

Kinetic Theory: from a vector calculus and exterior calculus (coordinate-free) perspectives

references (for vector calculus perspective):

Nicholson "Intro to Plasma Theory" 1983
http://tiny.cc/plasma_physics_nicholson

Ford & Uhlenbeck, "Lectures in Statistical Mechanics" 1963
http://tiny.cc/ford_uhlenbeck

digital version of lecture notes
http://tiny.cc/kinetics_lecture_notes

Course of Study

Vector Calculus

Liouville Equation

→ BBGKY Hierarchy ($\Gamma < 1$, Mayer Cluster Expansion) ref. F&U

→ Vlasov Equation (Collisionless Boltzmann Equation)

→ Fluid Equations (moment of distribution equations)

→ MHD (quasi-neutrality, and other approximations)

→ Master Equation (transition kernel) two-state

→ Boltzmann Equation (ref. F&U)

→ Fokker-Planck Equation (small step transitions)

Exterior Calculus (ref original research of these lecture notes)

- repeat line of derivation using exterior calculus for coordinate free

versions of equations

- "pull back" coordinate free versions to vector equivalents

Lesson #1 Liouville Equation (ref Nich ch. 4.2)

$\vec{x}_i$ = position particle i is measured

$\vec{v}_i$ = velocity particle i is measured

$\vec{X}_i(t)$ = positions occupied by particle i as function t

$\vec{V}_i(t)$ = velocities occupied by particle i as function t

↳ N_0 -dimensional phase space

density of systems $\equiv N(\vec{x}_1, \vec{v}_1, \dots, \vec{x}_{N_0}, \vec{v}_{N_0}; t)$

$$= \prod_{i=1}^{N_0} \delta[\vec{x}_i - \vec{X}_i(t)] \delta[\vec{v}_i - \vec{V}_i(t)]$$

Now take $\frac{\partial N}{\partial t}$ by using chain rule

$$\frac{\partial}{\partial t} \delta[\vec{x}_i - \vec{X}_i(t)] = -\frac{\partial \vec{X}_i}{\partial t} \cdot \vec{\nabla}_{\vec{x}_i} \delta[\vec{x}_i - \vec{X}_i(t)]$$

$$\frac{\partial}{\partial t} \delta[\vec{v}_i - \vec{V}_i(t)] = -\frac{\partial \vec{V}_i}{\partial t} \cdot \vec{\nabla}_{\vec{v}_i} \delta[\vec{v}_i - \vec{V}_i(t)]$$

to get

$$\begin{aligned} \frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \vec{V}_i(t) \cdot \vec{\nabla}_{\vec{x}_i} \prod_{j=1}^{N_0} \delta(\vec{x}_j - \vec{X}_j) \delta(\vec{v}_j - \vec{V}_j) \\ + \sum_{i=1}^{N_0} \frac{\partial \vec{V}_i}{\partial t} \cdot \vec{\nabla}_{\vec{v}_i} \prod_{j=1}^{N_0} \delta(\vec{x}_j - \vec{X}_j) \delta(\vec{v}_j - \vec{V}_j) = 0 \end{aligned}$$

using $a \delta(a-b) = b \delta(a-b)$ to replace $\vec{V}_i$ by $\vec{v}_i$ and $\frac{\partial \vec{V}_i}{\partial t}$ by $\dot{\vec{v}}_i$

and using the definition of N

$$\boxed{\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} N + \sum_{i=1}^{N_0} \dot{\vec{v}}_i \cdot \vec{\nabla}_{\vec{v}_i} N = 0}$$

Liouville Equation

combined with Lorentz force equation for $\dot{\vec{v}}_i$ and Maxwell equations for $\vec{B}$ and $\vec{E}$

Note that convective time derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{N_0} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} + \sum_{i=1}^{N_0} \dot{\vec{v}}_i \cdot \vec{\nabla}_{\dot{\vec{v}}_i}$$

so that the Liouville Equation can be simply written as

$$\boxed{\frac{DN}{Dt} = 0}$$

Now put the equation into conservative form by using the identity $\vec{\nabla} \cdot (a \vec{b}) = \vec{b} \cdot \vec{\nabla} a + a \vec{\nabla} \cdot \vec{b}$ so that

$$\vec{\nabla}_{\vec{x}_i} (N \vec{v}_i) = \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} N + N \vec{\nabla}_{\vec{x}_i} (\vec{v}_i) \quad \begin{matrix} \nearrow 0 \\ \text{because } \vec{x}_i \text{ and } \vec{v}_i \\ \text{independent} \end{matrix}$$

$$\vec{\nabla}_{\dot{\vec{v}}_i} (N \dot{\vec{v}}_i) = \dot{\vec{v}}_i \cdot \vec{\nabla}_{\dot{\vec{v}}_i} N + N \vec{\nabla}_{\dot{\vec{v}}_i} (\dot{\vec{v}}_i) \quad \nearrow 0$$

$$\text{because } \vec{\nabla}_{\dot{\vec{v}}_i} (\dot{\vec{v}}_i) = \frac{\partial}{\partial \dot{\vec{v}}_i} \left(\frac{q_{si}}{m_{si}} \left(\vec{E} + \frac{\vec{v}_i}{c} \times \vec{B} \right) \right) = 0$$

so Liouville Equation as

$$\boxed{\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \vec{\nabla}_{\vec{x}_i} (N \vec{v}_i) + \sum_{i=1}^{N_0} \vec{\nabla}_{\dot{\vec{v}}_i} (N \dot{\vec{v}}_i) = 0}$$

Now let's write this equation in terms of the probability density $f_{N_0}(\vec{x}_1, \vec{v}_1, \dots, \vec{x}_{N_0}, \vec{v}_{N_0}; t)$ such that probability that the system is at $(\vec{x}_1, \vec{v}_1, \dots, \vec{x}_{N_0}, \vec{v}_{N_0})$ is $f_{N_0} \prod_{i=1}^{N_0} d\vec{x}_i d\vec{v}_i$ such that $\int f_{N_0} \prod_{i=1}^{N_0} d\vec{x}_i d\vec{v}_i = 1$

$$\boxed{\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0} + \sum_{i=1}^{N_0} \dot{\vec{v}}_i \cdot \vec{\nabla}_{\dot{\vec{v}}_i} f_{N_0} = 0}$$

Liouville Equation for probability density

Lesson #2 BBGKY Hierarchy (Ref: Nicholson Sec 4.3)

$\Rightarrow$ integrate Liouville equation over $\int \prod_{i=1}^{N_0-n} d\vec{x}_i d\vec{v}_i$

Define the reduced probability distributions

$$f_n(\vec{x}_1, \vec{v}_1, \dots, \vec{x}_n, \vec{v}_n; t) \equiv V^n \int \prod_{i=n+1}^{N_0} d\vec{x}_i d\vec{v}_i f_{N_0}$$

normalized so that

$$N_0 = \frac{1}{V^n} \int \left(\prod_{i=1}^n d\vec{x}_i d\vec{v}_i \right) f_n$$

also note that

$$f_n = \frac{1}{V} \int d\vec{x}_{n+1} d\vec{v}_{n+1} f_{n+1}$$

For simplicity of derivation adopt a Coulomb model where (one species)

$$\dot{\vec{v}}_i(t) = \sum_{j=1}^{N_0} \vec{a}_{ij}$$

where

$$\vec{a}_{ij} = \frac{q^2}{m |\vec{x}_i - \vec{x}_j|^3} (\vec{x}_i - \vec{x}_j) \quad (\vec{a}_{ii} = 0) \quad \text{no self force}$$

is the acceleration of particle i due to the coulomb field of particle j

Liouville equation now becomes

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0} + \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0} = 0$$

We now integrate this equation $\int \prod_{i=n+1}^{N_0} d\vec{x}_i d\vec{v}_i$

let's start by deriving the equation for f_{N_0-1}

by integrating this equation over $\int d\vec{x}_{N_0} d\vec{v}_{N_0}$

obtaining

①

②

N_0

$$\int d\vec{x}_{N_0} d\vec{v}_{N_0} \frac{\partial f_{N_0}}{\partial t} + \int d\vec{x}_{N_0} d\vec{v}_{N_0} \sum_{i=1}^{N_0} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0} \\ + \int d\vec{x}_{N_0} d\vec{v}_{N_0} \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0} = 0 \quad \textcircled{3}$$

For term ① we can move $\partial/\partial t$ outside the integral so that

$$\textcircled{1} = \frac{\partial}{\partial t} \int d\vec{x}_{N_0} d\vec{v}_{N_0} f_{N_0} = V^{1-N_0} \frac{\partial}{\partial t} f_{N_0-1} \quad \text{using definition of } f_{N_0-1}$$

Term ② is also easy. For the first N_0-1 terms the operator $\vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i}$ is independent of the integration so that it can be moved outside of the integration and the definition of f_{N_0-1} used.

The last term with $i = N_0$ is

$$\int d\vec{x}_{N_0} d\vec{v}_{N_0} (v_{xN_0} \partial_{xN_0} + v_{yN_0} \partial_{yN_0} + v_{zN_0} \partial_{zN_0}) f_{N_0} \\ = \int d\vec{v}_{N_0} \left. \frac{d}{dx} (v_{xN_0} f_{N_0}) \right|_{xN_0=-\infty}^{xN_0=+\infty} \\ + 2 \text{ similar terms} \\ = 0$$

because f_{N_0} vanishes on the boundary

Thus

$$\textcircled{2} = V^{1-N_0} \sum_{i=1}^{N_0-1} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0-1}$$

Term ③ is much harder. Start by splitting the double sum

$$\sum_{i=1}^{N_0} \sum_{j=1}^{N_0} g_{ij} = \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} g_{ij} + \sum_{j=1}^{N_0-1} g_{N_0 j} + \sum_{i=1}^{N_0-1} g_{i N_0} + g_{N_0 N_0}$$

So now

$$\textcircled{3} = V^{1-N_0} \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-1} + \int d\vec{x}_{N_0} d\vec{v}_{N_0} \sum_{j=1}^{N_0-1} \vec{a}_{N_0 j} \cdot \vec{\nabla}_{\vec{v}_{N_0}} f_{N_0} \quad \text{after } d\vec{v}_{N_0} \text{ integration and evaluation on boundary} \\ + \int d\vec{x}_{N_0} d\vec{v}_{N_0} \sum_{i=1}^{N_0-1} \vec{a}_{i N_0} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0} + \vec{a}_{N_0 N_0} \cdot \vec{\nabla}_{\vec{v}_{N_0}} f_{N_0} \quad \textcircled{0}$$

= 0

leaves us with (multiplying by $V_0^{N_0-1}$)

$$\frac{\partial f_{N_0-1}}{\partial t} + \sum_{i=1}^{N_0-1} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0-1} + \sum_{i=1}^{N_0-1} \sum_{j=1}^{N_0-1} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-1} + \frac{1}{V} \sum_{i=1}^{N_0-1} \int d\vec{x}_{N_0} d\vec{v}_{N_0} \vec{a}_{iN_0} \cdot \vec{\nabla}_{\vec{v}_i} (V^{N_0} f_{N_0}) = 0$$

Now proceed to derive the equation for f_{N_0-2} by $\int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1}$

Term ① yields $V \frac{\partial}{\partial t} f_{N_0-2}$. Again term ② yields one term that vanishes upon integration leaving a sum from 1 to N_0-2 .

$$\textcircled{2} = V \sum_{i=1}^{N_0-2} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0-2}$$

Splitting the double sum in term ③ as before

$$\textcircled{3} = V \sum_{i=1}^{N_0-2} \sum_{j=1}^{N_0-2} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-2} + \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} \sum_{i=1}^{N_0-2} \vec{a}_{i,N_0-1} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-2} + \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} \sum_{j=1}^{N_0-2} \vec{a}_{N_0-1,j} \cdot \vec{\nabla}_{\vec{v}_{N_0-1}} f_{N_0-2} + \vec{a}_{N_0-1,N_0-1} \cdot \vec{\nabla}_{\vec{v}_{N_0-1}} f_{N_0-2} = 0$$

For term ④ we have

$$\textcircled{4} = V^{N_0-1} \sum_{i=1}^{N_0-1} \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} d\vec{x}_{N_0} d\vec{v}_{N_0} \vec{a}_{i,N_0} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0} = V^{N_0-1} \sum_{i=1}^{N_0-2} \int d\vec{x}_{N_0} d\vec{v}_{N_0} \vec{a}_{i,N_0} \cdot \vec{\nabla}_{\vec{v}_i} \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} f_{N_0} + 0$$

exchange $N_0 \leftrightarrow N_0-1$ as variables of integration and use the symmetry of f_{N_0} under that interchange to give

$$\begin{aligned}
 (4) &= V^{N_0-1} \sum_{i=1}^{N_0-2} \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} \vec{a}_{i, N_0-1} \cdot \vec{\nabla}_{\vec{v}_i} \underbrace{\int d\vec{x}_{N_0} d\vec{v}_{N_0} f_{N_0}}_{V^{1-N_0} f_{N_0-1}} \\
 &= \sum_{i=1}^{N_0-2} \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} \vec{a}_{i, N_0-1} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-1}
 \end{aligned}$$

which is the same as the second term of term (3)

This now gives combining (1), (2), (3), and (4) and dividing by V

$$\begin{aligned}
 \frac{d}{dt} f_{N_0-2} + \sum_{i=1}^{N_0-2} \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_{N_0-2} + \sum_{i=1}^{N_0-2} \sum_{j=2}^{N_0-2} \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-2} \\
 + \frac{2}{V} \sum_{i=1}^{N_0-2} \int d\vec{x}_{N_0-1} d\vec{v}_{N_0-1} \vec{a}_{i, N_0-1} \cdot \vec{\nabla}_{\vec{v}_i} f_{N_0-1} = 0
 \end{aligned}$$

By induction

$$\begin{aligned}
 \frac{d f_n}{dt} + \sum_{i=1}^n \vec{v}_i \cdot \vec{\nabla}_{\vec{x}_i} f_n + \sum_{i=1}^n \sum_{j=1}^n \vec{a}_{ij} \cdot \vec{\nabla}_{\vec{v}_i} f_n \\
 = - \frac{N_0-n}{V} \sum_{i=1}^n \int d\vec{x}_{n+1} d\vec{v}_{n+1} \vec{a}_{i, n+1} \cdot \vec{\nabla}_{\vec{v}_i} f_{n+1}
 \end{aligned}$$

BBGKY Hierarchy

define the functional (collisional operator) of $n+1$ particle with the first n particles

$$C[\] \equiv - \frac{N_0-n}{V} \sum_{i=1}^n \int d\vec{x}_{n+1} d\vec{v}_{n+1} \vec{a}_{i, n+1} \cdot \vec{\nabla}_{\vec{v}_i}$$

One gets the simple form of the BBGKY Hierarchy

$$\boxed{\frac{D f_n}{Dt} = C[f_{n+1}]}$$

Bogoliubov-Born-Green-Dirkwood-Yvon

note that $\int d\vec{v}_1 f_1(\vec{x}_1, \vec{v}_1; t) = 1$

$f_1(\vec{x}_1, \vec{v}_1, t) d\vec{x}_1 d\vec{v}_1$ is the probability that a particle finds itself in the region of phase space between $(\vec{x}_1, \vec{v}_1)$ and

$\vec{x}_1 + d\vec{x}_1, \vec{v}_1 + d\vec{v}_1$. f_1 is also the ensemble averaged number of particles per unit $\vec{x}_1$ and $\vec{v}_1$.

$f_2(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2; t)$ is the ensemble averaged number of particles per unit of $\vec{x}_1, \vec{x}_2, \vec{v}_1$ and $\vec{v}_2$. f_2 is also proportional to the probability that particle 1 finds itself at $(\vec{x}_1, \vec{v}_1)$ and particle 2 finds itself at $(\vec{x}_2, \vec{v}_2)$.

Lesson #3

Closure: the Mayer Cluster Expansion and the Ideas of Bogoliubov

This is the road to:

(1) Vlasov Equation $C[f_2] = C[f_1 f_1] \Rightarrow g_{12} = 0$

(2a) Boltzmann Equation $C[f_3] = 0$

(2b) Master Equation

(3) Vlasov-Fokker-Planck Equation

(small step size transport)



Mayer Cluster Expansion
weak correlation
 $\rho \ll 1$

$f_2 = f_1 f_1 + g_{12}$

$n b^3 \sim \rho^3 \ll 1$

$f_1 \rightarrow \bar{f}_1$

100 density

Idea of Bogoliubov

(ref. Nich ch. 4.3, App B; F&U ch. II.3-6, Ch. VII.3-6)

1871

First note that the rate of evolution of the left hand side of the f_i equation

$$\frac{Df_1}{Dt} = C[f_2]$$

is $\Omega \equiv \bar{v}/b$ where $\bar{v} \equiv \sqrt{k_B T/m}$ and $b \equiv \frac{e^2}{k_B T}$.
 while the rate of evolution of the
 right hand side is $d\Omega/dt/\Omega = nb^3(\frac{\bar{v}}{b})$

Now note the definitions: $\lambda = \frac{h\nu}{kT}$
collision rate

plasma parameter $\equiv g \equiv \frac{1}{n \lambda_D^3}$ where $\lambda_D = \sqrt{\frac{k_B T}{4\pi n e^2}}$

correlation parameter $\equiv \Gamma \equiv \frac{e^2/a}{k_B T}$ where $a = \left(\frac{4}{3}\pi n\right)^{-1/3}$
 $\omega_p = \left(\frac{4\pi n e^2}{m}\right)^{1/2}$

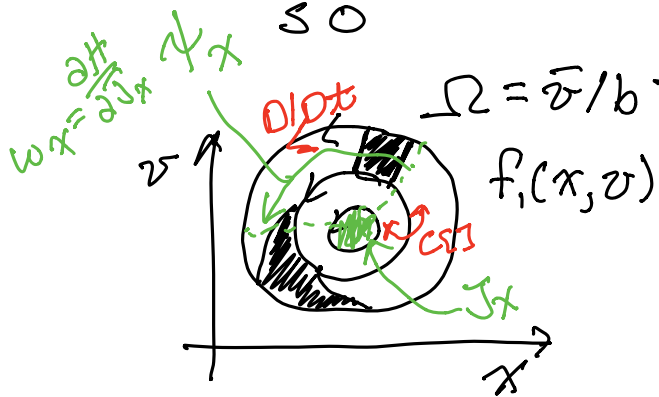
and

$$nb^3 \sim g^2 \sim \Gamma^3 \ll 1 \quad (\text{under dense plasma})$$

30

$$\Omega = \bar{\sigma}/b \gg n \bar{\sigma} b^2 = d\Omega/dt/\Omega$$

$$\omega_p \sim \sqrt{\frac{3/2 v}{n \sigma b^2}}$$



f_1 mixes along dynamical orbit at the dynamical rate, Ω , to $\bar{f}_1$. Then $\bar{f}_1$ evolves at the collision rate, $d\Omega/dt/\Omega$.

Continuing, f_2 relaxes at the collision rate to $\bar{f}_2$, then $\bar{f}_2$ evolves at the correlation rate $\equiv d^2\Omega/dt^2/\Omega^2 = n^2\sigma^2 v^2$ and so on. Also, $f_2[\bar{f}_1]$, $\ll d\Omega/dt/\Omega$

and so on. Also, $f_2[\bar{f}_1]$, f_2 is a functional of $\bar{f}_1(x)$ with no other t dependence

Mayer Cluster Expansion

Mayer Cluster Expansion

A linear graph theory approach to this perturbative expansion in Γ , analogous to Feynman Diagrams.

$$f_1(1) = f_1(1)$$

2-particle
correlation function

$$f_2(1,2) = f_1(1) f_1(2) + g(1,2)$$

$$f_3(1,2,3) = f_1(1) f_1(2) f_1(3) + f_1(1) g(2,3) + f_1(2) g(3,1) + f_1(3) g(1,2) + h(1,2,3)$$

3-particle
correlation function

in graphs:

$$① = \bullet = 1 \Rightarrow f_1$$

$$② = \bullet \text{---} \bullet = 1 + b_{12} \Rightarrow f_1 f_2 + g_{12}$$

$$③ = \bullet \text{---} \bullet \text{---} \bullet + \bullet \text{---} \bullet \text{---} \bullet + \bullet \text{---} \bullet \text{---} \bullet + (\text{triangle} + \text{triangle} + \text{triangle} + \text{triangle})$$

$$= 1 + b_{12} + b_{23} + b_{13} + (b_{12}b_{23} + b_{12}b_{13} + b_{13}b_{23} + b_{12}b_{23}b_{31})$$

$$\text{expansion of } \prod_{i < j=2}^N (1 + b_{ij})$$

$$\Rightarrow f_1 f_2 f_3 + f_1 g_{23} + f_2 g_{13} + f_3 g_{12} + h_{123}$$

$$\text{Note } \frac{g}{f_1^2} \sim \Gamma \sim (nb^3)^{1/3} \quad \frac{h}{f_1^3} \sim \Gamma^2$$

weaken convergence that Bogoliubov

Vlasov Equation (Collisionless Boltzmann Equation)

Start with $n=1$ BBGKY equation

$$0 = \partial_t f_1(\vec{x}_1, \vec{v}_1, t) + \vec{v}_1 \cdot \vec{\nabla}_{\vec{x}_1} f_1 + \frac{N_0 - 1}{V} \int d\vec{x}_2 d\vec{v}_2 \vec{a}_{12} \cdot \vec{\nabla}_{\vec{x}_1} f_2(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2, t)$$

Now assume that plasma is weakly correlated

(underdense) so that $f_2(1,2) \approx f_1(1) f_1(2)$

Note that we are explicitly making the Maxwell cluster expansion.

