e_1 and e_2 to make them match up with the initial values, but it can be done since M^3 is assumed orientable and so $e_1 e_2$ can not return with the opposite orientation. If $\mathcal{C}$ is a map $\mathcal{C}: [0,1] \to M^3$ (with of course $\mathcal{C}(0) = \mathcal{C}(1)$), consider the map defined by

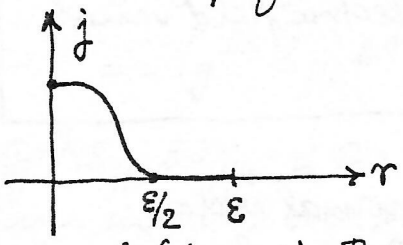
$$\Phi(t, r, \theta) = \exp_{\mathcal{C}(t)} \left(r(\cos\theta\, e_1 + \sin\theta\, e_2) \right)$$

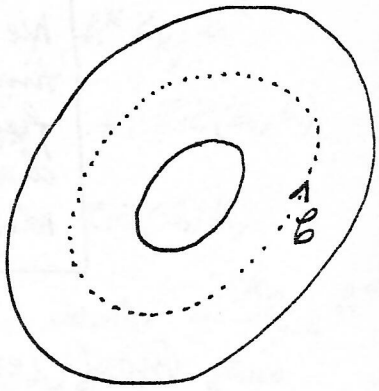
i.e $\Phi(t, r, \theta)$ is the point in M^3 that lies on the geodesic that leaves $\mathcal{C}$ at the point $\mathcal{C}(t)$, makes an angle of θ with the vector $e_1(t)$ in the plane $e_1(t) \wedge e_2(t)$, and has distance r from $\mathcal{C}(t)$



If r is sufficiently small, say $r \leq$ some number $\varepsilon > 0$, then we can use Φ to introduce "cylindrical coordinates" $x_\Phi(r, \theta, t)$ in a "tubular nbhd of radius ε" with "axis" the curve $\mathcal{C}$. The coordinate t is periodic, period 1, θ is periodic, period 2π.

We now construct a "current" 2-form j^2 in this tube as follows. Let $j(r)$ be a "bump function"





and then define, in the tube

$$j^2 = j(r)\, dr \wedge d\theta .$$

We may assume that $\int_0^{2\pi} d\theta \int_0^{\varepsilon} j(r)\, dr = 1$. (53

Since $j^2 = 0$ for $\varepsilon/2 \le r \le \varepsilon$ we may extend j^2 to all of M^3 by demanding $j^2 = 0$ outside the tube. j^2 is a smooth 2-form on M^2 and clearly

$$dj^2 = 0 \qquad (\text{i.e. } \operatorname{div} \underset{\sim}{J} = 0)$$

everywhere in M^3. j^2 is our mathematical model for a current. We now wish to find a magnetic field 1-form $\mathcal{H}^1$ such that

$$d*\mathcal{H}^1 \equiv d\mathcal{B}^2 = 0 \quad \& \quad d\mathcal{H}^1 = 4\pi j^2 .$$

This will correspond to a pure magnetic field ($\mathcal{E} = 0$, charge density vanishing).

Since $dj^2 = 0$, j^2 is a closed 2-form on the closed M^3. By the Hodge decomposition (49.9) and (53.1)

$$j^2 = d(d^* A^2) + h^2 \tag{53.7}$$

where A^2 is some 2-form and h^2 is harmonic. Look at the periods of the harmonic 2-form h^2; if V^2 is a closed orientable submanifold of M^3 then, from (53.7)

$$\int_{V^2} h^2 = \int_{V^2} j^2 . \tag{53.8}$$

We may restrict ourselves to looking at basis elements

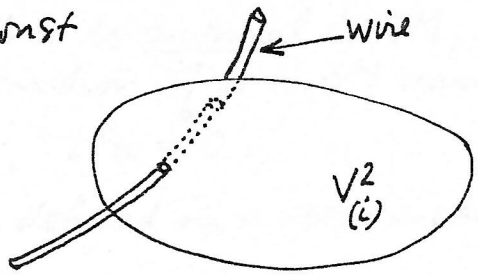
$$V^2_{(1)}, \ldots, V^2_{(b_2)}$$

of $H_2(M^3; \mathbb{R})$. If we choose ε small enough, then by moving

each $V^2_{(i)}$ slightly we may assume that the ε-tube about $\mathcal{C}$ hits each $V^2_{(i)}$ transversally in the sense that each "parallel" to $\mathcal{C}$ given by

$$r = \text{const}, \quad \theta = \text{const}$$

strikes each $V^2_{(i)}$ transversally. But then it is "clear" that (because of (53.6))



$$\int_{V^2_{(i)}} j^2 = \text{the intersection number of } \mathcal{C} \text{ with } V^2_{(i)}$$

See (22.2) !