This gives

$$0 = \partial_t f_1 + \vec{v}_1 \cdot \vec{\nabla}_{\vec{x}_1} f_1 + [n_0 \int d\vec{x}_2 d\vec{v}_2 \vec{a}_{12} f_1(\vec{x}_1, \vec{v}_1, t) \cdot \vec{\nabla}_{\vec{x}_1} f_1(\vec{x}_1, \vec{v}_1, t)]$$

Now identify

ensemble averaged acceleration experienced by particle
by all other particles

$$\equiv \vec{a}(\vec{x}_1, t) \equiv n_0 \int d\vec{x}_2 d\vec{v}_2 \vec{a}_{12} f(\vec{x}_2, \vec{v}_2; t)$$

Then

$$\left[\partial_t f_1 + \vec{v}_1 \cdot \vec{\nabla}_{\vec{x}_1} f_1 + \vec{a} \cdot \vec{\nabla}_{\vec{v}_1} f_1 = 0 \right]$$

Vlasov Equation

Boltzmann collision operator

$$= \mathcal{P}_B[f_1]$$

$$\sim \mathcal{O}(\Gamma^3) \cdot \mathcal{O}(v)$$

$$\frac{v}{b} \ll \omega_p \ll n v b^2$$

to be shown in next section

can be slower plasma nodes so not $\mathcal{O}_B[f]$ is needed $\Rightarrow$ VFP equation

Boltzmann and Master Equations

For the derivation of the Boltzmann, Master, and Fokker-Planck equations we will switch gears and turn our attention to the problem of 3-body recombination in guiding center plasmas that are strongly magnetized such that $r_{ce} \ll b$.

In such a plasma the perpendicular electron motion is an adiabatic invariant. The parallel motion has a characteristic frequency of $v_{\parallel}/b \gg \omega_{EB}$ where the electron bounces back and forth along a magnetic field line. It then moves slowly $\mathbf{E} \times \mathbf{B}$ drifts in the plane $\perp$ to the magnetic field. Such atoms are called Guiding Center Atoms.

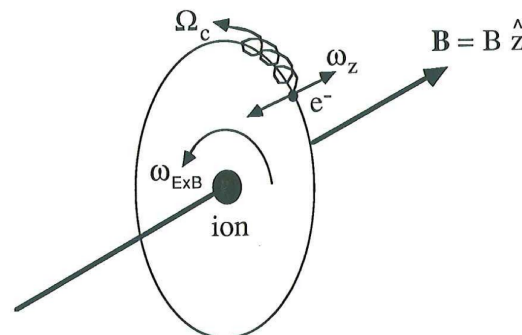
(ref Glinsky & O'Neil, Phys. Fluids B '91,

http://www.qitech.biz/papers_refereed/phys_plasma_3_body_91.pdf

Also use

Landau & Lifshitz
"Physical Kinetics"
Ch. I, Secs. 2-3.

For definition of
Boltzmann Equation



energy scale = $k_B T$, length scale $b = e^2 / k_B T e$,
 velocity scale $v_e = \sqrt{k_B T e / m_e}$, time scale b / v_e

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^N v_i \frac{\partial D_N}{\partial z_i} - \sum_{i=1}^N \sum_{j=0}^N \frac{\partial \phi_{ij}}{\partial z_i} \frac{\partial D_N}{\partial v_i} + \left(\frac{r_{ce}}{b} \right) \sum_{i=1}^N \sum_{j=0}^N \hat{z} \times \vec{\nabla}_i \phi_{ij} \cdot \vec{\nabla}_i D_N = 0$$

$\{ \} \approx$

Liouville's Equation $\{ \} \approx$ ExB drift

where $D_N(\vec{r}_1, v_1, \dots, \vec{r}_N, v_N; t) \equiv N$ -electron distribution function normalized such that

$$1 = \int d\vec{r}_1 dv_1 \dots d\vec{r}_N dv_N D_N$$

where $\vec{B} = B \hat{z}$, particle ϕ is the ion, $\phi_{i0} = \frac{-1}{|\vec{r}_i|}$,
 and $\phi_{ij} = \frac{1}{|\vec{r}_i - \vec{r}_j|}$ for $i, j = 1, \dots, N$

The s -electron distribution function is

$$f_s \equiv \frac{V^s}{b^3 s} \int d\vec{r}_{s+1} dv_{s+1} \dots d\vec{r}_N dv_N$$

where V is the plasma volume

Now the 1st and second order equations are

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial z_1} - \frac{\partial \phi_{10}}{\partial z_1} \frac{\partial f_1}{\partial v_1} + \left(\frac{r_{ce}}{b} \right) \hat{z} \times \vec{\nabla}_1 \phi_{10} \cdot \vec{\nabla}_1 f_1$$

$\sim O(nb^3)$ $\rightarrow$ by radial symmetry

$$= nb^3 \int d\vec{r}_2 dv_2 \left(\frac{\partial \phi_{12}}{\partial z_1} \frac{\partial}{\partial v_1} - \frac{r_{ce}}{b} \hat{z} \times \vec{\nabla}_1 \phi_{12} \cdot \vec{\nabla}_1 \right) f_2$$

$$\frac{\partial f_2}{\partial t} + \sum_{i=1}^2 \left[v_i \frac{\partial}{\partial z_i} - \frac{\partial \phi_{i0}}{\partial z_i} \frac{\partial}{\partial v_i} + \left(\frac{r_{ce}}{b} \right) \hat{z} \times \vec{\nabla}_i \phi_{i0} \cdot \vec{\nabla}_i \right] f_2$$

$$+ \left[- \frac{\phi_{12}}{\partial z_1} \left(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} \right) + \frac{r_{ce}}{b} \hat{z} \times \vec{\nabla}_1 \phi_{12} \cdot (\vec{\nabla}_1 - \vec{\nabla}_2) \right] f_2$$

$$= nb^3 \int d\vec{r}_3 dv_3 \sum_{i=1}^2 \left[\frac{\partial \phi_{i3}}{\partial z_i} \frac{\partial}{\partial v_i} - \frac{r_{ce}}{b} \hat{z} \times \vec{\nabla}_i \phi_{i3} \cdot \vec{\nabla}_i \right] f_3$$

closure

expand in density of gas, not weakness of correlation

two small parameters $r_{ce}/b \ll 1$ and $nb^3 \ll 1$.
We lose Debye shielding in this equation, but it is unimportant at the close particle distance of separation.

Make the change of variable from $v_i \rightarrow \varepsilon_i$

$$z_i \equiv -[v_i^2/2 + \phi_{i0}(z_i, \rho_i)] \quad i=1,2$$

which is the z -binding energy of the electron

Now the equations take the simple form

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial z_1} = -nb^3 \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} v_1 \frac{\partial f_2}{\partial \varepsilon_1} \quad (\text{Eq 1})$$

$$\frac{\partial f_2}{\partial t} + \left(v_1 \frac{\partial}{\partial z_1} + v_2 \frac{\partial}{\partial z_2} \right) f_2 + \frac{\partial \phi_{12}}{\partial z_1} \left(v_1 \frac{\partial}{\partial \varepsilon_1} - v_2 \frac{\partial}{\partial \varepsilon_2} \right) f_2 = 0 \quad (\text{Eq 2})$$

Now further using the ideas of Bougeolibou

f_1 evolves to be independent of z_1 to the

form $f_1 = \bar{f}_1(\rho_1, \varepsilon_1; t) + \delta f_1(z_1, \rho_1, \varepsilon_1; t)$,

where $\delta f_1 / \bar{f}_1 \sim (nb^3)$, at the rate $\bar{v}/b$.

The LHS of Eq 1 dominates for early time

so that $\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial z_1} = 0$. At a later time

$\bar{f}_1$ evolves at a rate $n\bar{v}b^2 = \frac{\bar{v}}{b}(nb^3)$ determined

by the RHS. We derive the equation for the evolution of $\bar{f}_1$ by substituting $f_1 = \bar{f}_1 + \delta f$

into Eq. 1 and keep the terms of nb^3 order.

$$\frac{\partial \bar{f}_1}{\partial t} + v_1 \frac{\partial (\delta f_1)}{\partial z_1} = -nb^3 \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} v_1 \frac{\partial f_2}{\partial \varepsilon_1}$$

We now integrate over a closed orbit of particle 1 $\oint_{\varepsilon_1} \frac{dz_1}{v_1}$

$$\left(\oint_{\varepsilon_1} \frac{dz_1}{v_1} \right) \frac{d\bar{f}_1}{dt} + \oint_{\varepsilon_1} dz_1 \frac{\partial(\delta P_1)}{\partial z_1} = -nb^3 \oint_{\varepsilon_1} dz_1 \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} \frac{df_2}{dz_1}$$

$\tau_1(p_1, \varepsilon_1) \equiv$ period of oscillatory motion of particle 1

$$\tau_1 \frac{d\bar{f}_1}{dt} = - \frac{d}{d\varepsilon_1} nb^3 \oint_{\varepsilon_1} dz_1 \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} f_2 \quad (\text{Eq. 3})$$

note that for $\varepsilon_1 < 0$, particle 1 is unbound and

$$f_1(p_1, \varepsilon_1 < 0) = f_{th}(\varepsilon_1) = e^{\varepsilon_1 / \sqrt{2v}}$$

Eq. 2 has terms of the same order, $\bar{v}/b$, when written in unscaled variables. f_2 will relax to become a functional of f_1 , that is $f_2[f_1]$.

After the relaxation has occurred, $df_2/dt \sim O(nb^3)$

Dropping df_2/dt and operating on the remaining terms with $\oint_{\varepsilon_1} \frac{dz_1}{v_1} \int d\vec{r}_2 dv_2$

$$\oint_{\varepsilon_1} \frac{dz_1}{v_1} \int d\vec{r}_2 dv_2 \left(v_1 \frac{\partial}{\partial z_1} + v_2 \frac{\partial}{\partial z_2} \right) f_2 = - \oint_{\varepsilon_1} \frac{dz_1}{v_1} \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} \left(v_1 \frac{\partial}{\partial z_1} - v_2 \frac{\partial}{\partial z_2} \right) f_2$$

where particle 1 is assumed to be bound, $\varepsilon_1 > 0$.

Do $\int dz_2$ integral on the LHS giving

$$\oint_{\varepsilon_1} \frac{dz_1}{v_1} \int d\vec{r}_2 dv_2 v_2 [f_2(z_2 = +\infty) - f_2(z_2 = -\infty)] = - \frac{d}{d\varepsilon_1} \oint_{\varepsilon_1} dz_1 \int d\vec{r}_2 dv_2 \frac{\partial \phi_{12}}{\partial z_1} f_2 = \frac{1}{nb^3} \text{RHS of Eq. 3}$$

Consider $v_2 > 0$ so $f_2(z_2 = -\infty)$ is incident particle 2 on bound particle 1 so

$$f_2(z_2 = \infty) = \bar{f}_1(p_1, \varepsilon_1) \bar{f}_1(p_2, \varepsilon_2)$$

Eq. 2 implies $f_2(p_1, \varepsilon'_1, p_2, \varepsilon'_2) = f_2(p_1, \varepsilon_1, p_2, \varepsilon_2)$ where $(p_1, \varepsilon'_1, p_2, \varepsilon'_2) \rightarrow (p_1, \varepsilon_1, p_2, \varepsilon_2)$ during the collision so that

$$f_2(z_2 = +\infty) = \bar{f}_1(p_1, \varepsilon'_1) \bar{f}_1(p_2, \varepsilon'_2)$$

so that

$$\frac{d\bar{f}_1}{dt} = nb^3 \frac{1}{\tau_1} \oint_{\mathbf{z}_1} \frac{d\mathbf{z}_1}{v_1} \int d\vec{r}_{12} dv_2 |v_2| [\bar{f}_1(\rho_1, \epsilon'_1) \bar{f}_1(\rho_2, \epsilon'_2) - \bar{f}_1(\rho_1, \epsilon_1) \bar{f}_1(\rho_2, \epsilon_2)]$$

$= \mathcal{C}_B[\bar{f}_1]$ Boltzmann Equation

where $|v_2|$ is needed to make valid for $v_2 < 0$, and the RHS is referred to as the Boltzmann collision operator. Expanding with $\int d\epsilon'_1 d\rho'_1 d\epsilon'_2 \delta(\rho'_1 - \rho_1) \delta(\epsilon'_1 - \epsilon_1) \delta(\rho'_2 - \rho_2) \delta(\epsilon'_2 - \epsilon_2)$, then regrouping (form of FSU)

$$\frac{df(\Gamma)}{dt} = nb^3 \int d\Gamma_1 d\Omega |v| \frac{d\sigma}{d\Omega} [f(\Gamma') f(\Gamma'_1) - f(\Gamma) f(\Gamma_1)] = \mathcal{C}_B[f]$$

$\swarrow$ differential cross section

where $d\Gamma = d\rho_1 d\epsilon_1$, $d\Gamma_1 = dv_2$, $d\Gamma'_1 = d\rho'_2 d\epsilon'_2$, $d\Gamma' = d\rho'_1 d\epsilon'_1$, $d\Omega = d\Gamma' d\Gamma'_1$

and $|v| \frac{d\sigma}{d\Omega} = \frac{1}{\tau} \oint_{\mathbf{z}_1} \frac{d\mathbf{z}_1}{v_1} \int \rho_2 d\rho_2 |v_2| \delta(\rho'_1 - \rho_1) \delta(\epsilon'_1 - \epsilon_1) \delta(\rho'_2 - \rho_2) \delta(\epsilon'_2 - \epsilon_2)$
 $= K(\rho_1, \epsilon_1, \rho_2, \epsilon_2 | \rho'_1, \epsilon'_1, \rho'_2, \epsilon'_2)$ Alternatively,

$$\frac{df(\Gamma)}{dt} = nb^3 \int d\Gamma' d\Gamma_1 d\Gamma'_1 K(\Gamma, \Gamma_1 | \Gamma', \Gamma'_1) [f(\Gamma') f(\Gamma'_1) - f(\Gamma) f(\Gamma_1)]$$

(form of LL' Physical Kinetics')

Now continue onto the Master Equation we rederive the Boltzmann equation as

$$\frac{d\bar{f}_1}{dt} = nb^3 \frac{1}{\tau_1} \oint_{\mathbf{z}_1} \frac{d\mathbf{z}_1}{v_1} \int d\vec{r}_{12} dv_2 |v_2| f_{th}(\epsilon_1) f_{th}(\epsilon_2)$$

$$\left[\frac{\bar{f}_1(\rho_1, \epsilon'_1) \bar{f}_1(\rho_2, \epsilon'_2)}{f_{th}(\epsilon'_1) f_{th}(\epsilon'_2)} - \frac{\bar{f}_1(\rho_1, \epsilon_1) \bar{f}_1(\rho_2, \epsilon_2)}{f_{th}(\epsilon_1) f_{th}(\epsilon_2)} \right]$$

using conservation of energy $\epsilon_1 + \epsilon_2 = \epsilon'_1 + \epsilon'_2$, and $f_{th}(\epsilon) = \frac{e^{-\epsilon}}{\Omega \tau}$.

In the post collision state particle 2 is free so that $\epsilon'_2 < 0$ and $f_1(\rho_2, \epsilon'_2) = f_{th}(\epsilon'_2)$. In the precollision state particle 1, particle 2, or both particle 1 and 2 are free. So we define $\epsilon = \max(\epsilon_1, \epsilon_2)$ and let ρ be the corresponding ρ_1 or ρ_2 . The other particle is free and is guaranteed to have a thermal distribution. So

$$\frac{d\bar{f}_1(\rho_1, \epsilon_1)}{dt} = f_{th}(\epsilon_1) nb^3 \frac{1}{\tau_1} \oint_{\mathbf{z}_1} \frac{d\mathbf{z}_1}{v_1} \int d\vec{r}_{12} dv_2 |v_2| f_{th}(\epsilon_2) \left[\frac{\bar{f}_1(\rho', \epsilon')}{f_{th}(\epsilon')} - \frac{\bar{f}_1(\rho, \epsilon)}{f_{th}(\epsilon)} \right]$$

define the forward transition rate

$$k_+(\rho_1, \varepsilon_1 | \bar{\rho}, \bar{\varepsilon}) \equiv nb^3 \frac{1}{\tau} \oint_{\varepsilon_1} \frac{dz_1}{v_1} \int \rho_2 d\rho_2 d\theta_2 d\sigma_2 |v_2| f_{th}(\varepsilon_2) \\ \times \delta[\bar{\varepsilon} - \varepsilon_+(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)] \\ \times \delta[\bar{\rho} - \rho_+(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)]$$

where plus is evolution forward in time from initial state $(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)$ where particle #1 is initially bound ($z > 0$) and θ_2 is measured relative to θ_1 , and z_1 specifies the initial phase of the bound z -motion.

And the backward transition rate

$$k_-(\rho_1, \varepsilon_1 | \bar{\rho}, \bar{\varepsilon}) \equiv nb^3 \frac{1}{\tau} \oint_{\varepsilon_1} \frac{dz_1}{v_1} \int \rho_2 d\rho_2 d\theta_2 d\sigma_2 |v_2| f_{th}(\varepsilon_2) \\ \times \delta[\bar{\varepsilon} - \varepsilon_-(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)] \\ \times \delta[\bar{\rho} - \rho_-(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)]$$

where the minus indicates evolution backward in time from an initial state $(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)$

The functions ε_- and ρ_- refer to the electron with the largest energy in the pre-collision state.

By time reversal symmetry

$$k_+(\rho_1, \varepsilon_1 | \bar{\rho}, \bar{\varepsilon}) = k_-(\rho_1, \varepsilon_1 | \bar{\rho}, \bar{\varepsilon}) \\ = k(\rho_1, \varepsilon_1 | \bar{\rho}, \bar{\varepsilon})$$

So now we write

$$\frac{d\bar{f}_i(\rho, \varepsilon)}{dt} = \int d\bar{\rho} d\bar{\varepsilon} f_{th}(\varepsilon) k(\rho, \varepsilon | \bar{\rho}, \bar{\varepsilon}) \left[\frac{\bar{f}_i(\bar{\rho}, \bar{\varepsilon})}{f_{th}(\bar{\varepsilon})} - \frac{\bar{f}_i(\rho, \varepsilon)}{f_{th}(\varepsilon)} \right]$$

Define

$$W(\rho, \varepsilon) \equiv nb^3 2\pi \rho \tau(\rho, \varepsilon) \bar{f}_i(\rho, \varepsilon)$$

where $nb^3 2\pi \rho \tau(\rho, \varepsilon)$ is the density of states for $d\rho d\varepsilon$

and $W_{th}(\rho, \varepsilon) \equiv nb^3 2\pi \rho \Gamma(\rho, \varepsilon) f_{th}(\varepsilon)$

so that

$$\frac{\partial W(\rho, \varepsilon)}{\partial t} = \int d\bar{\rho} d\bar{\varepsilon} W_{th}(\rho, \varepsilon) K(\rho, \varepsilon | \bar{\rho}, \bar{\varepsilon}) \left[\frac{W(\bar{\rho}, \bar{\varepsilon})}{W_{th}(\bar{\rho}, \bar{\varepsilon})} - \frac{W(\rho, \varepsilon)}{W_{th}(\rho, \varepsilon)} \right]$$

By using time reversal symmetry and Liouville's Theorem on the RHS equation must conserve particle number for arbitrary choice $W(\rho, \varepsilon)$, one gets detailed balance

$$W_{th}(\rho, \varepsilon) K(\rho, \varepsilon | \bar{\rho}, \bar{\varepsilon}) = W_{th}(\bar{\rho}, \bar{\varepsilon}) K(\bar{\rho}, \bar{\varepsilon} | \rho, \varepsilon)$$

Finally

$$\boxed{\frac{\partial W(\rho, \varepsilon)}{\partial t} = \int d\bar{\rho} d\bar{\varepsilon} [K(\bar{\rho}, \bar{\varepsilon} | \rho, \varepsilon) W(\bar{\rho}, \bar{\varepsilon}) - K(\rho, \varepsilon | \bar{\rho}, \bar{\varepsilon}) W(\rho, \varepsilon)]}$$

Master Equation

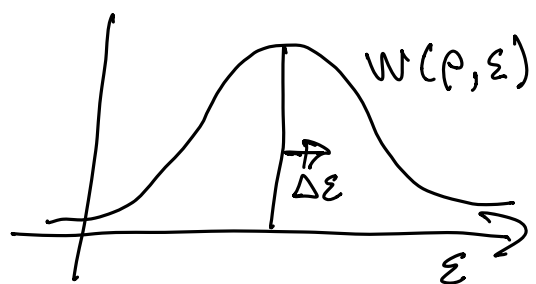
where

$$K(\rho, \varepsilon | \bar{\rho}, \bar{\varepsilon}) = nb^3 \frac{1}{\tau_1} \oint_{\varepsilon_1} \frac{dz_1}{v_1} \int \rho_2 d\rho_2 d\theta_2 dv_2 [v_2] f_{th}(\varepsilon_2) \\ \times \mathcal{P}[\bar{\varepsilon} - \varepsilon(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)] \\ \times \delta[\bar{\rho} - \rho(z_1, \rho_1, \varepsilon_1, \rho_2, \theta_2, \varepsilon_2)]$$

= transition rate (kernel) of scale $n\bar{v}b^2$

= collision rate = $\frac{\bar{v}}{v} nb^3$

Now to the Fokker-Planck collision operator
 Assume electron #1 is bound ($\varepsilon_1 > 0$) at a
 radius $\rho_1 < 1$ and electron #2 is incident
 from infinity $\varepsilon_2 < 0$ at a relatively large
 radius $\rho_2 \gtrsim \rho_d \gg 1$. We will solve Eq. 2
 perturbatively through an expansion in $1/\rho_2$ for f_2
 that is through the weakness of interaction
 ϕ_{12} and ϕ_{20} . We will also simplify the analysis
 by assuming that electron #1 is bound deeply
 enough that the oscillatory motion is
 simple harmonic. The result for f_2 will
 be substituted into Eq. 3. Pictorially



$$\langle \Delta \varepsilon \rangle \ll \frac{W}{dW/d\varepsilon}$$

note that $\Delta \rho = 0$ for $\rho_2 \gg 1$

Rewrite Eq. 3 as

$$(L^{(0)} + L^{(1)}) f_2 = 0$$

where

$$L^{(0)} \equiv \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial z_1} \Big|_{\varepsilon_1} + v_2 \frac{\partial}{\partial z_2} \Big|_{\varepsilon_2}$$

$$L^{(1)} \equiv -\frac{\partial \phi_{12}}{\partial z_1} v_1 \frac{\partial}{\partial \varepsilon_1} - \left(\frac{\partial \phi_{12}}{\partial z_2} + \frac{\partial \phi_{20}}{\partial z_2} \right) \frac{\partial}{\partial v_2}$$

Look for solution of form

$$f_2 = f_2^{(0)} + f_2^{(1)}$$

where

$$f_2^{(0)} = \bar{f}_1(\rho_1, \varepsilon_1) e^{-v_2^2/2} / \sqrt{2\pi}$$

$f_2^{(1)}$ determined by

$$L^{(0)} f_2^{(1)} = -L^{(1)} f_2^{(0)}$$

note $L^{(0)} = \frac{D}{Dt} \Big|_{0^{th} \text{ order orbits}}$

$$\begin{aligned} f_2^{(1)} &= \int_{-\infty}^t dt' L^{(1)} f_2^{(0)} \\ &= \int_{-\infty}^t dt' \left[\left(\frac{\partial \phi_{12}}{\partial z_1} v_1 \right)' \frac{\partial \bar{f}_1}{\partial \varepsilon_1} \frac{e^{-v_2^2/2}}{\sqrt{2\pi}} \right. \\ &\quad \left. - \left(\frac{\partial \psi_{12}}{\partial z_2} + \frac{\partial \phi_{20}}{\partial z_2} \right)' v_2 \frac{e^{+v_2^2/2}}{\sqrt{2\pi}} \bar{f}_1 \right] \end{aligned}$$

where the 0th order orbits are

$$z_1(t') = z_1, \quad v_2(t') = v_2, \quad z_2(t') = z_2 + v_2(t' - t)$$

$$z_1(t') = a_1 \cos[\psi_1(t')] \quad \psi_1(t') = \psi_1 + \omega(t' - t)$$

where $\omega_1^2 a_1^2 / 2 = (1/\rho_1 - \varepsilon_1)$ and $\omega_1^2 = 1/\rho_1^3$

Substituting this into Eq. 3 for f_2 yields

$$r_1 \frac{\partial \bar{f}_1}{\partial t} = - \frac{\partial}{\partial \varepsilon_1} \left(B \bar{f}_1 + A \frac{\partial \bar{f}_1}{\partial \varepsilon_1} \right)$$

where

$$\begin{aligned} A &\equiv nb^3 r_1 \int_0^{2\pi} \frac{d\psi_1}{2\pi} \int d\vec{r}_2 \int_{-\infty}^{\infty} dv_2 \int_{-\infty}^t dt' \frac{e^{-v_2^2/2}}{\sqrt{2\pi}} \left(v_1 \frac{\partial \phi_{12}}{\partial z_1} \right) \left(v_1 \frac{\partial \phi_{12}}{\partial z_1} \right)' \\ B &\equiv -nb^3 r_1 \int_0^{2\pi} \frac{d\psi_1}{2\pi} \int d\vec{r}_2 \int_{-\infty}^{\infty} dv_2 \int_{-\infty}^t dt' \frac{e^{-v_2^2/2}}{\sqrt{2\pi}} v_2 \left(v_1 \frac{\partial \phi_{12}}{\partial z_1} \right)' \\ &\quad \times \left(\frac{\partial \psi_{12}}{\partial z_2} + \frac{\partial \phi_{20}}{\partial z_2} \right)' \end{aligned}$$

To evaluate A

$$\frac{\partial \phi_{12}}{\partial z_1} \sim \frac{z_2}{(z_2^2 + \rho_2^2)^{3/2}} = \frac{z}{\pi} \int_0^{\infty} dk \, k \, K_0(k\rho_2) \sin k z_2$$

$$A = nb^3 r_1 \int_0^{2\pi} \frac{d\psi_1}{2\pi} \int_{-\infty}^{\infty} dz_2 \int_{\rho_d}^{\infty} 2\pi \rho_2 d\rho_2 \int_{-\infty}^{\infty} dv_2 \int_{-\infty}^t dt' \left(\frac{z}{\pi} \right)^2 \int_0^{\infty} dk \, k$$

$$\int_0^\infty d\varphi \varphi \frac{e^{-v_z^2/2}}{\sqrt{2\pi}} K_0(k\rho_z) K_0(q\rho_z) (a, \omega_1)^2 \sin \psi,$$

do z_2, ψ, t, k
integrals

$$\sin[\omega_1(t'-t) + \psi] \sin k z_2 \sin[q z_2 + q \sigma_1(t'-t)]$$

use asymptotic
expansion

$K_1(\xi) \rightarrow \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}$
and saddle point

$$= nb^3 \tau_1 (a, \omega_1)^2 \frac{8\pi}{\sqrt{2\pi}} \int_0^\infty \frac{d\sigma_z}{\sigma_z} e^{-v_z^2/2} \int_{\omega_1 \rho_d / v_z}^\infty d\xi \xi K_0^2(\xi) \\ = nb^3 \tau_1 (a, \omega_1)^2 \frac{2\pi^2}{\sqrt{3}} \frac{e^{-3/2} (2\omega_1 \rho_d)^{2/3}}{(2\omega_1 \rho_d)^{1/3}}$$

It also can be shown that $B=A$ (see ref) so

$$\tau_1 \frac{d\bar{f}_1}{dt} = -\frac{d}{d\varepsilon_1} \left[A \left(\bar{f}_1 + \frac{d\bar{f}_1}{d\varepsilon_1} \right) \right]$$

Fokker-Planck Equation

note that this approximation is fundamentally violated for this problem Δp is always large when it is not $\emptyset$ because there must be a particle interchange for it to change. Furthermore A is very sensitive to ρ_d when $\rho_d \sim 1$ and $\Delta\varepsilon$ is large. Also note that $A \rightarrow \infty$ as $\rho_d \rightarrow \infty$ since ε_1 will be an adiabatic invariant. Finally note that if $\bar{f}_1(p_1, \varepsilon_1) = e^{-\varepsilon_1} f_{th}$ then $d\bar{f}_1/dt = 0$.

Since $\Delta\varepsilon \sim \varepsilon$ and $\Delta p \sim 1$ the FP approximation can not be made and $k(\bar{p}, \bar{\varepsilon} | p, \varepsilon)$ needs to

be estimated by a Monte Carlo method. First note the ordering of time scales

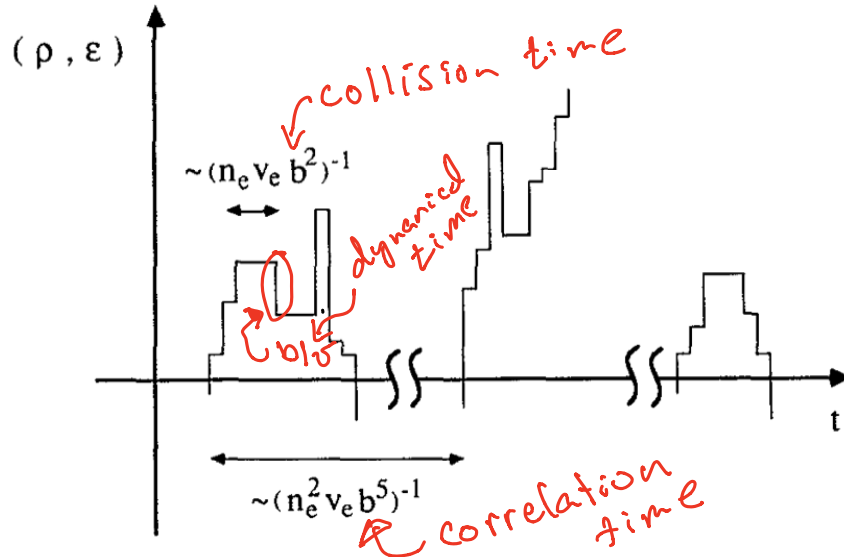
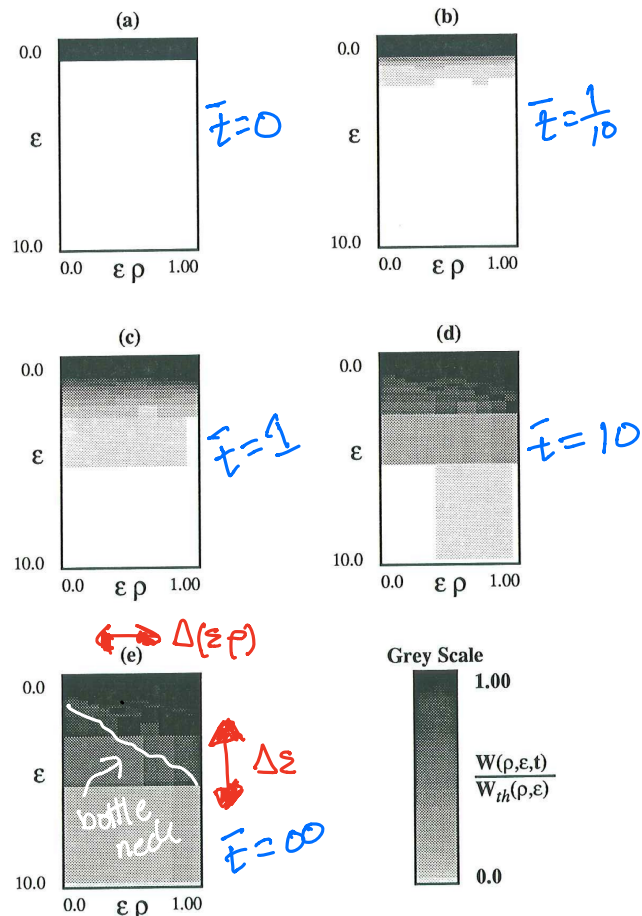


FIG. 5. An example of a time history showing the order of the time scales. The state of the atom (ρ, ϵ) is plotted as a function of time. The square corners show the duration collision [i.e., $t \sim (v_e/b)^{-1}$] as being effectively instantaneous on the time scale of an evolution. Three evolutions are shown. The duration of an evolution is of order $(n_e v_e b^2)^{-1}$, and the time between evolutions is of order $(n_e^2 v_e b^5)^{-1}$.

Now for the time evolution of $W(\epsilon, \rho, \epsilon; t)$ to steady state

$$\bar{t} = n_e v_e b^2 t$$

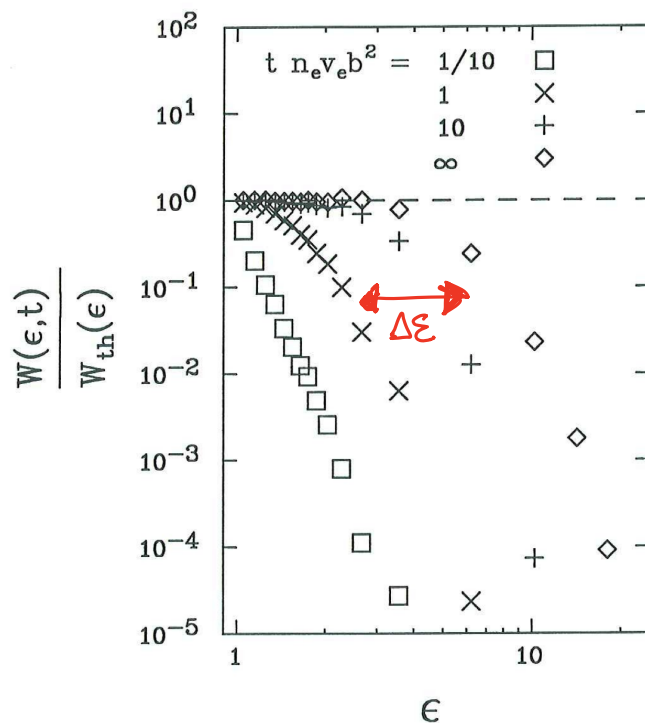
Figure 3.7: Time dependent distribution function divided by its thermal value. Shown are its value at four different times (a) $t = 0$ (initial condition), (b) $t = (1/10)(n_e v_e b^2)^{-1}$, (c) $t = (n_e v_e b^2)^{-1}$, (d) $t = 10(n_e v_e b^2)^{-1}$ and its steady state value (e). All scales are linear.



1/100 of collisions change $\epsilon\rho$

flux through bottleneck
0.070(10)
 $n^2 v_e b^5$

and the $\int d(\varepsilon p) W(\varepsilon p, \varepsilon; t) = W(\varepsilon; t)$



Lesson #4

Fluid Equations (ref. Nicholson 27.2)

Start with the Vlasov Equation for species, s .

$$\frac{\partial}{\partial t} f_s(\vec{x}, \vec{v}; t) + \vec{v} \cdot \vec{\nabla}_{\vec{x}} f_s + \frac{q_s}{m_s} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \cdot \vec{\nabla}_{\vec{v}} f_s = 0$$

Use normalization remember that collisions are neglected $n \bar{v} b^2 \ll \Omega$

$$n_s(\vec{x}, t) = \int d\vec{v} f_s(\vec{x}, \vec{v}, t) \quad \text{if } \omega = \omega_p$$

and the definition

$$\vec{V}_s \equiv \frac{1}{n_s} \int d\vec{v} \vec{v} f_s(\vec{x}, \vec{v}, t)$$

Operate on the Vlasov Equation with $\int d\vec{v}$

$$\textcircled{1} = \frac{\partial}{\partial t} n_s$$

$$\textcircled{2} = \int d\vec{v} \vec{v} \cdot \vec{\nabla}_{\vec{x}} f_s = \vec{\nabla}_{\vec{x}} \cdot \int d\vec{v} \vec{v} f_s = \vec{\nabla}_{\vec{x}} \cdot (n_s \vec{V}_s)$$

$$\vec{v} \cdot \vec{\nabla}_x f_s = \vec{\nabla}_x \cdot (\vec{v} f_s) - f_s \vec{\nabla}_x \cdot \vec{v}$$

$$\textcircled{3} = \textcircled{4} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt} n_s + \vec{\nabla} \cdot (n_s \vec{V}_s) = 0}$$

continuity Equation

where $\vec{v} = \vec{V}_x$

Operate on the Vlasov Equation with $\int d\vec{v} \vec{v}$

$$\frac{d}{dt} \int d\vec{v} \vec{v} f_s + \int d\vec{v} \vec{v} \vec{v} \cdot \vec{\nabla}_x f_s + \frac{q_s}{m_s} \int d\vec{v} \vec{v} \left[\left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) \cdot \vec{\nabla}_v f_s \right] = 0$$

$$\textcircled{1} = \frac{d}{dt} (n_s \vec{V}_s)$$

$$\textcircled{2} = \int d\vec{v} \vec{v} \vec{v} \cdot \vec{\nabla}_x f_s = \vec{\nabla}_x \cdot \left(\int d\vec{v} (\vec{v} \vec{v} f_s) \right) = \vec{\nabla}_x \cdot (n_s \langle \vec{v} \vec{v} \rangle)$$

$$= \vec{v} \cdot \vec{\nabla}_x (\vec{v} f_s) = \vec{\nabla}_x \cdot (\vec{v} \vec{v} f_s)$$

$$\langle g \rangle = \frac{\int d\vec{v} g f_s}{\int d\vec{v} f_s} = \frac{1}{n_s} \int d\vec{v} g f_s$$

$$\textcircled{3} = -(q_s/m_s) \vec{E} n_s$$

by
integration
by parts

$$\textcircled{4} = \frac{q_s}{m_s c} \int d\vec{v} \vec{v} (\vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f_s = \frac{q_s}{m_s c} \int d\vec{v} \vec{v} \nabla_v \cdot [(\vec{v} \times \vec{B}) f_s]$$

$$= - \frac{q_s}{m_s c} \int d\vec{v} (\vec{v} \times \vec{B}) f_s = - \frac{q_s}{m_s c} n_s (\vec{V}_s \times \vec{B})$$

by
integration
by parts

Therefore,

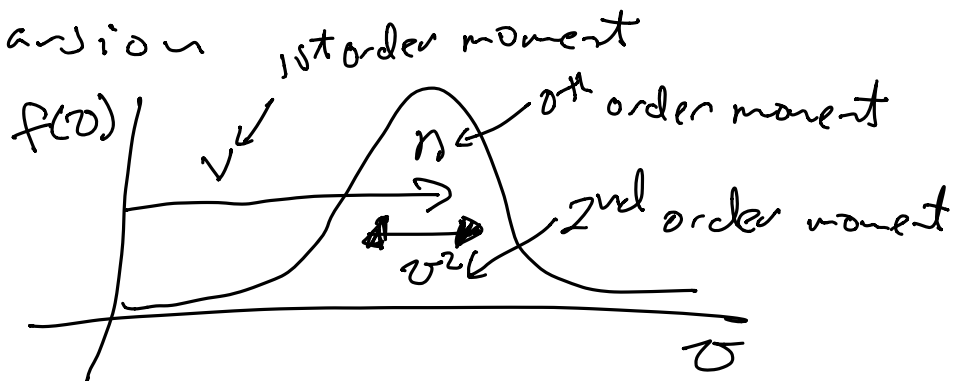
$$\boxed{\frac{d}{dt} (n_s \vec{V}_s) + \vec{\nabla} \cdot (n_s \langle \vec{v} \vec{v} \rangle) = \frac{q_s}{m_s} n_s \left(\vec{E} + \frac{\vec{V}_s}{c} \times \vec{B} \right)}$$

Momentum Eq or Fluid Force Eq

Note that like the BBGKY hierarchy where there is a coupling to the next order distribution, the fluid equations are coupled to the next order velocity moment. The 0th order involves $\langle \vec{v} \rangle$. The 1st order involves $\langle \vec{v} \vec{v} \rangle$ and so on.