Now it is a fact from differential topology (and a sketch of the proof is not difficult) that <u>since</u> $\mathcal{C} \sim 0$, <u>the intersection number of</u> $\mathcal{C}$ <u>with any 2-cycle</u> $V^2_{(i)}$ <u>must be</u> 0 ! From (53.8) we conclude that all the periods of the harmonic part h^2 of j^2 must vanish. From (53.4) we conclude

$$h^2 = 0$$

and so

$$j^2 = d(d^* A^2).$$

Define then the magnetic 1-form by

$$\mathcal{H}^1 = d^* A^2.$$

We have

$$j^2 = d\mathcal{H}^1.$$

Furthermore

$$d^* \mathcal{H}^1 = d^* d^* A^2 = 0$$

and so $\operatorname{div} B = 0$. If we define charge density $\rho \equiv 0$, then charge conservation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} J = 0$$

i.e

$$\frac{\partial}{\partial t}(*\rho) + d j^2 = 0$$

is trivially satisfied. Maxwell's equations are satisfied if we put $\mathcal{E} \equiv 0$. □

Lecture 54. Hodge Theory for Manifolds with Boundary

(1) *Tangential and Normal Differential Forms.* Let M^n be compact with boundary ∂M.

A form α^p on M^n is said to be *normal* to ∂M, or briefly, simply, normal provided the restriction of α to ∂M vanishes

$$i: \partial M \to M^n \text{ is inclusion}, \quad i^*\alpha = 0.$$

If we suppose that ∂M is locally defined in a coordinate system $x^1, \cdots, x^n$ by

$$x^n = 0$$

then α^p is normal iff α^p is of the form

$$\alpha^p = dx^n \wedge \gamma$$

(for some form γ) at points of ∂M.

For example a 1-form α^1 is normal provided

$$\alpha^1 = a_n \, dx^n$$

at points of ∂M. If $\underset{\sim}{T}$ is a tangent vector to ∂M, then

$$\alpha^1(\underset{\sim}{T}) = \langle \underset{\sim}{A}, \underset{\sim}{T} \rangle$$

where $\underset{\sim}{A}$ is the contravariant version of α^1. α^1 is normal iff

$$\alpha^1(\underset{\sim}{T}) = \langle \underset{\sim}{A}, \underset{\sim}{T} \rangle = 0, \text{ all } T \text{ tangent to } \partial M$$

ie

$$\boxed{\alpha^1 \text{ normal iff } \underset{\sim}{A} \text{ is a normal vector to } \partial M.}$$

If β^{n-1} is an $(n-1)$ form, $\beta^{n-1} = i_{\underset{\sim}{B}} \, vol^n$ then β^{n-1} is normal provided $vol^n(\underset{\sim}{B}, \underset{\sim}{T}_2, \cdots, \underset{\sim}{T}_n) = 0$ for all tangent vectors $T_2, \cdots, T_n$ to ∂M. Thus

$$\boxed{\beta^{n-1} = i_{\underset{\sim}{B}} vol^n \text{ is normal iff } \underset{\sim}{B} \text{ is tangent to } \partial M}$$

A form α^p on M^n is said to be <u>tangent</u> to ∂M, or simply tangent provided

$$*\alpha^p \text{ is normal to } \partial M$$

If the coordinates $x^1, \dots, x^n$ such that ∂M is defined by

$$x^n = 0$$

are <u>orthogonal</u> at $p \in \partial M$ then clearly

α^p is tangent iff $\alpha = a_J\, dx^J$ never contains a term involving dx^n

α^1 is tangent means $*\alpha^1 = i_{\underset{\sim}{A}} \text{vol}^n$ is normal, i.e

$\boxed{\alpha^1 \text{ is tangent iff } \underset{\sim}{A} \text{ is a tangent vector to } \partial M.}$

On the other hand

$\beta^{n-1} = i_{\underset{\sim}{B}} \text{vol}^n$ is tangent means $*\beta^{n-1} = \pm\, B_i\, dx^i$ is normal, ie

$\boxed{\beta^{n-1} \text{ is tangent iff } \underset{\sim}{B} \text{ is normal to } \partial M}$

(2) <u>Tangent Harmonic Forms</u>.