The hierarchy of equations needs to be closed. In general, the equation for the time derivative of $\langle \vec{v} \vec{v} \rangle$ is called the energy equation.

What we are doing is approximating the velocity distribution with a moment expansion



Here are some ways of closing the equations and obtaining $\langle \vec{v} \vec{v} \rangle$:

(1) Cold plasma

$$f_s(\vec{x}, \vec{v}, t) = n_s(\vec{x}, t) \delta(\vec{v} - \vec{V}_s)$$

$$\text{thus } \langle \vec{v} \vec{v} \rangle = \frac{1}{n_s} \int d\vec{v} n_s \vec{v} \vec{v} \delta(\vec{v} - \vec{V}_s)$$

$$= \vec{V}_s \vec{V}_s$$

(2) Isotropic velocity distribution

$$\langle \vec{v} \vec{v} \rangle = v_s^2(\vec{x}) \mathbf{I} \quad T_s \equiv m_s v_s^2$$

now the second term in momentum

Equation takes the form

$$\nabla \cdot (n_s \langle \vec{v} \vec{v} \rangle) = \nabla \cdot (n_s v_s^2 \frac{\vec{v} \vec{v}}{v_s^2}) = \nabla P_s / m_s$$

where $P_s \equiv n_s m_s v_s^2 = n_s T_s$

(3) Drifting Maxwellian

$$\langle \vec{v} \vec{v} \rangle \equiv \vec{V}_s \vec{V}_s + \frac{P_s}{m_s n_s} \frac{\vec{v} \vec{v}}{v_s^2}$$

Now

$$\begin{aligned} \nabla \cdot (n_s \langle \vec{v} \vec{v} \rangle) &= \nabla \cdot (n_s \vec{V}_s \vec{V}_s) + \frac{1}{m_s} \nabla P_s \\ &= (\nabla \cdot \vec{V}_s) (n_s \vec{V}_s) + (\vec{V}_s \cdot \nabla) (n_s \vec{V}_s) + \frac{1}{m_s} \nabla P_s \end{aligned}$$

The force equation now becomes

$$\begin{aligned} \partial_t (m_s n_s \vec{V}_s) + (\nabla \cdot \vec{V}_s) (m_s n_s \vec{V}_s) + (\vec{V}_s \cdot \nabla) (m_s n_s \vec{V}_s) \\ = -\nabla P_s + q_s n_s (E + \frac{\vec{V}_s}{c} \times \vec{B}) \end{aligned}$$

subtract continuity equation multiplied by $\vec{V}_s$ from LHS:

$$\boxed{m_s n_s \partial_t \vec{V}_s + m_s n_s (\vec{V}_s \cdot \nabla) \vec{V}_s = -\nabla P_s + q_s n_s (E + \frac{\vec{V}_s}{c} \times \vec{B})}$$

momentum equation w/ drifting Maxwellian

we now also need an expression for the pressure P_s , EOS along with other possible assumption. For instance, if isothermal (slow)

$P_s = n_s T_s$
EOS

$$\nabla P_s = \nabla (n_s T_s) = n_s \nabla T_s + T_s \nabla n_s = T_s \nabla n_s$$

or adiabatic (fast)

$$\nabla P_s = \nabla (n_s T_s) = n_s \nabla T_s + T_s \nabla n_s = \gamma_s T_s \nabla n_s$$

ratio of specific heats $\equiv \gamma_s = (2+D)/D$ where
 $D \equiv$ number of dimensions which there in increased temperature

To complete the system define

$$\text{charge density} \equiv \rho(\vec{x}, t) \equiv \sum_s q_s n_s(\vec{x}, t)$$

$$\text{total current density} \equiv \vec{J}(\vec{x}, t) \equiv \sum_s q_s n_s(\vec{x}, t) \vec{v}_s(\vec{x}, t)$$

and use Maxwell's Equations

$$\nabla \cdot \vec{E}(\vec{x}, t) = 4\pi\rho$$

$$\nabla \cdot \vec{B}(\vec{x}, t) = 0$$

$$\nabla \times \vec{E}(\vec{x}, t) = -\frac{1}{c} \partial_t \vec{B}$$

$$\nabla \times \vec{B}(\vec{x}, t) = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \partial_t \vec{E}$$

Lesson #5

MHD

(ref Nicholson 28.1)

What we wish to do is find a form of the "fluid" equations which would be valid if $\Omega \ll n v b^2$ that is for low frequency.

In this case, we need to start with the collisional Vlasov equation

$$\partial_t f_s + \vec{v} \cdot \vec{\nabla}_x f_s + \frac{q_s}{m_s} (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \cdot \vec{\nabla}_v f_s = C[f_s]$$

using the Ideas of Bogoliubov and the fact that Ω is small so that $f_s = \bar{f}_s$. Note that this is not the Vlasov-Fokker-Planck Equation because we have not made the diffusive small step approximation. In general $C[f_s]$ is a Boltzmann collision operator.

We now proceed as we did in the derivation of the fluid equations and make the moment expansion.

For the 0^{th} order equation we get, when we operate on the collisional Vlasov equation with $\int d\vec{v}$

$$\partial_t n_s + \nabla \cdot (\vec{V}_s n_s) = \int d\vec{v} C[f_s]$$

Since the change due to collisions is mostly in velocity

$$\boxed{\partial_t n_s + \nabla \cdot (\vec{V}_s n_s) = 0} \text{ as before for continuity eq.}$$

Next operating on the collisional Vlasov equation with $\int d\vec{v} \vec{v}$ and assuming a drifting Maxwellian we get for the momentum equation

$$\boxed{m_s n_s \partial_t \vec{V}_s + m_s n_s (\vec{V}_s \cdot \nabla) \vec{V}_s = -\nabla P_s + q_s n_s (\vec{E} \times \frac{1}{c} \vec{V}_s \times \vec{B}) + \vec{K}_s}$$

where $\vec{K}_s \equiv m_s \int d\vec{v} \vec{v} C[f_s] = \text{change in momentum of species } s \text{ due to collisions}$

We now assume that we have a two-fluid model where the fluids are ions and electrons that is $s = \{i, e\}$.

We now proceed to the one-fluid model, also known as the magnetohydrodynamics (MHD) equations. Define the

$$\text{mass density} = \rho_M \equiv m_e n_e + m_i n_i \approx m_i n_i$$

$$\text{charge density} = \rho_c \equiv q_e n_e + q_i n_i$$

$$\text{center of mass velocity} = \vec{V} \equiv \frac{1}{\rho_M} (m_i n_i \vec{V}_i + m_e n_e \vec{V}_e)$$

$$\text{current density} = \vec{J} \equiv q_i n_i \vec{V}_i + q_e n_e \vec{V}_e$$

$$\text{total pressure} = P \equiv P_e + P_i$$

We now wish have four equations (mass conservation, charge conservation, momentum, and generalized Ohm's Law) for the four unknowns $\rho_M, \rho_c, \vec{V}$, and $\vec{J}$.

Multiply ion continuity by m_i , and the electron continuity equation by m_e and add together to get

$$\boxed{\frac{d\rho_M}{dt} + \nabla \cdot (\rho_M \vec{V}) = 0} \text{ Mass conservation}$$

Next multiply the ion continuity equation by q_i and the electron continuity equation by q_e and add together to get

$$\boxed{\frac{d\rho_c}{dt} + \nabla \cdot \vec{J} = 0} \quad \text{charge conservation law}$$

Add together the momentum equations for ions and electrons. Regard V_s and ∂n_s as small quantities and neglect their products. Remember $\vec{K}_i = -\vec{K}_e$. One gets

$$\boxed{\rho_m \frac{d\vec{V}}{dt} = -\nabla P + \rho_e \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}} \quad \text{force eq.}$$

Now onto the equation for the time derivative of the current, the generalized Ohm's law. We form $\sum_s \frac{q_s}{m_s}$ momentum eq., assume that ∂n_s and $\vec{V}_s$ are small quantities (eliminating quadratic terms), assume that $Z=1$ so that $q_i = -q_e = e$, and $\vec{K}_e = -\vec{K}_i$, we get

$$\begin{aligned} \frac{d\vec{J}}{dt} = & -\frac{e}{m_i} \nabla P_i + \frac{e}{m_e} \nabla P_e + \left(\frac{e^2 n_e}{m_e} + \frac{e^2 n_i}{m_i} \right) \vec{E} + \frac{e^2 n_e}{m_e c} \vec{V}_e \times \vec{B} \\ & + \frac{e^2 n_i}{m_i c} \vec{V}_i \times \vec{B} + \left(\frac{e}{m_i} + \frac{e}{m_e} \right) \vec{K}_i \end{aligned}$$

Notice that

$$\begin{aligned} \frac{n_e e^2}{m_e c} \vec{V}_e &= \frac{e}{m_e c} (n_e e \vec{V}_e - n_i e \vec{V}_i) + \frac{e^2}{m_e m_i c} (m_i n_i \vec{V}_i) \\ &= \frac{-e}{m_e c} \vec{J} + \frac{e^2}{m_e m_i c} (m_i n_i \vec{V}_i + m_e n_e \vec{V}_e) \\ &= \frac{-e}{m_e c} \vec{J} + \frac{e^2}{m_e m_i c} (\rho_m \vec{V}) \end{aligned}$$

small

Now substituting this into the previous equation using $m_i^{-1} \ll m_e^{-1}$, $P_e \approx P_i \approx P/2$, and $n_i \approx n_e$

$$\frac{d\vec{J}}{dt} = \frac{e}{2m_e} \nabla P + \frac{e^2 \rho_m}{m_e m_i} \left(\vec{E} + \frac{1}{c} \vec{V} \times \vec{B} \right) - \frac{e}{m_e c} \vec{J} \times \vec{B} + \frac{e}{m_e} \vec{K}_i$$

Recall $\vec{K}_i$ represents the change in ion momentum due

to collisions with the electrons. Assuming that K_i is a function of $\vec{V}_i - \vec{V}_e$ and keeping only the first term in the Taylor expansion

$$\vec{K}_i = C_1 (\vec{V}_i - \vec{V}_e) = C_2 \vec{J} = - \frac{\rho_m e}{m_i \tau} \vec{J}$$

where we have defined

$$\sigma \equiv - \frac{\rho_m e}{m_i C_2}$$

so that the constant has dimensions of conductivity. We now have the generalized Ohm's Law

$$\frac{m_e m_i}{\rho_m e^2} \frac{d\vec{J}}{dt} = \frac{m_i}{2\rho_m e} \nabla P + \vec{E} + \frac{1}{c} \vec{V} \times \vec{B} - \frac{m_i}{\rho_m e c} \vec{J} \times \vec{B} - \frac{1}{\sigma} \vec{J}$$

The name Ohm's Law comes from the form of the equation if only the second and fifth terms are significant $\vec{J} = \sigma \vec{E}$, where σ is now obviously the conductivity. This is practically the current momentum equation.

Now in summary the MHD equations are:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{V}) = 0$$

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{J} = 0$$

$$\rho_m \frac{d\vec{V}}{dt} = -\nabla P + \rho_c \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}$$

$$\frac{m_e m_i}{\rho_m e^2} \frac{d\vec{J}}{dt} = \frac{m_i}{2\rho_m e} \nabla P + \vec{E} + \frac{1}{c} \vec{V} \times \vec{B} - \frac{m_i}{\rho_m e c} \vec{J} \times \vec{B} - \frac{1}{\sigma} \vec{J}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{d\vec{E}}{dt}$$

in 14 unknowns $\rho_m, \rho_e, \vec{V}, \vec{J}, \vec{E}, \vec{B}$. It is assumed that $\rho = \rho(\rho_m)$. The other assumptions that we have used in the derivation are that we are dealing with slow phenomena where their rates are less than the collision rate (i.e., $\Omega \ll n\sigma b^2$), $m_i \gg m_e$, $n_e \approx n_i$, $\vec{R}_i = \vec{R}_e \approx C_1(\vec{V}_i - \vec{V}_e)$. The rate assumption is what allowed us to neglect the quadratic $d_t n_s$ and $\vec{V}_s$. This is the fundamental assumption.

For very low frequencies there are further assumptions that can be made and significant simplifications can be made to the system of equations. First for Ohm's Law, neglect the $d_t \vec{J}$, $\nabla \rho$, $\vec{J} \times \vec{B}$ terms so that

$$0 = \vec{E} + \frac{1}{c} \vec{V} \times \vec{B} - \frac{\vec{J}}{\sigma}$$

The plasma also becomes quasineutral so that $\rho_e = 0$ implying $\nabla \cdot \vec{J} = 0$ and the $\rho_e \vec{E}$ term can be dropped in the momentum equation. One can also drop the $d_t \vec{E}$ term in the $\nabla \times \vec{B}$ equation. The weakest of these assumptions is dropping the $\vec{J} \times \vec{B}$ Hall term in Ohm's Law. This leaves us with the ideal resistive MHD equations

$$\begin{aligned} \frac{d\rho_m}{dt} + \nabla \cdot (\rho_m \vec{V}) &= 0 \\ \rho_m \frac{d\vec{V}}{dt} &= \frac{\vec{J} \times \vec{B}}{c} - \nabla p \\ \vec{E} + \frac{\vec{V} \times \vec{B}}{c} &= \frac{\vec{J}}{\sigma} \\ \nabla \times \vec{E} &= -\frac{1}{c} \frac{d\vec{B}}{dt} \\ \nabla \times \vec{B} &= \frac{4\pi \vec{J}}{c} \\ (\nabla \cdot \vec{J} &= 0) \end{aligned}$$

Finally if the plasma is considered to be collisionless $\sigma \rightarrow \infty$, and we are left with the ideal MHD equations

$$\begin{aligned}\frac{d\rho_m}{dt} + \nabla \cdot (\rho_m \vec{V}) &= 0 \\ \rho_m \frac{d\vec{V}}{dt} &= \frac{\vec{J} \times \vec{B}}{c} - \nabla P \\ \nabla \times (\nabla \times \vec{B}) &= \frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \frac{4\pi \vec{J}}{c}\end{aligned}$$

Lesson #6

Liouville Equation and BBGKY Hierarchy from coordinate free exterior calculus perspective

This lesson starts the development of kinetic theory from the coordinate-free perspective, using exterior calculus. Tutorial notes, a mini-course on exterior calculus can be found at:

http://tiny.cc/exterior_calculus

Also recommend the textbook:

Frankel "The Geometry of Physics" 2nd edition 2004
It also explicitly uses the symplectic, canonical, structure of physical dynamics. We will start with a coordinate-free formulation of the Liouville equation. It will continue with the coordinate-free development of the BBGKY hierarchy. The BBGKY hierarchy will then be pulled back to canonical coordinates. This pull-back, further restricted to canonical action angle coordinates, will

be used to derive the Ulasov Equation, Boltzmann Equation, Master Equation, and Fokker-Planck Equation, in that order. To make it more tangible, it will be applied to a reduced guiding center Hamiltonian (which will be derived in Appendix B).

Finally a coordinate-free version of the Variational Theory of Reaction Rates will be developed, then pulled-back and evaluated for guiding center atoms using the Hamiltonian of App. B

Consider Hamiltonian flow on phase space of N identical particles on the same base manifold, M , with N particle Hamiltonian function

$$H^{(N)}: T^*M^N \rightarrow \mathbb{R}$$

where T^*M^N is product space of n symplectic cotangent bundles, (T^*M, ω) , and ω is a non-degenerate two form on T^*M that is exact $\omega = d\lambda$. For instance if there is locally a coordinate q on M and the corresponding coordinates (q, p) on T^*M , then $\lambda = p dq$, and $\omega = dp \wedge dq$.

Flow on T^*M^N is given by the vector field U defined by:

$$i_{U_N} \omega^{(N)} = -dH^{(N)}$$

where $\omega^{(N)} = \sum_{i=1}^N \omega_i$ and ω_i acts on T^*M of i^{th}

particle. Restrict $H^{(N)}$ to binary interactions

which is the fundamental assumption so that we can write:

$$H^{(N)} = \sum_{i=1}^N H_1(x_i) + \sum_{i < j}^N H_2(x_i, x_j)$$

where x_i is a point in T^*M of i^{th} particle and H_1 and H_2 are arbitrary functions

$$H_1: T^*M \rightarrow \mathbb{R}$$

$$H_2: T^*M^2 \rightarrow \mathbb{R}$$

Wish to describe evolution of "ensemble" of N particle system specified by $2N$ -form on T^*M^N , symmetric with respect to particle interchange

$$N\text{-particle distribution form} \equiv \rho^{(N)} = f_N(x_1, \dots, x_N) \tau^{(N)}$$

where

$$\tau^{(N)} \equiv \prod_{i=1}^N \omega_i$$

and f_N is symmetric with respect to $x_i \leftrightarrow x_j$ interchange.

$f_N(x_1, \dots, x_N) \equiv N\text{-particle distribution function}$
 $=$ probability of finding a system of N particles in a particular state specified by a point on T^*M^N

Normalization of $\rho^{(N)}$ chosen to expose ordering of the terms in the resulting BBGKY hierarchy:

$$N = \frac{1}{V^N} \int_{T^*M^N} \rho^{(N)}$$

where V is a real number identified as the

volume of M .

Assuming that there is no physical mechanism that can create or destroy particles ($dN/dt=0$), $\rho^{(N)}$ must be invariant under flow generated by U

$$\frac{d\rho^{(N)}}{dt} + \mathcal{L}_U \rho^{(N)} = 0$$

Generalized Liouville Equation

where $\mathcal{L}_U$ is the Lie derivative.

To move towards the BBGKY hierarchy introduce the n -particle distribution form

$$\rho^{(n)} \equiv \frac{1}{V^{N-n}} \int_{T^*M^{N-n}} \rho^{(N)} = f_n(x_1, \dots, x_n) \rho^{(n)}$$

where f_n is the n -particle distribution function.

Now integrate the Generalized Liouville Equation over the cotangent bundles of $N-n$ particles.

$$0 = \frac{1}{V^{N-n}} \int_{T^*M^{N-n}} \left[\frac{d\rho^{(N)}}{dt} + \mathcal{L}_U \rho^{(N)} \right]$$

Commuting d_t with integration and using the definition of $\rho^{(n)}$

$$\frac{d\rho^{(n)}}{dt} = - \frac{1}{V^{N-n}} \int_{T^*M^{N-n}} \mathcal{L}_U \rho^{(N)}$$

Now split RHS into 4 terms by splitting U into four terms and using distributive property of Lie derivative, by defining four vector fields $u_i, u_{ij}, u^{(n)}$ and $u_{int}^{(n)}$

$$i_{u_i} \omega^{(N)} = -dH_1(x_i)$$

$$i_{u_{ij}} \omega^{(n)} = -dH_2(x_i, x_j)$$

$$i_{u^{(n)}} \omega^{(n)} = -dH^{(n)}(x_1, \dots, x_n)$$

$$i_{u_{int}^{(n)}} \omega^{(n)} = -dH_{int}^{(n)}(x_1, \dots, x_{n+1})$$

where

$$H_{int}^{(n)} \equiv \sum_{i=1}^n H_2(x_i, x_{n+1})$$

Now write

$$-\frac{\partial \rho^{(n)}}{\partial t} = \frac{1}{V^{N-n}} \left[\int_{T^*M^{N-n}} \mathcal{L}_{u^{(n)}} \rho^{(n)} + (N-n) \int_{T^*M^{N-n}} \mathcal{L}_{u_{n+1}} \rho^{(n)} \right. \\ \left. + (N-n) \int_{T^*M^{N-n}} \mathcal{L}_{u_{int}} \rho^{(n)} + \frac{(N-n)(N-n-1)}{2} \int_{T^*M^{N-n}} \mathcal{L}_{u_{n+1, n+2}} \rho^{(n)} \right]$$

where the symmetry of $\rho^{(n)}$ under particle interchange used to consolidate last three terms on R.H.S.

Commute parts of integrals with Lie derivatives which do not involve those particles and use the definition of $\rho^{(n)}$.

$$-\frac{\partial \rho^{(n)}}{\partial t} = \mathcal{L}_{u^{(n)}} \rho^{(n)} + \frac{N-n}{V} \int_{T^*M} \mathcal{L}_{u_{n+1}} \rho^{(n+1)} + \frac{N-n}{V} \int_{T^*M} \mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)} \\ + \frac{(N-n)(N-n-1)}{2V^2} \int_{T^*M^2} \mathcal{L}_{u_{n+1, n+2}} \rho^{(n+2)}$$

Using Cartan's formula it can be shown that

$$\mathcal{L}_{\underline{w}} \rho^{(n)} = d(i_{\underline{w}} \rho^{(n)}) \quad \forall \text{ vector fields } \underline{w}$$

Using this fact and Stoke's Thm, second and fourth terms can be written

$$-\frac{\partial \rho^{(n)}}{\partial t} = \mathcal{L}_{u^{(n)}} \rho^{(n)} + \frac{N-n}{V} \int_{\partial(T^*M)} i_{u_{n+1}} \rho^{(n+1)} + \frac{N-n}{V} \int_{T^*M} \mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)} \\ + \frac{(N-n)(N-n-1)}{2V^2} \int_{\partial(T^*M^2)} i_{u_{n+1, n+2}} \rho^{(n+2)}$$

These terms can be set equal to zero if

$$0 = i u_{n+1} \rho^{(n+1)} \Big|_{\partial(T^*M)} = i u_{n+1, n+2} \rho^{(n+2)} \Big|_{\partial(T^*M^2)}$$

Condition is met if no particles on the boundary, or no flow across the boundary, or there is no boundary. Defining $n_0 \equiv (N-n)/V$, one obtains

$$\frac{\partial \rho^{(n)}}{\partial t} + \mathcal{L}_{u^{(n)}} \rho^{(n)} = -n_0 \int_{T^*M} \mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)}$$

Generalized BBGKY Hierarchy

There is a straight forward geometric interpretation of this equation. What appears on the LHS is the full time derivative of $\rho^{(n)}$ with respect to the vector field $u^{(n)}$ flow generated by the n -particle Hamiltonian ($D\rho^{(n)}/Dt$). If the RHS were zero the n -particle distribution would evolve as if the $N-n$ did not exist. The term on the RHS is the perturbation to the n -particle flow due to interactions with the $n+1$ particle. It is the functional collision operator, $\mathcal{C}[\rho^{(n+1)}]$.

Assume that locally one can define a coordinate q , on M and the associated system (q, p) on T^*M where $\lambda = p dq$ and $\omega = dp \wedge dq$. Pulling back the 1st and 2nd equations in the Generalized BBGKY hierarchy one gets (see <http://tiny.cc/kinetics-lecture-notes>, App. A for details)

$$\frac{df_1}{dt} + \{ \tilde{H}_1, f_1 \} = -n_0 \int dp_2 dq_2 \{ \tilde{H}_{12}, f_2 \},$$

$$\frac{d f_2}{d t} + \{ \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_{12}, f_2 \}_{12} = -n_0 \int d\rho_3 d\mathbf{q}_3 \{ \tilde{H}_{13} + H_{23}, f_3 \}_{12}$$

Sum over particle 1 and 2

BBGKY pull back

where $\tilde{H}_i(\rho_i, \mathbf{q}_i) \equiv H_1(x_i)$,

$\tilde{H}_{ij}(\rho_i, \mathbf{q}_i, \rho_j, \mathbf{q}_j) \equiv H_{12}(x_i, x_j), (\rho_i, \mathbf{q}_i) \in \mathbb{R}^2$
and

Poisson bracket of g and $h \equiv \{g, h\} = \sum_{i=1}^N \frac{\partial g}{\partial \rho_i} \frac{\partial h}{\partial \mathbf{q}_i} - \frac{\partial g}{\partial \mathbf{q}_i} \frac{\partial h}{\partial \rho_i}$

Lesson #7

Closure of
Generalized BBGKY.

Vlasov, Boltzmann, Master and VFP
(Vlasov-Fokker-Planck) Equations

Need to make an expansion that can allow BBGKY Hierarchy to be truncated. There are two methods

(1) $n_0 b^3$, short range of interaction,

Ideas of Bogoliubov

(2) $\Gamma = \frac{e^2/a}{k_B T_e} \sim (n b^3)^{1/3}$, weakness of correlation, Mayer Cluster Expansion

Vlasov Equation

Make the Mayer Cluster expansion (#2) for

$$f_2(\rho_1, \mathbf{q}_1, \rho_2, \mathbf{q}_2) = f_1(\rho_1, \mathbf{q}_1) f_1(\rho_2, \mathbf{q}_2) + g(\rho_1, \mathbf{q}_1, \rho_2, \mathbf{q}_2)$$

$$\approx f_1(p_1, q_1) f_1(p_2, q_2) + O(\epsilon)$$

and substitute into the first equation in the BBGKY hierarchy

$$\frac{df_1}{dt} + \{\tilde{H}_1, f_1\} = -n_0 \int dp_2 dq_2 \{\tilde{H}_{12}, f_2\},$$

to get

$$\begin{aligned} \frac{df_1(p_1, q_1)}{dt} + \{\tilde{H}_1(p_1, q_1), f_1(p_1, q_1)\} \\ = -n_0 \int dp_2 dq_2 \{\tilde{H}_{12}, f_1(p_1, q_1) f_1(p_2, q_2)\} \\ = -n_0 \int dp_2 dq_2 f_1(p_2, q_2) \{\tilde{H}_{12}, f_1(p_1, q_1)\} \\ = - \underbrace{\left\{ n_0 \int dp_2 dq_2 f_1(p_2, q_2) \tilde{H}_{12}, f_1(p_1, q_1) \right\}}_{\equiv \langle \tilde{H}_{12} \rangle_2(p_1, q_1)} \\ = \{f_1, \langle \tilde{H}_{12} \rangle_2\} \end{aligned}$$

$$\Rightarrow \boxed{\frac{df_1}{dt} + \{\tilde{H}_1 + \langle \tilde{H}_{12} \rangle_2, f_1\} = 0}$$

Generalized Vlasov Equation

for instance if $\tilde{H}_1 = \frac{p_1^2}{2}$ and $\tilde{H}_{12} = V(q_1 - q_2)$

$$\frac{df_1}{dt} + \left(\underbrace{\frac{\partial \tilde{H}_1}{\partial p_1}}_{p_1 = v_1} \frac{df_1}{dq_1} - \cancel{\frac{\partial \tilde{H}_1}{\partial q_1}} \frac{df_1}{dp_1} \right) + \left(\cancel{\frac{\partial \langle \tilde{H}_{12} \rangle_2}{\partial p_1}} \frac{df_1}{dq_1} - \underbrace{\frac{\partial \langle \tilde{H}_{12} \rangle_2}{\partial q_1}}_{-\frac{\partial V(q_1)}{\partial q_1} = \dot{p} = \langle \vec{a}_{12} \rangle} \frac{df_1}{dp_1} \right) = 0$$

$$\tilde{H}_{1, \text{eff}} = \frac{p_1^2}{2} + V(q_1) \quad \frac{df_1}{dt} + p_1 \frac{df_1}{dq_1} - \frac{\partial V}{\partial q_1} \frac{df_1}{dp_1} = 0$$

$$\tilde{H}_{12, \text{eff}} = 0 \quad \frac{df_1}{dt} + \vec{v} \cdot \vec{\nabla}_x f_1 + \langle \vec{a}_{12} \rangle \cdot \vec{\nabla}_v f_1 = 0$$

$$\langle \vec{a}_{12} \rangle \equiv -\frac{\partial}{\partial q_1} \left(n_0 \int d\rho_2 dq_2 f_1(\rho_2, q_2) V(q_1 - q_2) \right) \equiv -\frac{\partial \bar{V}(q_1)}{\partial q_1}$$

as before for the electrostatic case utilizing vector calculus. The advantage of this canonical exterior calculus viewpoint is the generality of the formalism, the simplicity and geometric interpretability of the result, and the ease of calculation.

Boltzmann Equation

We use the ideas of Bogoliubov to close the equations. We start with the first two equations of the BBGKY, neglect the f_3 , third order term on the RHS of the second equation and the $\partial_t f_2$ term on the LHS.

$$\frac{\partial f_1}{\partial t} + \{ \tilde{H}_1, f_1 \} = -n_0 \int d\Gamma_2 \{ \tilde{H}_{12}, f_2 \},$$

$$\{ \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_{12}, f_2 \}_{12} = 0$$

where $d\Gamma_2 \equiv d\rho_2 dq_2$

Now substitute $\{ \tilde{H}_{12}, f_2 \}_{12}$ from the second equation into the first equation.

$$\frac{\partial f_1}{\partial t} + \{ \tilde{H}_1, f_1 \} = n_0 \int d\Gamma_2 [\{ \tilde{H}_1, f_2 \}_1 + \{ \tilde{H}_2, f_2 \}_2]$$

Transform $(\rho_i, q_i) \rightarrow (\psi_i, J_i)$ into action-angle variables where $\tilde{H}_i = \tilde{H}_i(J_i) \Rightarrow J_i = \text{constant}$ and $\psi_i = \omega_i t + \psi_{i0}$, since $\omega_i = \partial \tilde{H}_i / \partial J_i = \text{constant}$ and $\partial \tilde{H}_i / \partial \psi_i = 0$. Now $d\Gamma_i = dJ_i d\psi_i$

Now,

$$\frac{\partial f_1}{\partial t} + \omega_1 \frac{\partial f_1}{\partial \psi_1} = n_0 \int d\Gamma_2 \left(\omega_1 \frac{\partial f_2}{\partial \psi_1} + \omega_2 \frac{\partial f_2}{\partial \psi_2} \right)$$

$$\frac{df_1}{dt} + \frac{d}{d\psi_1}(\omega_1 f_1) = n_0 \int d\mathcal{J}_2 \left(\frac{d}{d\psi_1}(\omega_1 f_2) + \frac{d}{d\psi_2}(\omega_2 f_2) \right)$$

assume $\bar{v}/b \sim \omega_1 \gg \bar{v}_c \sim n\bar{v}b^2 = (nb^3)\bar{v}/b$. Average over ψ_1 by $\int \frac{d\psi_1}{2\pi}$ and define

$$\bar{f}_1(\mathcal{J}_1) \equiv \int \frac{d\psi_1}{2\pi} f_1(\mathcal{J}_1, \psi_1)$$

$$\begin{aligned} \frac{d\bar{f}_1(\mathcal{J}_1)}{dt} &= n_0 \int d\mathcal{J}_2 d\psi_2 \int \frac{d\psi_1}{2\pi} \frac{d}{d\psi_2}(\omega_2 f_2) \\ &= n_0 \int d\mathcal{J}_2 \int \frac{d\psi_1}{2\pi} \int_{dV} v_c(\mathcal{J}_2) d\sigma_S (f_{2\#} - f_{20}) \end{aligned}$$

do $d\psi_2$ integral

$$f_{20} = \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2) \text{ before collision}$$

$$f_{2\#} = \bar{f}_1(\mathcal{J}_{1\#}) \bar{f}_1(\mathcal{J}_{2\#}) \text{ after collision}$$

$$\mathcal{J}_{1\#}(\mathcal{J}_1, \psi_1, \mathcal{J}_2)$$

$$\mathcal{J}_{2\#}(\mathcal{J}_1, \psi_1, \mathcal{J}_2)$$

So

$$\frac{d\bar{f}_1(\mathcal{J}_1)}{dt} = n_0 \int d\mathcal{J}_2 \int \frac{d\psi_1}{2\pi} \int_{dV} v_c(\mathcal{J}_2) d\sigma_S [\bar{f}_1(\mathcal{J}_{1\#}(\mathcal{J}_1, \psi_1, \mathcal{J}_2)) \bar{f}_1(\mathcal{J}_{2\#}(\mathcal{J}_1, \psi_1, \mathcal{J}_2)) - \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)]$$

now multiply by identity

$$\int d\mathcal{J}_1' \int d\mathcal{J}_2' \delta(\mathcal{J}_1' - \mathcal{J}_{1\#}) \delta(\mathcal{J}_2' - \mathcal{J}_{2\#})$$

$$\begin{aligned} \frac{d\bar{f}_1(\mathcal{J}_1)}{dt} &= n_0 \int d\mathcal{J}_2 \int d\mathcal{J}_2' \int d\mathcal{J}_1' \int \frac{d\psi_1}{2\pi} \int v_c(\mathcal{J}_2) d\sigma_S \\ &\delta(\mathcal{J}_1' - \mathcal{J}_{1\#}) \delta(\mathcal{J}_2' - \mathcal{J}_{2\#}) [\bar{f}_1(\mathcal{J}_1') \bar{f}_1(\mathcal{J}_2') - \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)] \end{aligned}$$

define

$$|v| \frac{d\sigma}{d\mathcal{J}_1' d\mathcal{J}_2'} \equiv K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') = |v| \frac{d\sigma}{d\Omega} \text{ where } d\Omega \equiv d\mathcal{J}_1' d\mathcal{J}_2'$$

differential
cross
section

$$\equiv n_0 \int \frac{d\psi_1}{2\pi} \int v_c(\mathcal{J}_2) d\sigma \delta(\mathcal{J}_1' - \mathcal{J}_{1\#}) \delta(\mathcal{J}_2' - \mathcal{J}_{2\#})$$

To rewrite the equation

$$\frac{\partial \bar{f}_1(\mathcal{J}_1)}{\partial t} = \int d\mathcal{J}_2 d\Omega \frac{1}{v} \frac{d\sigma}{d\Omega} [\bar{f}_1(\mathcal{J}_1') \bar{f}_1(\mathcal{J}_2') - \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)]$$

or

$$\frac{\partial \bar{f}_1(\mathcal{J}_1)}{\partial t} = \int d\mathcal{J}_1' d\mathcal{J}_2 d\mathcal{J}_2' K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') [\bar{f}_1(\mathcal{J}_1') \bar{f}_1(\mathcal{J}_2') - \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)]$$

Generalized Boltzmann Equation

Master Equation

Start by writing the Boltzmann Equation as

$$\begin{aligned} \frac{\partial \bar{f}_1(\mathcal{J}_1)}{\partial t} &= \int d\mathcal{J}_2 d\mathcal{J}_1' d\mathcal{J}_2' K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') [\bar{f}_1(\mathcal{J}_1') \bar{f}_1(\mathcal{J}_2') - \bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)] \\ &= \int d\mathcal{J}_2 d\mathcal{J}_1' d\mathcal{J}_2' \bar{f}_{th}(\mathcal{J}_1) \bar{f}_{th}(\mathcal{J}_2) K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') \\ &\quad \left[\frac{\bar{f}_1(\mathcal{J}_1') \bar{f}_1(\mathcal{J}_2')}{\bar{f}_{th}(\mathcal{J}_1') \bar{f}_{th}(\mathcal{J}_2')} - \frac{\bar{f}_1(\mathcal{J}_1) \bar{f}_1(\mathcal{J}_2)}{\bar{f}_{th}(\mathcal{J}_1) \bar{f}_{th}(\mathcal{J}_2)} \right] \end{aligned}$$

where we have

used that for $\bar{f}_{th}(\mathcal{J}) = e^{-\tilde{H}_1(\mathcal{J})/k_B T}$, $\frac{\partial \bar{f}_{th}(\mathcal{J})}{\partial t} = 0$,

since $\tilde{H}_1(\mathcal{J}_1') + \tilde{H}_1(\mathcal{J}_2') = \tilde{H}_1(\mathcal{J}_1) + \tilde{H}_1(\mathcal{J}_2)$

by conservation of energy

and $\bar{f}_{th}(\mathcal{J}_1') \bar{f}_{th}(\mathcal{J}_2') = \bar{f}_{th}(\mathcal{J}_1) \bar{f}_{th}(\mathcal{J}_2)$

Since particle #2 is free $\bar{f}_1(\mathcal{J}_2) = \bar{f}_{th}(\mathcal{J}_2)$

and $\bar{f}_1(\mathcal{J}_2') = \bar{f}_{th}(\mathcal{J}_2)$ so

$$\frac{\partial \bar{f}_1(\mathcal{J}_1)}{\partial t} = \int d\mathcal{J}_1' \left[\int d\mathcal{J}_2' d\mathcal{J}_2 \bar{f}_{th}(\mathcal{J}_2) K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') \right] \bar{f}_{th}(\mathcal{J}_1) \left[\frac{\bar{f}_1(\mathcal{J}_1')}{\bar{f}_{th}(\mathcal{J}_1')} - \frac{\bar{f}_1(\mathcal{J}_1)}{\bar{f}_{th}(\mathcal{J}_1)} \right]$$

define

$$k(\mathcal{J}_1 | \mathcal{J}_1') \equiv \int d\mathcal{J}_2 d\mathcal{J}_2' \bar{f}_{th}(\mathcal{J}_2) K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2')$$

substituting this into the equation gives

$$\frac{\partial \bar{f}_1(\mathcal{J}_1)}{\partial t} = \int d\mathcal{J}_1' k(\mathcal{J}_1 | \mathcal{J}_1') \bar{f}_{th}(\mathcal{J}_1') \left[\frac{\bar{f}_1(\mathcal{J}_1')}{\bar{f}_{th}(\mathcal{J}_1')} - \frac{\bar{f}_1(\mathcal{J}_1)}{\bar{f}_{th}(\mathcal{J}_1)} \right]$$

by TP symmetry (time reversal and spatial inversion) $K(\mathcal{J}_1, \mathcal{J}_2 | \mathcal{J}_1', \mathcal{J}_2') = K(\mathcal{J}_1', \mathcal{J}_2' | \mathcal{J}_1, \mathcal{J}_2)$ and energy conservation, we have detailed balance (see LBL "Physical Kinetics" Ch I sec 2-3)

$$\bar{f}_{th}(\mathcal{J}) k(\mathcal{J} | \mathcal{J}') = \bar{f}_{th}(\mathcal{J}') k(\mathcal{J}' | \mathcal{J})$$

using this one gets

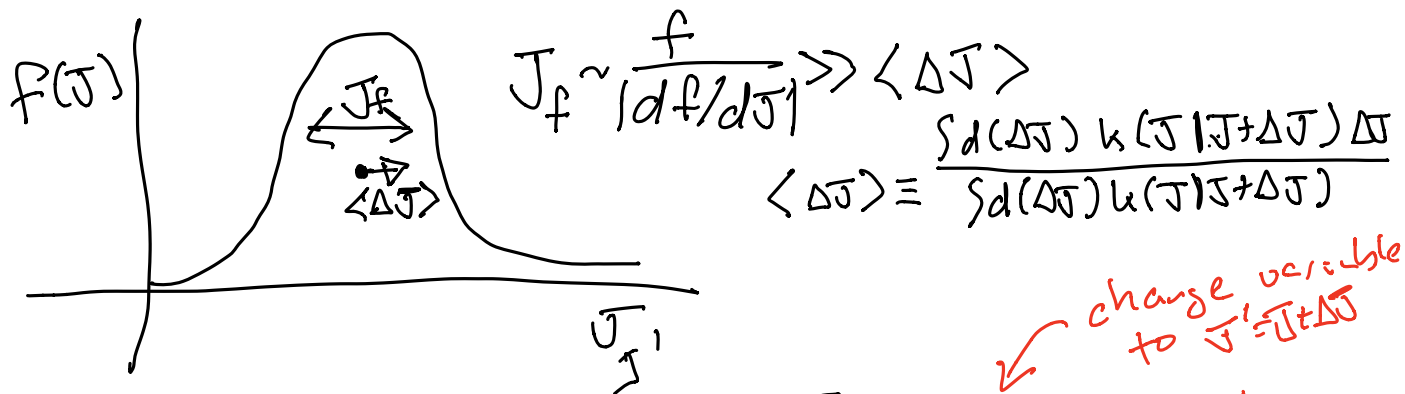
$$\frac{\partial \bar{f}_1(\mathcal{J})}{\partial t} = \int d\mathcal{J}' [k(\mathcal{J}' | \mathcal{J}) \bar{f}_1(\mathcal{J}') - k(\mathcal{J} | \mathcal{J}') \bar{f}_1(\mathcal{J})]$$

Generalized Master Equation

Fokker-Planck

We start with the Generalized Master Equation above and make a moment expansion in terms of the smallness of