<u>Theorem</u>. Let M^n be a compact Riemannian manifold with boundary ∂M. Let $Z_p^{(1)}, \dots, Z_p^{(b_p)}$ be a basis for the p^{th} homology vector space $H_p(M^n; \mathbb{R})$.

Then $\exists$ a unique <u>tangent</u> harmonic p-form α^p

$$d\alpha^p = 0 = d*\alpha$$

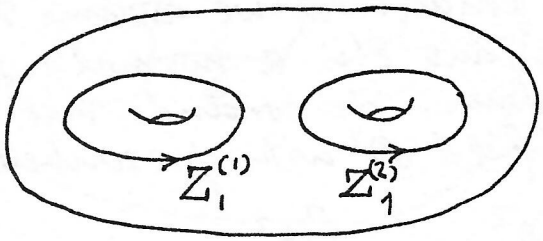
with prescribed periods

$$\int_{Z_p^{(i)}} \alpha^p.$$

Example (1) Let $\underset{\sim}{v}$ be the velocity field for an incompressible, irrotational fluid flow inside a closed surface of genus 2, M^2_g. M^2_g is the boundary of a compact manifold M^3

$$\partial M^3 = M^2_g$$

and the velocity $\underset{\sim}{v}$ is tangent to M^2_g. $\nu \equiv v_i dx^i$ is then a harmonic 1-form on M^3.

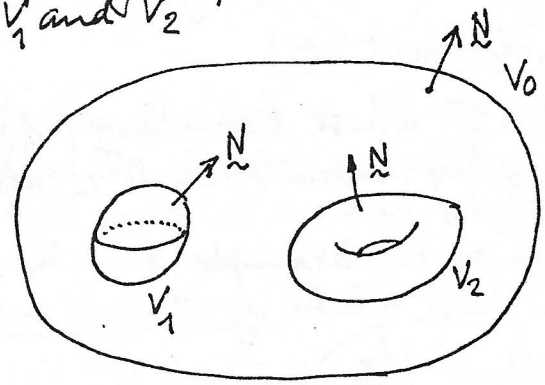


$\Sigma_1^{(1)}$ and $\Sigma_1^{(2)}$ give, plausibly, a basis for $H_1(M^3; \mathbb{R})$. We then have Lord Kelvin's theorem; $\exists$ a unique incompressible, irrotational flow in M^3 (tangent to M^2) with given prescribed circulations

$$\oint_{\Sigma_1^{(i)}} \nu = \oint_{\Sigma_1^{(i)}} \underset{\sim}{v} \cdot d\underset{\sim}{x} \quad \text{given}$$

In particular, if we have a flow inside a 2-sphere S^2 then (since $H_1(S^2; \mathbb{R}) = 0$) the flow vanishes identically.

Example (2) Let $M^3 \subset \mathbb{R}^3$ be a region in $\mathbb{R}^3$ inside a closed conducting surface V_0 and outside closed conducting surfaces V_1 and V_2



Then $\partial M^3 = V_0 - V_1 - V_2$ and a basis for $H_2(M^3; \mathbb{R})$ is plausibly given by V_1 and V_2. Suppose we consider an electrostatic

problem and suppose that there are no charges inside M^3. Then the electric field satisfies, in M^3

$$d\mathcal{D}^2 = 4\pi\rho = 0$$

$$d*\mathcal{D}^2 = d\mathcal{E}' = -\frac{\partial \mathcal{B}^2}{\partial t} = 0$$

and so $\mathcal{D}^2$ is harmonic in M^3. Since the bounding surfaces V_0, V_1 and V_2 are conductors we assume that $\underset{\sim}{E}$ is normal to these surfaces. Thus $\mathcal{E}'$ is a normal 1-form and $\mathcal{D}^2$ is a tangent 2-form. We conclude that there exists a unique electric field $\mathcal{D}^2$ with prescribed "fluxes"

$$\int_{V_1} \mathcal{D}^2 \quad , \quad \int_{V_2} \mathcal{D}^2$$

i.e. with prescribed charges inside V_1 and V_2. Of course the flux of $\mathcal{D}^2$ thru V_0 is (why?)

$$\int_{V_0} \mathcal{D}^2 = \int_{V_1+V_2} \mathcal{D}^2$$

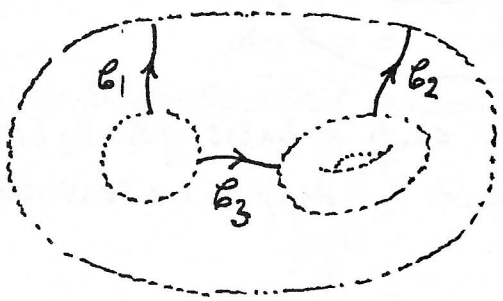
Furthermore, if there are no charges inside V_1 and V_2 then $\underset{\sim}{E} = 0$ in M^3 (electric "shielding")

(3) Normal Harmonic Forms and Relative Homology. The "cycles" we have been describing so far are called absolute cycles. Given a compact manifold M^n with boundary ∂M we can define a

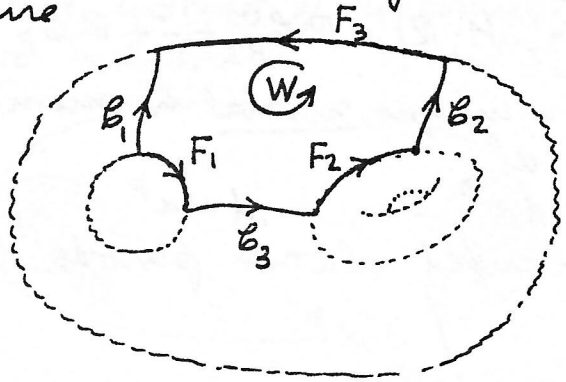
relative p-cycle (mod ∂M)

to be a p-chain on M^n whose boundary (if there is one) lies on ∂M. Of course every (absolute) cycle is also a relative cycle.

In the figure drawn in Example 2



the curves $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are all relative 1-cycles (mod ∂M). We shall systematically throw away any chain that lies on ∂M. That is why we can think of a relative cycle as a cycle; we may discard its boundary since it lies on ∂M. In the above figure



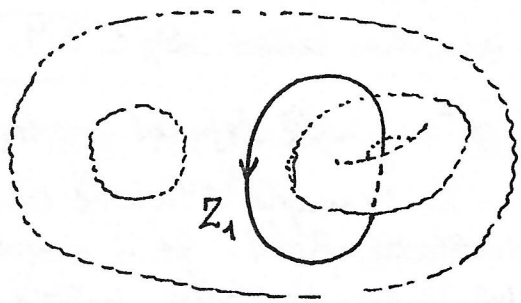
it is clear that $\partial W = -\mathcal{C}_1 + F_1 + \mathcal{C}_3 + F_2 + \mathcal{C}_2 + F_3$. But F_1, F_2, F_3 all lie on ∂M^3, hence

$$\partial W \equiv -\mathcal{C}_1 + \mathcal{C}_3 + \mathcal{C}_2 \quad (\text{mod } \partial M)$$

We would then say that

$$\mathcal{C}_3 \sim \mathcal{C}_1 - \mathcal{C}_2. \quad (\text{mod } \partial M)$$

Thus only $\mathcal{C}_1$ and $\mathcal{C}_2$ are <u>independent</u> relative 1-cycles. Are there any more? Consider the absolute (hence relative) 1-cycle Z_1



that threads thru the hole in V_2. As an <u>absolute</u> 1-cycle it does not bound in M^3, but as a relative 1-cycle (mod ∂M) it does become trivial since it is easily deformed to lie on the torus $V_2 \subset \partial M$. It can be shown that $\mathcal{C}_1$ and $\mathcal{C}_2$ indeed form a basis for the <u>relative homology</u> vector space

$$H_1(M^3, \partial M; \mathbb{R}) = \mathbb{R}\mathcal{C}_1 + \mathbb{R}\mathcal{C}_2$$

Theorem. Let M^n be a compact Riemannian manifold with boundary ∂M. Let $\mathcal{C}_p^{(1)}, \dots, \mathcal{C}_p^{(k)}$ be a basis for the relative p-cycles of $M^3 \pmod{\partial M}$

$$H_p(M^3, \partial M; \mathbb{R}) = \mathbb{R}\,\mathcal{C}_p^{(1)} + \dots + \mathbb{R}\,\mathcal{C}_p^{(k)}$$