ΔJ when compared to the scale on which the distribution function changes



$$\begin{aligned} \frac{\partial \bar{f}_1(J)}{\partial t} &= \int d(\Delta J) [k(J+\Delta J|J) \bar{f}_1(J+\Delta J) - k(J|J+\Delta J) \bar{f}_1(J)] \\ &= \int d(\Delta J) [k(J|J-\Delta J) \bar{f}_1(J) + \Delta J \cdot \frac{\partial}{\partial J} (k(J|J-\Delta J) \bar{f}_1(J)) \\ &\quad + \frac{1}{2} \Delta J \Delta J \cdot \frac{\partial^2}{\partial J \partial J} (k(J|J-\Delta J) \bar{f}_1(J)) - k(J|J-\Delta J) \bar{f}_1(J)] \end{aligned}$$

define

$$\begin{aligned} \tilde{A} &\equiv \int d(\Delta J) k(J|J+\Delta J) \Delta J \\ B &\equiv \frac{1}{2} \int d(\Delta J) k(J|J+\Delta J) \Delta J \Delta J \end{aligned}$$

now substitute into equation and use integration by parts

$$\begin{aligned} \frac{\partial f(J)}{\partial t} &= \frac{\partial}{\partial J} (f \tilde{A}) + \frac{\partial^2}{\partial J \partial J} (f B) \\ &= \frac{\partial}{\partial J} \cdot [f \tilde{A} + \frac{\partial}{\partial J} (f B)] \\ &= \frac{\partial}{\partial J} \cdot [f \tilde{A} + f \frac{\partial B}{\partial J} + B \frac{\partial f}{\partial J}] \\ &= \frac{\partial}{\partial J} \cdot [f A + B \frac{\partial f}{\partial J}] \\ &= - \frac{\partial S}{\partial J} \end{aligned}$$

where $A \equiv \tilde{A} + f \frac{dB}{dJ}$ $S \equiv -fA - B \cdot \frac{df}{dJ}$

for equilibrium $\frac{df_{eq}}{dt} = -\frac{dS}{dJ} \Big|_{eq} = 0$

$\Rightarrow S_{eq} = 0 \Rightarrow f_{eq} A + B \frac{df_{eq}}{dJ} = 0$

$\Rightarrow f_{eq} A = -B \frac{df_{eq}}{dJ}$

$\Rightarrow \frac{df}{dt} = \frac{d}{dJ} \cdot \left[-\frac{f}{f_{eq}} B \cdot \frac{df_{eq}}{dJ} + B \cdot \frac{df}{dJ} \right]$
 $= \frac{d}{dJ} \cdot \left[B \cdot \left(\frac{df}{dJ} - \frac{f}{f_{eq}} \frac{df_{eq}}{dJ} \right) \right]$

$\uparrow$ diffusion tensor

$\left\langle \frac{\Delta J \Delta J}{\Delta t} \right\rangle$

so

$$\frac{d\bar{f}_1(J)}{dt} = \frac{d}{dJ} \cdot \left[\left\langle \frac{\Delta J \Delta J}{\Delta t} \right\rangle \cdot \left(\frac{d\bar{f}_1}{dJ} - \frac{\bar{f}_1}{f_{eq}} \frac{df_{eq}}{dJ} \right) \right]$$

Generalized Fokker-Planck Equation

let $f_{eq} = e^{-V(J)} \Rightarrow \frac{1}{f_{eq}} \frac{df_{eq}}{dJ} = -\frac{dV}{dJ}$

$\nearrow$ restoring force

so

$\frac{df}{dt} = \frac{d}{dJ} \cdot \left[B \cdot \left(\frac{df}{dJ} + \frac{dV}{dJ} f \right) \right]$

Vlasov Fokker Planck (VFP)

We now consider correlated modes where $\omega_1 \ll \omega_2 = \omega_c = n \bar{v} b^2$, the fastest modes of the collisionless Vlasov equation are of order $\omega_p \sim O(n^{3/2} \frac{\bar{v}}{b}) \sim O(n^{3/2} n \bar{v} b^2)$ so that for these modes one can neglect

the effects of collisions: For low frequency modes one can include the FP collision operator with the LHS of the Vlasov equation

$$\frac{\partial f_1}{\partial t} + \{ \tilde{H}_1 + \langle \tilde{H}_{12} \rangle_2, f_1 \} = \frac{d}{dJ} \cdot \left[B \cdot \left(\frac{d\tilde{f}_1}{dJ} + \frac{dV}{dJ} \tilde{f}_1 \right) \right]$$

Vlasov Fokker Planck Equation

where the averaging in $\tilde{f}_1$ are taken over the collision frequency $n\bar{v}b^2$.

Many times the gradients in the distribution function f_1 are too large for the VFP equation to be used. In that case one needs to drop back to the form of the collision operator given by the Master Equation

$$\frac{\partial f_1}{\partial t} + \{ \tilde{H}_1 + \langle \tilde{H}_{12} \rangle_2, f_1 \} = \int dJ' [k(J'|J) \tilde{f}_1(J') - k(J|J') \tilde{f}_1(J)]$$

$\theta(\omega_p) \sim \theta(\Gamma^{3/2})$ $\theta(v) \sim \theta(\Gamma^3)$

Here, instead of the operator being summarized by the diffusion in the canonical momentum (for instance action, J , or velocity, v) determined by the diffusion tensor $B = \langle \frac{\Delta J \Delta J}{\Delta t} \rangle$, it is characterized by the scattering kernel $k(J|J')$. One can also use the Boltzmann operator if $f(J_2) \neq f_{th}(J_2)$, using $K(J_1, J_2 | J'_1, J'_2)$.

Examples

Consider two examples

① free particle w/ electrostatic interaction

$$H_1 = \frac{\vec{p}_1^2}{2} \quad H_{12} = \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

The Boltzmann Equation will be starting with

$$\frac{\partial \bar{f}_1(\vec{J}_1)}{\partial t} = \int d\vec{J}_2 d\Omega |\vec{v}| \frac{d\sigma}{d\Omega} [\bar{f}_1(\vec{J}_1') \bar{f}_1(\vec{J}_2') - \bar{f}_1(\vec{J}_1) \bar{f}_1(\vec{J}_2)]$$

$$\int d\vec{J}_1' d\vec{J}_2 d\vec{J}_2' K(\vec{J}_1, \vec{J}_2 | \vec{J}_1', \vec{J}_2') [\bar{f}_1(\vec{J}_1') \bar{f}_1(\vec{J}_2') - \bar{f}_1(\vec{J}_1) \bar{f}_1(\vec{J}_2)]$$

$$\text{where } |\vec{v}| \frac{d\sigma}{d\Omega} \equiv |\vec{v}| \frac{d\sigma}{d\vec{J}_1' d\vec{J}_2'} = K(\vec{J}_1, \vec{J}_2 | \vec{J}_1', \vec{J}_2')$$

$$= n_0 \int \frac{d^3k}{2\pi} \int v_c(\vec{J}_2) d\vec{\sigma} \delta(\vec{J}_1' - \vec{J}_{1\#}) \delta(\vec{J}_2' - \vec{J}_{2\#})$$

we can identify

$$\bar{f}_1(\vec{J}_1) \rightarrow f(\vec{p}_1, t) \quad d\vec{J}_1' d\vec{J}_2' \rightarrow d^3\vec{p}_1' d^3\vec{p}_2' \quad d\vec{J}_2 \rightarrow d^3\vec{p}_2$$

$$\text{so } K(\vec{p}_1, \vec{p}_2 | \vec{p}_1', \vec{p}_2') d^3\vec{p}_1' d^3\vec{p}_2' = |\vec{v}_2 - \vec{v}_1| d\sigma$$

and

$$\boxed{\frac{\partial f(\vec{p}_1, t)}{\partial t} = \int |\vec{v}_2 - \vec{v}_1| [f(\vec{p}_1') f(\vec{p}_2') - f(\vec{p}_1) f(\vec{p}_2)] d\sigma d^3\vec{p}_2} \quad \text{[Boltzmann Eq]}$$

Now for the Master Equation

$$\frac{\partial \bar{f}_1(\vec{J})}{\partial t} = \int d\vec{J}' [k(\vec{J}' | \vec{J}) \bar{f}_1(\vec{J}') - k(\vec{J} | \vec{J}') \bar{f}_1(\vec{J})]$$

where

$$k(\vec{J}_1 | \vec{J}_1') \equiv \int d\vec{J}_2 d\vec{J}_2' f_{th}(\vec{J}_2) K(\vec{J}_1, \vec{J}_2 | \vec{J}_1', \vec{J}_2')$$

again, identifying

$$\mathcal{J} \rightarrow \vec{p} \quad d\mathcal{J}' \rightarrow d^3\vec{p}' \quad d\mathcal{J}_2 d\mathcal{J}_2' \rightarrow d^3\vec{p}_2 d^3\vec{p}_2'$$

$$f_{th}(\mathcal{J}_2) = f_{th}(\vec{p}_2) = e^{-(\vec{p}_2)^2/2} / (2\pi)^{3/2}$$

$$K(\vec{p}_1, \vec{p}_2 | \vec{p}_1', \vec{p}_2') d^3\vec{p}_1' d^3\vec{p}_2' = |\vec{v}_2 - \vec{v}_1| d\sigma$$

so

$$\frac{\partial f(\vec{p}, t)}{\partial t} = \int d^3\vec{p}' [k(\vec{p}' | \vec{p}) f(\vec{p}') - k(\vec{p} | \vec{p}') f(\vec{p})]$$

where

$$k(\vec{p} | \vec{p}') = \int |\vec{v}_2 - \vec{v}_1| f_{th}(\vec{p}_2) \delta(\vec{p}' - \vec{p} + \vec{p}_2) \frac{d\psi_1}{2\pi} d\sigma_5 d^3\vec{p}_2$$

[Master Eq]

Finally for the Fokker-Planck Equation

$$\frac{\partial \bar{f}_1(\mathcal{J})}{\partial t} = \frac{\partial}{\partial \mathcal{J}} \cdot \left[\left\langle \frac{\Delta \mathcal{J} \Delta \mathcal{J}}{\Delta t} \right\rangle \cdot \left(\frac{\partial \bar{f}_1}{\partial \mathcal{J}} - \frac{\bar{f}_1}{f_{eq}} \frac{\partial f_{eq}}{\partial \mathcal{J}} \right) \right]$$

where

$$\left\langle \frac{\Delta \mathcal{J} \Delta \mathcal{J}}{\Delta t} \right\rangle \equiv \frac{1}{2} \int d(\Delta \mathcal{J}) k(\mathcal{J} | \mathcal{J} + \Delta \mathcal{J}) \Delta \mathcal{J} \Delta \mathcal{J}$$

Identify

$$\bar{f}_1(\mathcal{J}) \rightarrow f(\vec{p}, t) \quad \mathcal{J} \rightarrow \vec{p} \quad f_{eq}(\mathcal{J}) \rightarrow f_{eq}(\vec{p}) = \frac{e^{-\vec{p}^2/2}}{(2\pi)^{3/2}}$$

so

$$\frac{1}{f_{eq}} \frac{\partial f_{eq}}{\partial \vec{p}} = -\vec{p}$$

$$\frac{\partial f(\vec{p}, t)}{\partial t} = \frac{\partial}{\partial \vec{p}} \cdot \left[\left\langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \right\rangle \cdot \left(\frac{\partial f}{\partial \vec{p}} + \vec{p} f \right) \right]$$

where

$$\left\langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \right\rangle = \frac{1}{2} \int d(\Delta \vec{p}) \Delta \vec{p} \Delta \vec{p} k(\vec{p} | \vec{p} + \Delta \vec{p})$$

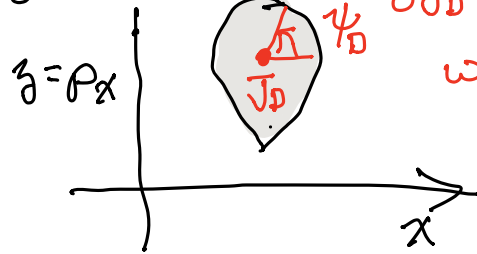
Fokker-Planck Eq

② Guiding Center Drift Atom (GCDA)

$$H_2(R, J_z, \psi, \bar{p}_\perp) = \Omega_i \left[\frac{R^2}{2} - R \sqrt{2m_e \Omega} \bar{p}_\perp \cos \psi + \bar{p}_\perp \right] + H_1(J_z, \bar{p}_\perp)$$

which is integrable to give

$$H_D \equiv H_2(J_z, J_D)$$



$$\omega_D = \frac{\partial H_D}{\partial J_D}$$

$$\omega_z = \frac{\partial H_D}{\partial J_z}$$

$$\text{now } \bar{H}_1 = H_D(J_z, J_D)$$

$$\bar{H}_{12} = \frac{1}{|\vec{r}_2 - \vec{r}_1|}$$

See App B & E for details.

J_z, J_D constants

$$\psi_z = \omega_z t + \psi_{z0}$$

$$\psi_D = \omega_D t + \psi_{D0}$$

$$\bar{f}_1(J_1) \rightarrow f(J_z, J_D, t) \quad dJ_1' dJ_2' \rightarrow dJ_z' dJ_D' d\rho_{2z}'$$

$$dJ_2 \rightarrow d\rho_{2z}$$

$$\text{so } K(J_z, J_D, \rho_{2z} | J_z' J_D', \rho_{2z}') dJ_z' dJ_D' d\rho_{2z}'$$

$$= |v_{2z}| d\sigma$$

$$\frac{\partial f(J_z, J_D, t)}{\partial t} = \int |v_{2z}| [f(J_z', J_D') f(\rho_{2z}') - f(J_z, J_D) f(\rho_{2z})] d\sigma d\rho_{2z}$$

Boltzman Eq

$$\frac{\partial f(J_z, J_D, t)}{\partial t} = \int dJ_z' dJ_D' k(J_z', J_D' | J_z, J_D) f(J_z', J_D') - k(J_z, J_D | J_z', J_D') f(J_z, J_D)$$

where

$$k(J_z, J_0 | J_z', J_0') = \int (v_{zz}) f_{th}(\rho_{zz}) \delta(J_z' - J_{zf}(J_z, J_0, \rho_{zz}, \psi_z, \psi_0, \sigma_s)) \delta(J_0' - J_{of}(J_z, J_0, \rho_{zz}, \psi_z, \psi_0, \sigma_s)) \frac{d\psi_z}{2\pi} \frac{d\psi_0}{2\pi} d\sigma_s d\rho_{zz}$$

where

$$f_{th}(\vec{p}_2) = e^{-(\vec{p}_2)^2/2} / (2\pi)^{3/2}$$

Master Eq

$$\text{Now define } \vec{J} = (J_z, J_0) \quad f_{eq} = e^{-H(\vec{J})}$$

$$\frac{1}{f_{eq}} \frac{\partial f_{eq}}{\partial \vec{J}} = - \frac{\partial H(\vec{J})}{\partial \vec{J}} = - \vec{\omega}$$

where $\vec{\omega} = (\omega_z, \omega_0)$

So

$$\frac{df(\vec{J}, t)}{dt} = \frac{\partial}{\partial \vec{J}} \cdot \left[\left\langle \frac{\Delta \vec{J} \Delta \vec{J}}{\Delta t} \right\rangle \cdot \left(\frac{\partial f}{\partial \vec{J}} + \vec{\omega} f \right) \right]$$

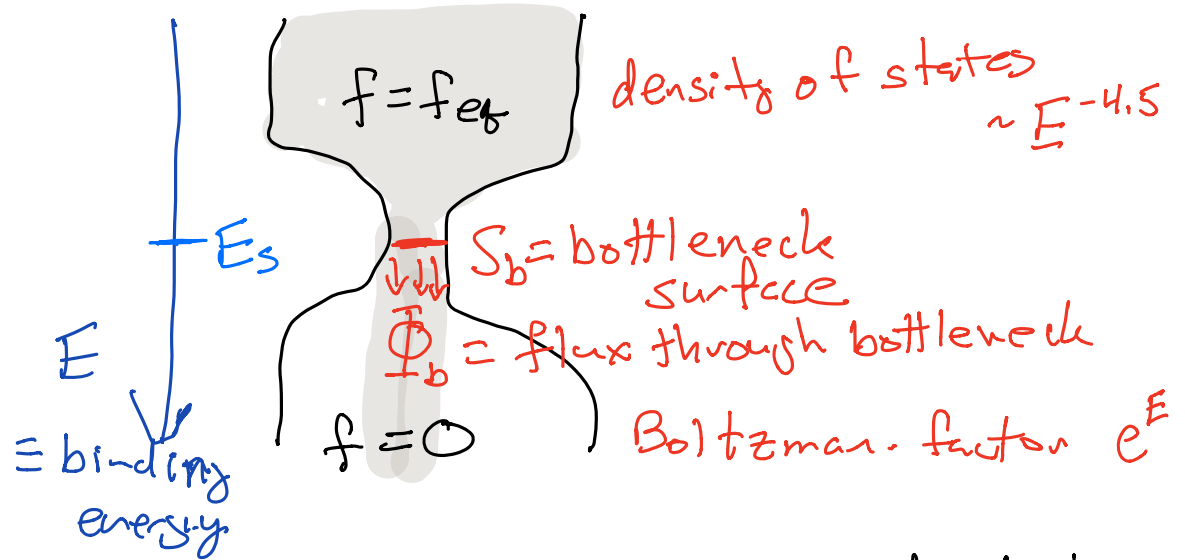
$$\left\langle \frac{\Delta \vec{J} \Delta \vec{J}}{\Delta t} \right\rangle = \frac{1}{2} \int d(\Delta \vec{J}) \Delta \vec{J} \Delta \vec{J} k(\vec{J} | \vec{J} + \Delta \vec{J})$$

Fokker-Planck Eq

Lesson #8

Variational Theory
of Reaction Rates

This is based on a theory of a kinetic bottleneck so that the overall rate will be the rate through the bottleneck. The bottle will be filled to thermal equilibrium above the bottleneck and depleted below the bottleneck.



So we need to do a variational calculation by finding Φ_b as a function of the binding energy which defines a family of surfaces $S_b(E)$ and the distribution is assumed to be thermal above ($E < E_s$) the bottleneck and Φ below. The rate can be estimated as the $\min_{S_b(E)}(\Phi_b)$. The details of this calculation can be found in App. E. What follows is an outline of this calculation.

Let's start by returning to the coordinate free perspective and use the geometric interpretation of the BBGKY hierarchy. We start by asking what is the flow out of a region $R \subset T^*M^n$ if we are given $\tilde{\rho}^{(n+1)}$. This flow is

$$\frac{dN^{(n)}}{dt} = \frac{1}{V^n} \int_R \frac{d\rho^{(n)}}{dt} + \mathcal{L}_{u^{(n)}}(\rho^{(n)})$$

$$= -\frac{n_0}{V^n} \int_{R \times T^*M} \mathcal{L}_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}$$

$$= -\frac{n_0}{V^n} \int_{\mathcal{D}(R \times T^*M)} i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}$$

Making the assumption

$$i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)} \Big|_{R \times \mathcal{D}(T^*M)} = 0$$

We find that the flux is

$$\frac{dN^{(n)}}{dt} = -\frac{n_0}{V^n} \int_{\mathcal{D}R \times T^*M} i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}$$

Assume that T^*M is foliated by a set of leaves, $\{S\}$. Let

$$\begin{aligned} \bar{\Phi} &= \frac{n_0}{V^n} \int_{S_b \times T^*M} |i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}| \\ &\leq \frac{n_0}{V^n} \int_{S \times T^*M} |i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}| \quad \forall S \end{aligned}$$

If $\tilde{\rho}^{(n+1)}$ is an upper bound on $\rho^{(n)}$ then

$$\boxed{\Phi \leq \bar{\Phi} = \frac{n_0}{V^n} \int_{S_b \times T^*M} |i_{u_{int}^{(n)}} \tilde{\rho}^{(n+1)}|}$$

For most applications one uses $\tilde{\rho}^{(2)}$ and

S_E such that $H^{(1)} = E$. For this case

$$\boxed{\Phi_b = \frac{n_0}{V} \int_{S_E \times T^*M} |i_{u_{12}} \tilde{\rho}^{(2)}|}$$

Pulling this back (see App. C) one gets

$$\Phi_b = \frac{n_0}{V} \int dq_1 dp_2 dq_2 \tilde{F}_2 \left[\{H_{12}, H_1\} \frac{\partial \rho_1}{\partial H_1} \right]_{E=H_1(p_1, q_1)}$$

To understand this formula first recognize that $\{H_1, H_{12}\} = \frac{dH_1}{dt}$ due to the Hamiltonian flow given by the interaction vector field u_{12} ($i u_{12} \omega = -dH_{12}$). So $\tilde{F}_2 \frac{dH_1}{dt} \frac{\partial \rho_1}{\partial H_1}$ is the flux through the surface S_E .