Then $\exists$ a unique normal harmonic p-form α^p

$$d\alpha^p = 0 = d * \alpha^p$$

with prescribed relative periods

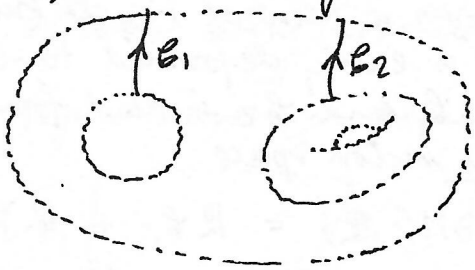
$$\int_{\mathcal{C}_p^{(i)}} \alpha^p$$

Note that if $\mathcal{C}_p' \sim \mathcal{C}_p \pmod{\partial M}$, then $\mathcal{C}_p' - \mathcal{C}_p = \partial \mathcal{C}_{p+1} + \mathcal{D}_p$ where $\mathcal{D}_p$ is a chain that lies on ∂M. Then if β^p is closed and normal

$$\int_{\mathcal{C}_p' - \mathcal{C}_p} \beta^p = \int_{\partial \mathcal{C}_{p+1} + \mathcal{D}_p} \beta^p = \int_{\mathcal{D}_p} \beta^p = 0$$ since the restriction

$i^* \beta^p$ of β^p to ∂M vanishes and $\mathcal{D}_p \subset \partial M$. Thus the relative periods of α^p are well defined on $H_p(M^3, \partial M; \mathbb{R})$.

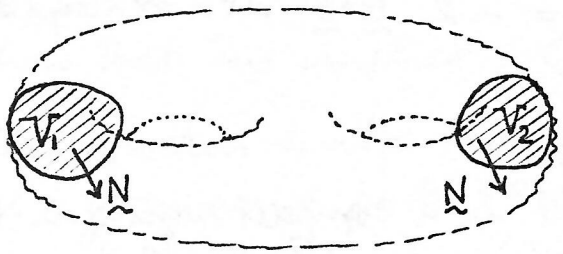
Example (2'). In example 2 above consider the electric 1-form $\mathcal{E}'$ for the electrostatic field. It is a normal harmonic 1-form on M^3. Thus we may prescribe, instead of the charges inside V_1 and V_2, the line integrals



$$\int_{\mathcal{C}_1} \mathcal{E}' \quad \text{and} \quad \int_{\mathcal{C}_2} \mathcal{E}'$$

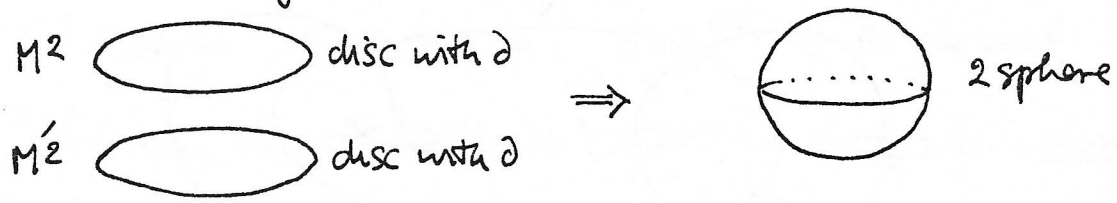
This means that we may prescribe the electrostatic potential differences between the "inside" and the "outside" conductors!

Example (1'). In example (1) above we may consider the 2-form $\beta^2 = i_{\underset{\sim}{v}} vol^3 = *(v_i dx^i)$ associated with the velocity $\underset{\sim}{v}$. β^2 is harmonic and normal to ∂M. Hence we may prescribe, instead of the circulations, the fluxes of $\underset{\sim}{v}$ thru the two "membranes" V_1 and V_2

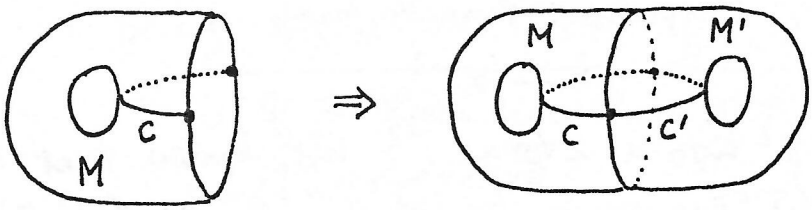


$$\int_{V_1} \underset{\sim}{v} \cdot d\underset{\sim}{S} \quad \text{and} \quad \int_{V_2} \underset{\sim}{v} \cdot d\underset{\sim}{S}$$

(4) Remarks. The proofs of these two versions of "Hodge theory for manifolds with boundary" cannot be given here. While the topological aspects are quite simple the analytical details are quite subtle. The crucial device used is to take a manifold with boundary M^n and "double it" to get a compact manifold without boundary



One then notes that the double of a relative cycle becomes an absolute cycle on the double



One then applies Hodge's theory to the closed double. For details one may consult the article by K. Friedrichs, Differential Forms on Manifolds with Boundary, Comm. on Pure and Appl. Maths, 8 (1955), pp 551-590.