Now for our application

3-body Recombination in magnetic field

where the magnetic field is large enough that the guiding center approximation can be made for the electron.

Let us start out by considering the "one particle" Hamiltonian of the electron and the ion. In appendix B, we show that the Hamiltonian can be reduced to two degrees of freedom (z and $E \times B$ dynamics)

$$\begin{aligned} H_2(R; \psi, p_\psi, z, p_z) \\ = \Omega_i \left[m_e \Omega_e \frac{R^2}{2} - R \sqrt{2 m_e \Omega_e p_\psi} \cos \psi + p_\psi \right] \\ + \frac{p_z^2}{2M} - \frac{e^2}{\sqrt{\frac{2 p_\psi}{m_e \Omega_e} + z^2}} \end{aligned}$$

This can be further "averaged" over z to give

$$H_2(R, J_z; \bar{\psi}, \bar{p}_\psi) = \Omega_i \left[m_e \Omega_e \frac{R^2}{2} - R \sqrt{2 m_e \Omega_e \bar{p}_\psi} \cos \bar{\psi} + \bar{p}_\psi \right] + H_1(J_z, \bar{p}_\psi)$$

defining $\rho^2 \equiv \frac{2 \bar{p}_\psi}{m_e \Omega_e}$ $\psi \equiv \bar{\psi}$ and

normalizing $\vec{p}, \vec{R}$ by $\bar{b}$ $\vec{p} \equiv (\bar{x}, \bar{y}, 0)$

H_1 by $k_B T_e$

J_z by $m \bar{v}_e \bar{b}$

$$\frac{1}{a} \equiv \left(\frac{b}{r_{ce}} \right)^2 \frac{m_e}{m_i}$$

$$= \frac{\Omega_i}{\omega_{exB}} = \bar{\tau}_{s1}^3$$

$$\frac{R}{a} \equiv \left(\frac{b}{r_{ce}} \right) \frac{v_{i\perp}}{v_e} = \bar{\tau}_{s2}^{-2} = \varepsilon^2$$

$$\bar{H}(\bar{J}_z, \vec{R}, a; \vec{p}) = \frac{1}{2} (\vec{p} - \vec{R})^2 + a H_1(\bar{J}_z, \rho)$$

$$\bar{H}(\bar{J}_z, \frac{R}{a}, \frac{1}{a}; \rho, \theta) = \left(\frac{1}{a} \right) \frac{\rho^2}{2} + \left(\frac{R}{a} \right) \rho \sin \theta + H_1(\bar{J}_z, \rho)$$

if $1/a, R/a \ll 1$ then $\bar{H} = H_1(\bar{J}_z, \rho)$ and ρ is cyclic so that $\rho = \text{const}$ along with J_z .

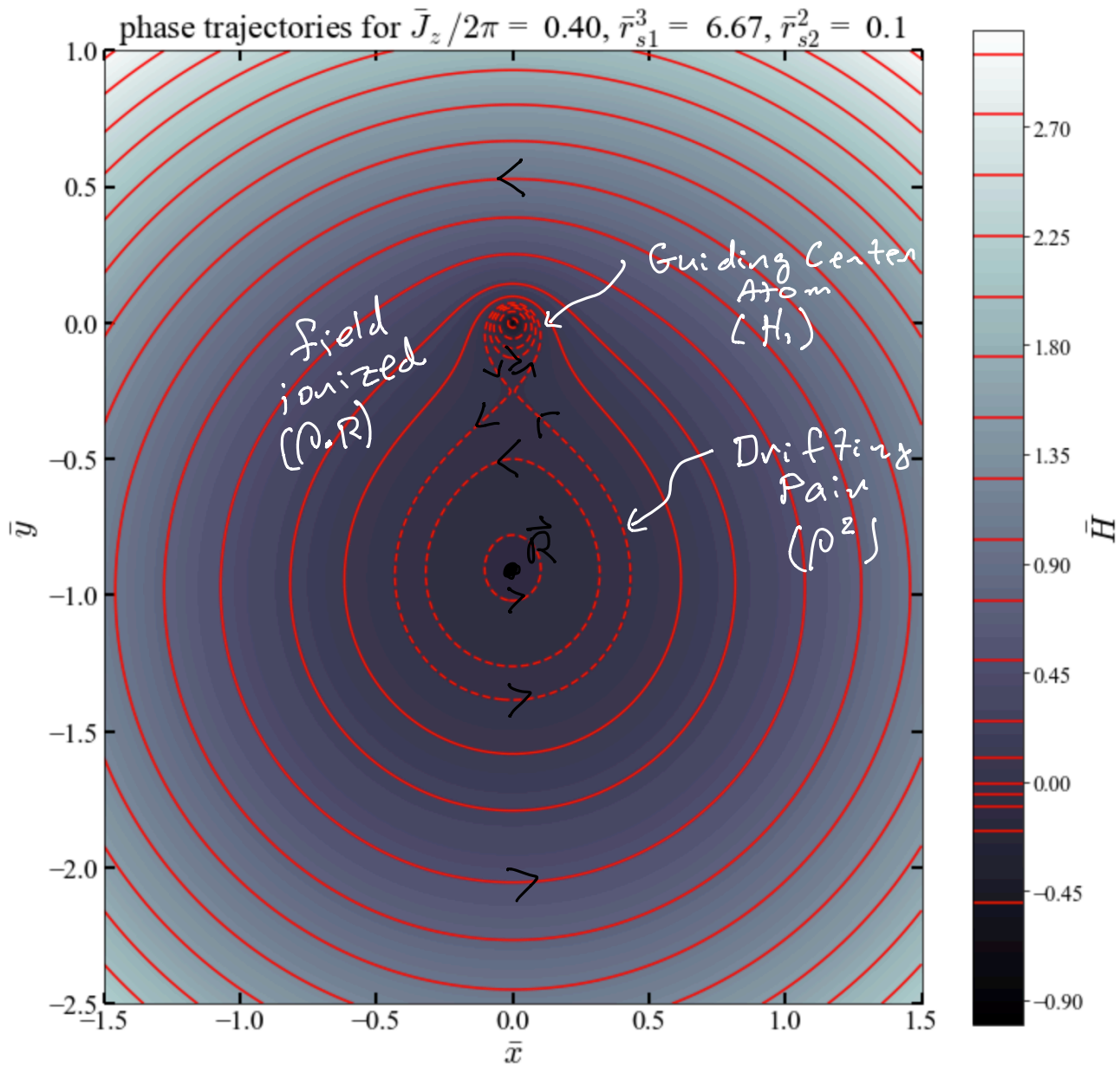
$$1/a \ll 1 \Rightarrow m_i \rightarrow \infty$$

$$R/a \ll 1 \Rightarrow v_{i\perp} \rightarrow 0$$

If neither of these conditions are met then one gets trajectories in phase space like are shown in the following figure. The circular $\rho = \text{const}$ trajectories are within the separatrix around the origin of size $\bar{\tau}_{s2} = (R/a)^{-1/2}$. Electrons outside the separatrix are not bound and are carried to ρ values of order the ion cyclotron radius.

$$a = 0.15 \quad \vec{R} = -\hat{y}$$

$$1/a = 6.67, R/a = 10$$



The character of the separatrix can be examined in more detail by assuming that $m_i \rightarrow \infty$, that is $1/a \rightarrow 0$, while allowing $R \rightarrow \infty$ so that R/a is finite and v_{it} is finite. This effectively takes the ion cyclotron radius to infinity.

The parameter $R/a = \bar{r}_{s2}^{-2} = \Sigma^2$ can be scaled out of the problem giving

$$H(\bar{J}_z, p, \theta) = p \sin \theta + H_1(\bar{J}_z, p)$$

where we scale:

H by ϵ ($k_B T_e$ units)

ρ by ϵ^{-1} ($\bar{b}$ units)

J_z by $\epsilon^{-1/2}$ ($m_e \bar{v} \bar{b}$ units)

ρ_z by $\epsilon^{1/2}$ ($m \bar{v}$ units)

ρ_x by $(\frac{b}{r_{ce}}) \epsilon^{-1}$ ($m \bar{v}$ units)

ω_z by $\epsilon^{3/2}$ ($\bar{v}/b$ units) $\Rightarrow \frac{\omega_D}{\omega_z} \sim \frac{r_{ce}}{b} \epsilon^{3/2}$

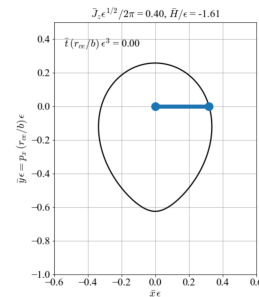
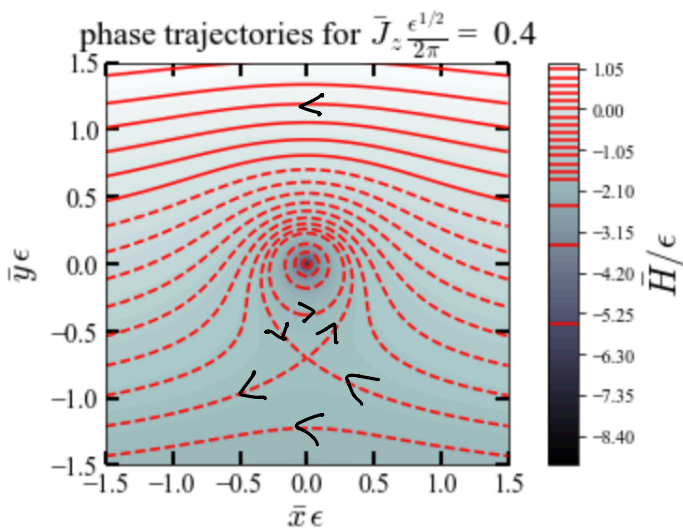
ω_D by $(\frac{r_{ce}}{b}) \epsilon^3$

J_D by $(\frac{b}{m_e}) \epsilon^{-2}$ ($m \bar{v} \bar{b}$ units)

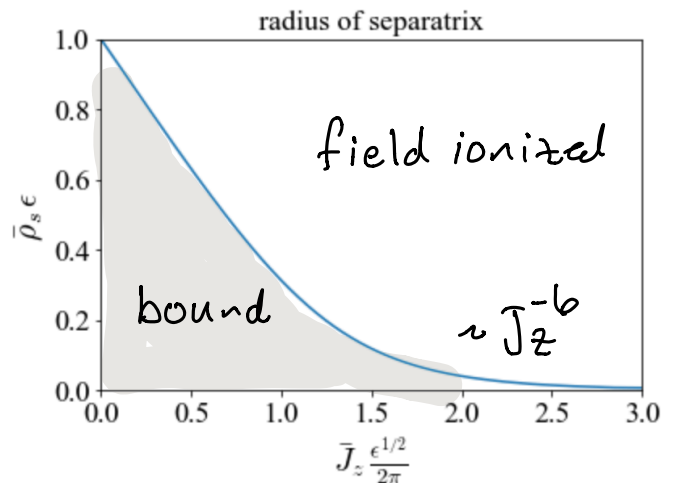
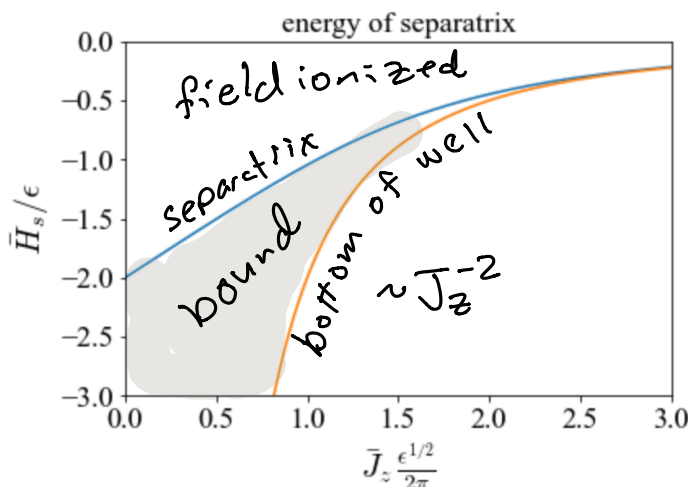
An example of phase space trajectories is

animation of a trajectory can be found at:

http://qitech.biz/videos/phase_space.html



The effect of finite ion thermal velocity is to field ionize atoms that bound with $\rho > \epsilon^{-1}$ and $|H| < \epsilon$ where $\epsilon \gg 1$



Recombination for $m_i \rightarrow \infty$, $v_{i\perp} \rightarrow 0$, stationary ion

The effect of this field ionization and subsequent transport to radii of order the ion cyclotron radius will be examined after we calculate the rate for the $m_i \rightarrow \infty$ $v_{i\perp} \rightarrow 0$ (that is $1/a \rightarrow 0$ and $R/a \rightarrow 0$ limit)

In this limit we have the following forms for H_1 and H_{12} :

$$H_1(z, p_z, x, p_x) = \frac{p_z^2}{2m} - \sqrt{x^2 + \left(\frac{p_x}{m\Omega}\right)^2 + z^2} \Big\} \equiv r_1$$

$$H_{12} = \sqrt{(x_1 - x_2)^2 + \left(\frac{p_{x1} - p_{x2}}{m\Omega}\right)^2 + (z_1 - z_2)^2} \Big\} \equiv r_{12}$$

where $\vec{B} = (mc\Omega/e)\hat{z}$

We set $\tilde{f}_z$ equal to $\emptyset$ for $H_1 < E$ and equal to a thermal distribution for $H_1 > E$

$$\tilde{f}_z \sim \frac{N}{2\pi(m\bar{v})^2(m\Omega)^2} e^{-(H_1 + H_{12} + H_2)}$$

Constraints are:

- (1) $H_1 = -E_S$ (at bottleneck or separatrix)
- (2) $H_1 \geq -E_2$ (particle #2 is unbound)
- (3) $i_{2z} \text{ vol} \geq 0$ ($\tilde{f}_z = 0$ below bottleneck)
- (4) $\bar{\rho}_{12} \leq \rho_0$ (adiabatic cutoff to limit multiple crossings)

Now we calculate:

$$\frac{d\rho_{12}}{dH_1} = \frac{m}{\rho_{12}}$$

$$\{H_{12}, H_1\} = \frac{e^2}{m} \frac{p_{12}(z_1 - z_2)}{r_{12}^3} + \left(\frac{e^2}{m\Omega}\right)^2 \left(\frac{x_2 p_{1x} - x_1 p_{2x}}{r_1^3 r_{12}^3} \right)$$

$$= (\vec{p}_1 \cdot \vec{F}_{12})_{1 \text{ to } B} + (\vec{p}_1 \cdot \vec{F}_{12})_{1 \text{ to } B \text{ (ie } E \times B)} \\ = \frac{dK_{11}}{dt} + \frac{dK_1}{dt}$$

Now,

$$\Phi_b = \frac{n}{V} \int_{E_s = -H_1} \frac{N}{2\pi (m\bar{v})^2 (m\Omega)^2} e^{-(H_1 + H_2 + H_{12})} (2)(2) \\ \left[\frac{e^2 (z_1 - z_2)}{r_{12}^3} + \frac{me}{p_{12}} \left(\frac{e^2}{m\Omega} \right)^2 \left(\frac{x_2 p_{1x} - x_1 p_{2x}}{r_1^3 r_{12}^3} \right) \right] \\ dx_1 dz_1 d\rho_{1x} d\rho_{1z} dx_2 dz_2 d\rho_{2x} d\rho_{2z}$$

make a change of variable (dimensionless)

$$\left\{ \begin{array}{lll} \varepsilon_s \equiv -\frac{E_s}{m\bar{v}^2} & \rho_1 \equiv r_1/b & \bar{x} \equiv x/b \\ & \rho_2 \equiv r_2/b & \bar{y} \equiv \rho_x / m\Omega b \\ \varepsilon_2 \equiv \frac{\rho_{2z}^2}{2m\bar{v}^2} & \rho_{12} \equiv r_{12}/b & \bar{z} \equiv z/b \\ \mu_2 \equiv \frac{1}{\rho_2} - \frac{1}{\rho_{12}} & b = \frac{e^2}{m\bar{v}^2} & \vec{r}_{21} = \vec{r}_1 - \vec{r}_2 \\ r_{ce}^2 = \left(\frac{\bar{v}}{\Omega} \right)^2 & & \end{array} \right.$$

$$= n^2 \bar{v} b^5 \frac{1}{2\pi\sqrt{2}} e^{\varepsilon_s} \int_{\varepsilon_s}^{\varepsilon_s} e^{-\varepsilon_2 + \mu_2} \left[\frac{\bar{z}_1 - \bar{z}_2}{\rho_{12}^3} + \left(\frac{r_{ce}}{\sqrt{2}b} \right) \frac{\bar{x}_2 \bar{y}_1 - \bar{x}_1 \bar{y}_2}{\rho_1^3 \rho_{12}^3 \sqrt{1 - \varepsilon_s}} \right] \\ \varepsilon_2^{-1/2} d\varepsilon_2 d\bar{x}_1 d\bar{y}_1 d\bar{z}_1 d\bar{x}_2 d\bar{y}_2 d\bar{z}_2$$

doing 4 of the integrals leaves us with

$$\Phi_b = R_0 (2\pi)^{3/2} \frac{e^{\varepsilon_s}}{\varepsilon_s^4} \int_{\bar{\rho}_1=0}^1 \bar{\rho}_1^2 d\bar{\rho}_1 \int_{\cos\xi=-1}^1 d(\cos\xi) (1+a) \int_{\bar{\rho}_{12}=0}^{\bar{\rho}_1} d\bar{\rho}_{12} F(\mu_2)$$

$$\left\{ \begin{array}{l} \text{where} \\ R_0 = n^2 \bar{v} b^5 \\ a \equiv \varepsilon_s^{3/2} \left(\frac{r_{ce}}{\sqrt{2}b} \right) \frac{\sin\xi}{\bar{\rho}_1 \sqrt{\bar{\rho}_1 (1-\bar{\rho}_1)}} \end{array} \right.$$

$$\left\{ \begin{aligned} \mu_2 &\equiv \varepsilon_s \left(\frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_{12}} \right) = \varepsilon_s \left(\frac{\bar{\rho}_{12} - \bar{\rho}_2}{\bar{\rho}_2 \bar{\rho}_{12}} \right) \\ \bar{\rho}_2^2 &= \bar{\rho}_1^2 + \bar{\rho}_{12}^2 - 2\bar{\rho}_1 \bar{\rho}_{12} \cos \xi \\ F(\mu_2) &\equiv e^{-\mu_2} \operatorname{erfc}(\sqrt{\max(0, \mu_2)}) \end{aligned} \right.$$

$$= R_0 \frac{e^{\varepsilon_s}}{\varepsilon_s^4} f\left(\frac{r_{ce}}{\sqrt{2}b}, \varepsilon_s, \rho_0\right)$$

$$= R_0 \frac{e^{\varepsilon_s}}{\varepsilon_s^4} \left[g(\varepsilon_s, \rho_0) + \left(\frac{r_{ce}}{\sqrt{2}b}\right) \varepsilon_s^{3/2} h(\varepsilon_s, \rho_0) \right]$$

where

$$g(\varepsilon_s, \rho_0) \equiv (2\pi)^{3/2} \int_{\bar{\rho}=0}^1 \bar{\rho}_1^2 d\bar{\rho}_1 \int_{\bar{\rho}_{12}=0}^{\rho_0} d\bar{\rho}_{12} \int_{x=-1}^1 dx F(\mu_2)$$

$$h(\varepsilon_s, \rho_0) = (2\pi)^{3/2} \int_{\bar{\rho}_1=0}^1 \frac{\bar{\rho}_1 d\bar{\rho}_1}{\sqrt{\bar{\rho}_1(1-\bar{\rho}_1)}} \int_{\bar{\rho}_{12}=0}^{\rho_0} d\bar{\rho}_{12} \int_{x=-1}^1 dx \sqrt{1-x^2} F(\mu_2)$$

Now, finally

$$\Phi_b = R_0 \rho_0 \frac{e^{\varepsilon_s}}{\varepsilon_s^4} 2(2\pi)^{3/2} \left[\bar{g}(\bar{\varepsilon}, \bar{\bar{\varepsilon}}) + \varepsilon \bar{h}(\bar{\varepsilon}, \bar{\bar{\varepsilon}}) \right]$$