Some Problems

Problem F.1. Let $\mathcal{F} = E\,dt + B$ be the electromagnetic 2-form in Minkowski space M^4_0. Show that

$$\mathcal{F} \wedge *\mathcal{F} = (|\underset{\sim}{B}|^2 - |\underset{\sim}{E}|^2)\, dt\wedge dx\wedge dy\wedge dz$$

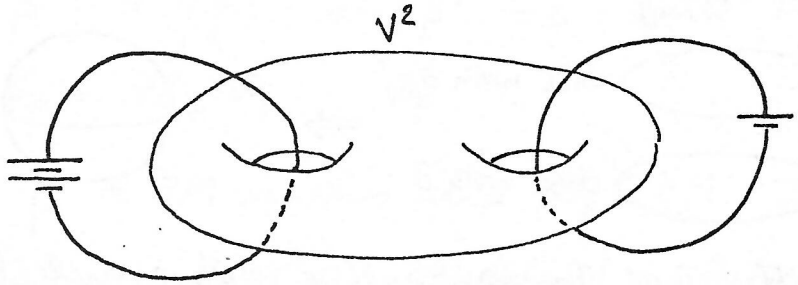
$$\mathcal{F} \wedge \mathcal{F} = -2\, \underset{\sim}{E}\cdot\underset{\sim}{B}\; dt\wedge dx\wedge dy\wedge dz.$$

Problem F.2. Let M^3 be a compact region in R^3 bounded by a conducting closed surface $V^2 = \partial M$ of genus 2. M^3 contains no charges & currents.

In magnetostatics we assume that (since V^2 is a conductor)

$\underset{\sim}{B}$ is tangent to V^2.

Show that the magnetic field in M^3 is uniquely determined by the currents passing thru the two "holes"



Problem F.3. Consider a smooth map

$$F : S^2 \longrightarrow T^2$$

of a 2 sphere S^2 into a 2 torus T^2. We know that a basis for $de\,R^1(T^2)$ is given by $\alpha' =$ "$d\theta$" and $\beta' =$ "$d\varphi$"; (See p. 145).

(i) Show that $F^*\alpha'$ and $F^*\beta'$ are exact.

(ii) Show that the Brouwer degree vanishes

$$\deg(F) = 0$$

Problem F.4 A "Killing" vector field X on a manifold M^n is a vector field whose flow φ_t is a 1-parameter group of <u>isometries</u>. Thus $(\varphi_t)_*$ preserves scalar products.

Let Y and Z be vector fields defined along an orbit of φ_t and let Y and Z be <u>invariant</u> under the flow φ_t. Thus

$$X \cdot \langle Y, Z \rangle = 0$$

(i) Show that this implies $\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$.

(ii) This equation is also true if Y and Z are any vectors at a given point $p \in M^n$; <u>why</u>?

(iii) Show then that X satisfies

$$\boxed{\begin{array}{l} X_{i|j} + X_{j|i} = 0 \\ \text{Killing's equation} \end{array}}$$

(iv) Show $\operatorname{div} X = 0$ (does this surprise you?). Hence, putting

$$\chi = X_i\, dx^i$$

show $\langle \Delta\chi, \chi \rangle = \|d\chi\|^2$ if M closed

(v) from the calculation on p. 151, we have, for a <u>closed</u> M^n

$$(\Delta\chi, \chi) = \int_M X_{j|k} X^{j|k} * 1 + \int_M \operatorname{Ric}(X,X) * 1$$

Being careful with factors of $\frac{1}{2}$, show that

$$\boxed{\begin{array}{l} \text{If the Ricci curvature of the closed} \\ M^n \text{ is negative} \\ \quad \operatorname{Ric}(X,X) \le 0, \quad = 0 \text{ only if } X = 0 \\ \text{then any Killing field } X \text{ on } M^n \\ \text{must vanish identically} \\ (\text{Another } \underline{\text{Theorem}} \text{ of Bochner}) \end{array}}$$

Thus negative Ricci curvature $\Rightarrow$ no "continuous groups of isometries".