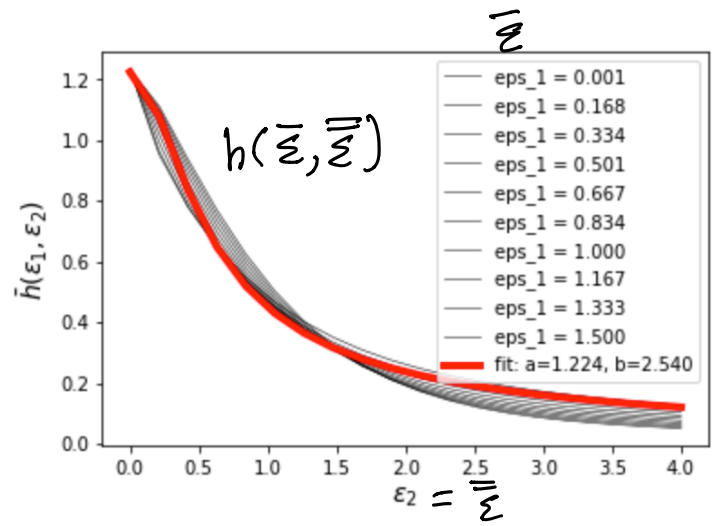
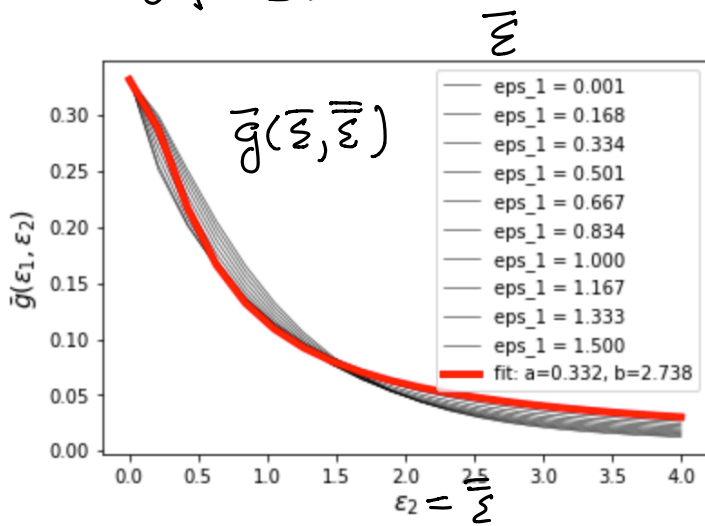
where $\varepsilon \equiv \frac{\varepsilon_s^{3/2}}{\sqrt{2}} \left(\frac{r_{ce}}{b} \right)$, $\bar{\varepsilon} \equiv 1/\rho_0$, $\bar{\bar{\varepsilon}} = \varepsilon_s^{1/2}/\rho_0$

$$\bar{g}(\bar{\varepsilon}, \bar{\bar{\varepsilon}}) = \int_{\xi=0}^{\pi/2} d\xi \sin \xi \cos^5 \xi \int_{\eta=-1}^1 d\eta \int_{z=0}^1 dz F\left(\bar{\bar{\varepsilon}}^2 \frac{(z-\eta)/\bar{\varepsilon}}{\chi z}\right)$$

$$\bar{h}(\bar{\varepsilon}, \bar{\bar{\varepsilon}}) = \int_{\xi=0}^{\pi/2} d\xi \cos^2 \xi \int_{\eta=-1}^1 d\eta \sqrt{1-\eta^2} \int_{z=0}^1 dz F\left(\bar{\bar{\varepsilon}}^2 \frac{(z-\eta)/\bar{\varepsilon}}{\chi z}\right)$$

$$\chi^2 = \bar{\varepsilon}^2 \cos^4 \xi + z^2 - 2\bar{\varepsilon} \eta z \cos^2 \xi$$

Functions $\bar{g}(\bar{\epsilon}, \bar{\epsilon})$ and $\bar{h}(\bar{\epsilon}, \bar{\epsilon})$ can both be evaluated numerically and table interpolated. It was found that both are weak functions of $\bar{\epsilon}$.



$$\approx \frac{1/3}{\sqrt{1 + A^2 \frac{\epsilon_s}{\rho_0^2}}} \quad A=2.7$$

$$\approx \frac{\pi^2/8}{\sqrt{1 + B^2 \frac{\epsilon_s}{\rho_0^2}}} \quad B=2.5$$

$$\Phi_b = \Phi_z + \Phi_D$$

$$\Phi_z \approx R_0 \rho_0 \frac{e^{\epsilon_s}}{\epsilon_s^4} \frac{2(2\pi)^2}{3} (1 + A^2 \epsilon_s / \rho_0^2)^{-1/2}$$

$$\xrightarrow{\rho_0 \rightarrow \infty} R_0 \rho_0 \frac{e^{\epsilon_s}}{\epsilon_s^4} \frac{2(2\pi)^2}{3} \quad (\epsilon_s)_{\min} = 4$$

$$\xrightarrow{\rho_0 \rightarrow 0} R_0 \rho_0^2 \frac{e^{\epsilon_s}}{\epsilon_s^{4.5}} \frac{2(2\pi)^2}{3A} \quad (\epsilon_s)_{\min} = 4.5$$

$$\Phi_D \approx R_0 \rho_0 \frac{e^{\epsilon_s}}{\epsilon_s^{5/2}} \frac{(2\pi)^{3/2}}{4\sqrt{2}} \pi^2 \left(\frac{r_{ce}}{b}\right) (1 + B^2 \epsilon_s / \rho_0^2)^{-1/2}$$

$$\xrightarrow{\rho_0 \rightarrow \infty} R_0 \rho_0 \frac{e^{\epsilon_s}}{\epsilon_s^{5/2}} \frac{(2\pi)^{3/2}}{4\sqrt{2}} \pi^2 \left(\frac{r_{ce}}{b}\right) \quad \epsilon_{sm} = 4$$

$$\xrightarrow{\rho_0 \rightarrow 0} R_0 \rho_0^2 \frac{e^{\epsilon_s}}{\epsilon_s^3} \frac{(2\pi)^{3/2}}{4\sqrt{2}} \frac{\pi^2}{B} \left(\frac{r_{ce}}{b}\right) \quad \epsilon_{sm} = 4.5$$

(Note that $\Phi_D / \Phi_z \sim \omega_0 / \omega_z \sim (r_{ce}/b) \epsilon_s^{3/2}$

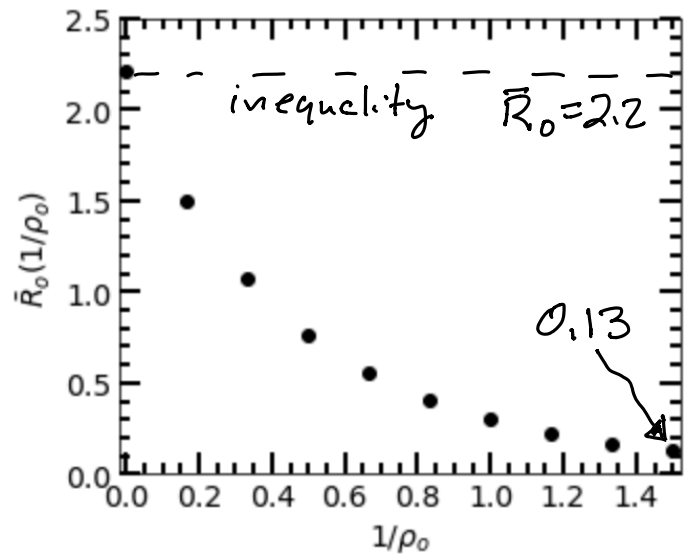
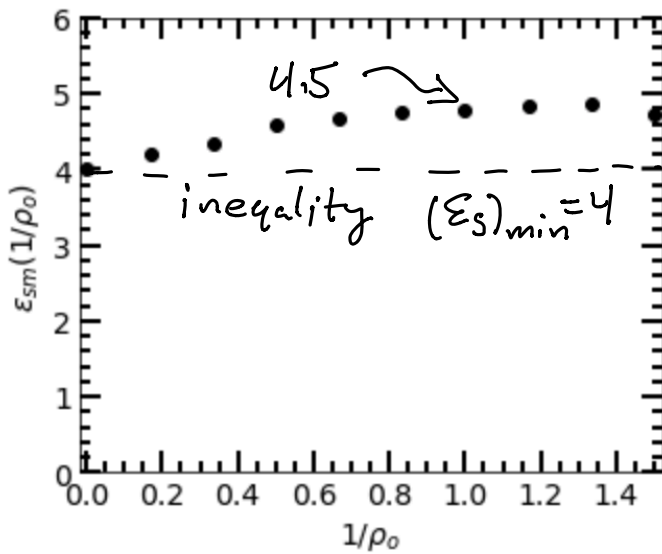
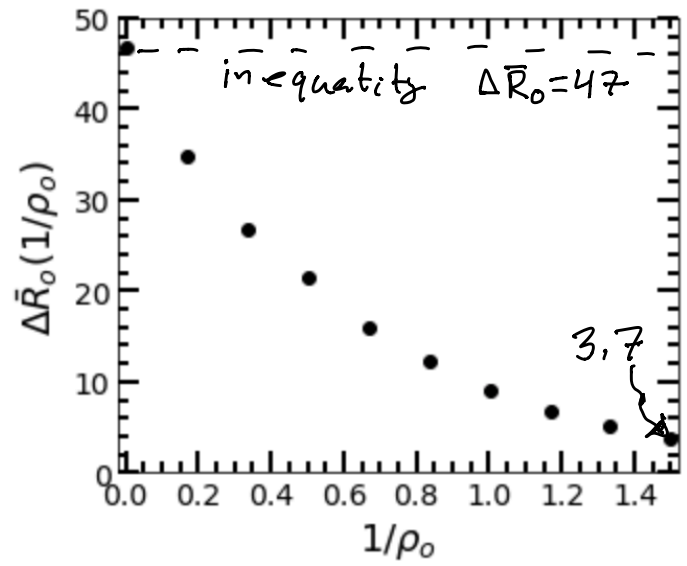
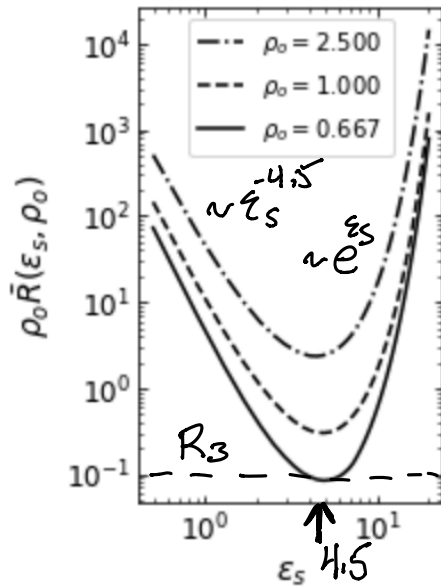
$$\Phi_b \leq R_0 \rho_0 [(2.2) + (47)(r_{ce}/b)]$$

equality for $\rho_0 \rightarrow \infty$

$$= n^2 \bar{\nu} b^5 [(0.08) + (1.7)(r_{ce}/b)]$$

for $\rho_0 = 2/3$

Numerically,



$$\Phi_b = R_0 \rho_0 \left[\bar{R}_0 \left(\frac{1}{\rho_0} \right) + \left(\frac{r_{ce}}{b} \right) \Delta \bar{R}_0 \left(\frac{1}{\rho_0} \right) \right]$$

$$= n^2 \bar{\nu} b^5 [(0.085) + (r_{ce}/b)(2.5)]$$

@ $\rho_0 = 2/3$

Recombination with finite m_i and $v_{i\perp}$

Now let us turn our attention to the case that there is finite $v_{i\perp}$ (i.e., R/a is not much less than 1) and the mass is also finite (i.e., $1/a$ is not much less than 1). In this case there is a separatrix at $\bar{r}_{s2} = (R/a)^{-1/2}$ and electrons at radius greater than that (or energies less than $1/\bar{r}_{s2}$) will go to larger radius until they reach the ion cyclotron radius $\bar{r}_{ci}$. Therefore $e^{\epsilon_s} \rightarrow e^{1/\bar{r}_{ci}}$ and $\epsilon_s^{-9/2} \rightarrow \bar{r}_{s2}^{9/2}$ in the formula for Φ . See App. B for many more details.

Remembering that $(\epsilon_s)_{\min}$ asymptotes to 4.5, and that Φ_b can not be faster than R_0 , we get

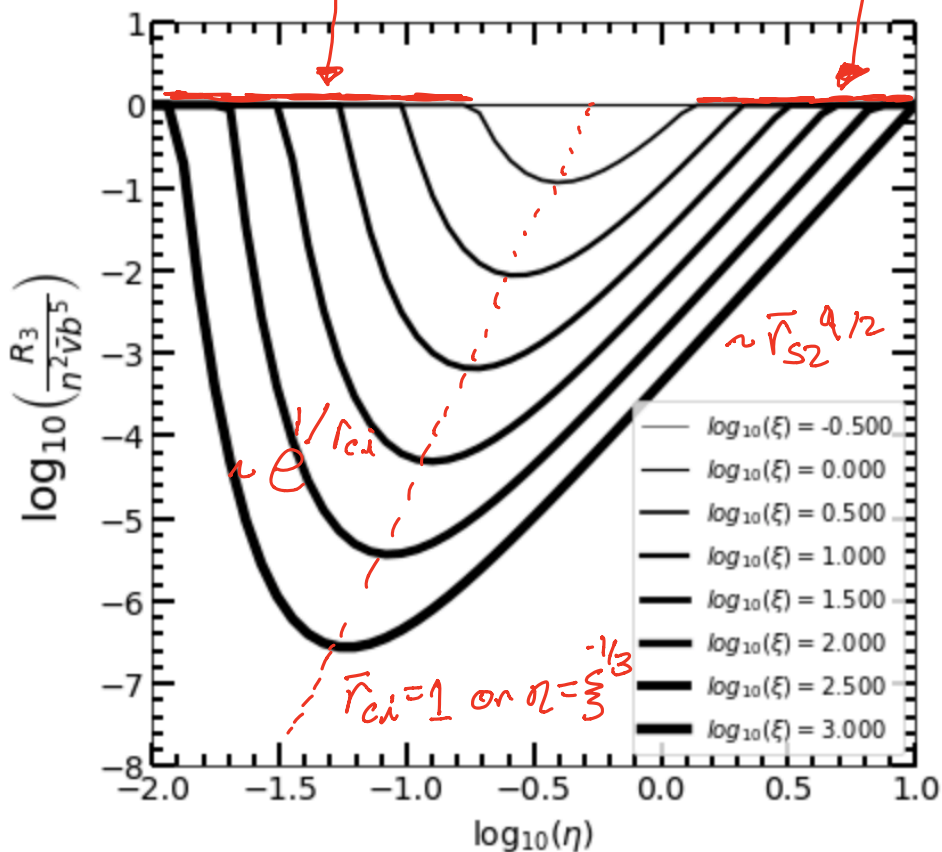
$$\left(\Phi_b\right)_{\bar{r}_{s2}} \sim R_0 \max\left(1, e^{1/\bar{r}_{ci}} \left(\mathcal{O}[\bar{r}_{s2}^{9/2}] + \mathcal{O}\left[\underbrace{\left(\frac{r_{ce}}{b}\right)^{-3/2}}_{\frac{\omega_{D2}}{\omega_{Z1}}} \bar{r}_{s2}^{9/2}\right] \right) \right)$$

A similar result is shown in App B where the separatrix is at

$$\bar{r}_{s2} = \left(\frac{v_{i\perp}}{v_{e\perp}} \frac{b}{r_{ce}}\right)^{-1/2} = \left(\frac{T_i}{T_e}\right)^{-1/4} \left[\left(\frac{m_e}{m_i}\right)^{1/2} \frac{b}{r_{ce}}\right]^{1/2} = (R/a)^{-1/2}$$

Details of this calculation can be found in App D.

$$e^{1/\bar{r}_{ci}} \bar{r}_{s2}^{a/2} = 1 \quad \text{or} \quad \eta = \xi^{-1/3} \left[\frac{a}{4} \ln \xi \right]^{2/3} \quad \left(\bar{r}_{s2} = 1 \quad \text{or} \quad \eta = \xi^{1/3} \right)$$



$$\frac{(\Phi)_{\tilde{R}_s}}{R_0} = \frac{R_3}{n^2 b \sqrt{v}}$$

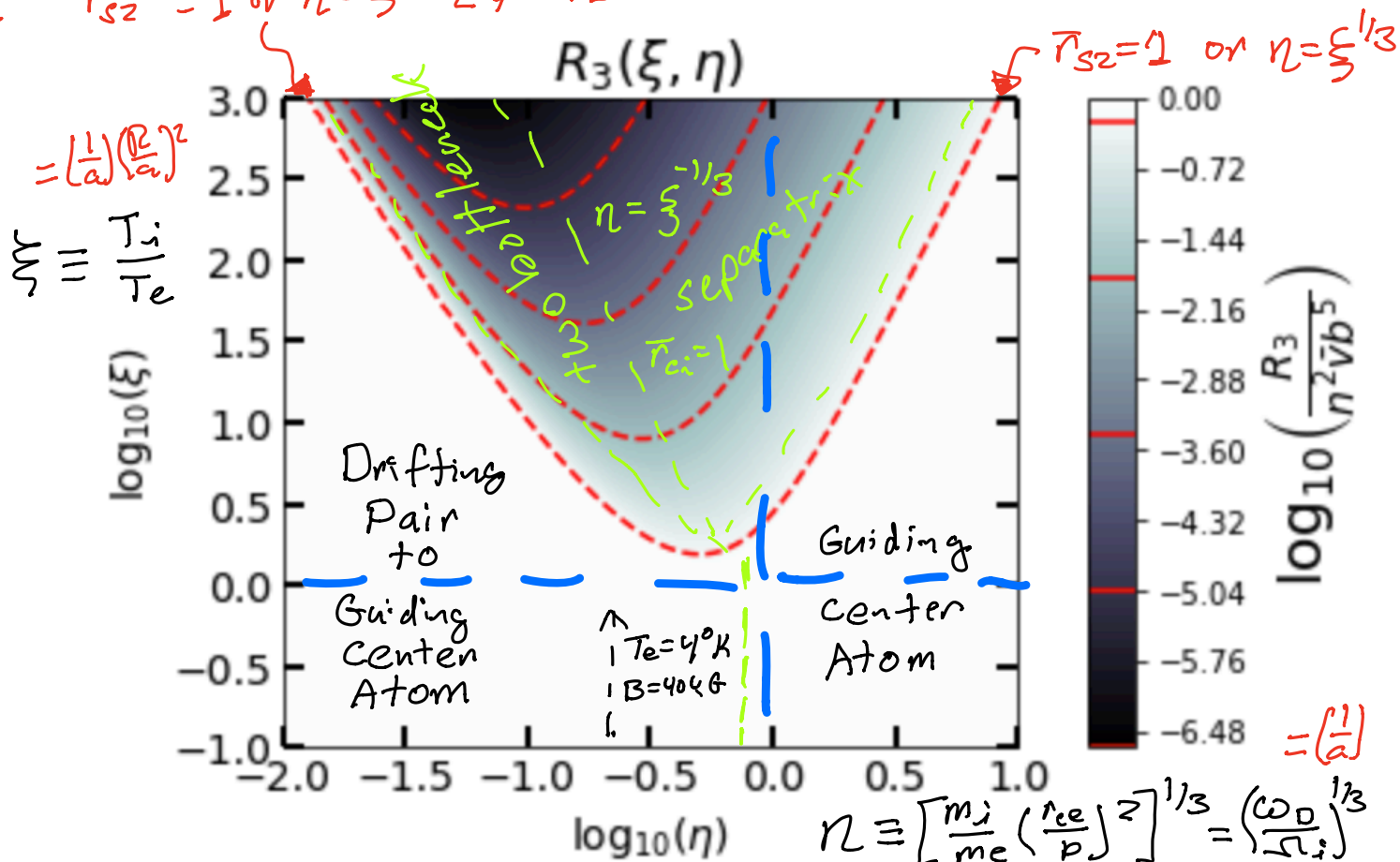
$$\xi = \frac{T_{it}}{T_e} = \left(\frac{1}{a}\right) \left(\frac{R}{a}\right)^2$$

$$n = \left[\left(\frac{r_{ce}}{b} \right)^2 \frac{m_i}{m_e} \right]^{1/3} = \left(\frac{\omega_p}{\Omega_i} \right)^{1/3}$$

$$= \left(\frac{1}{a} \right)$$

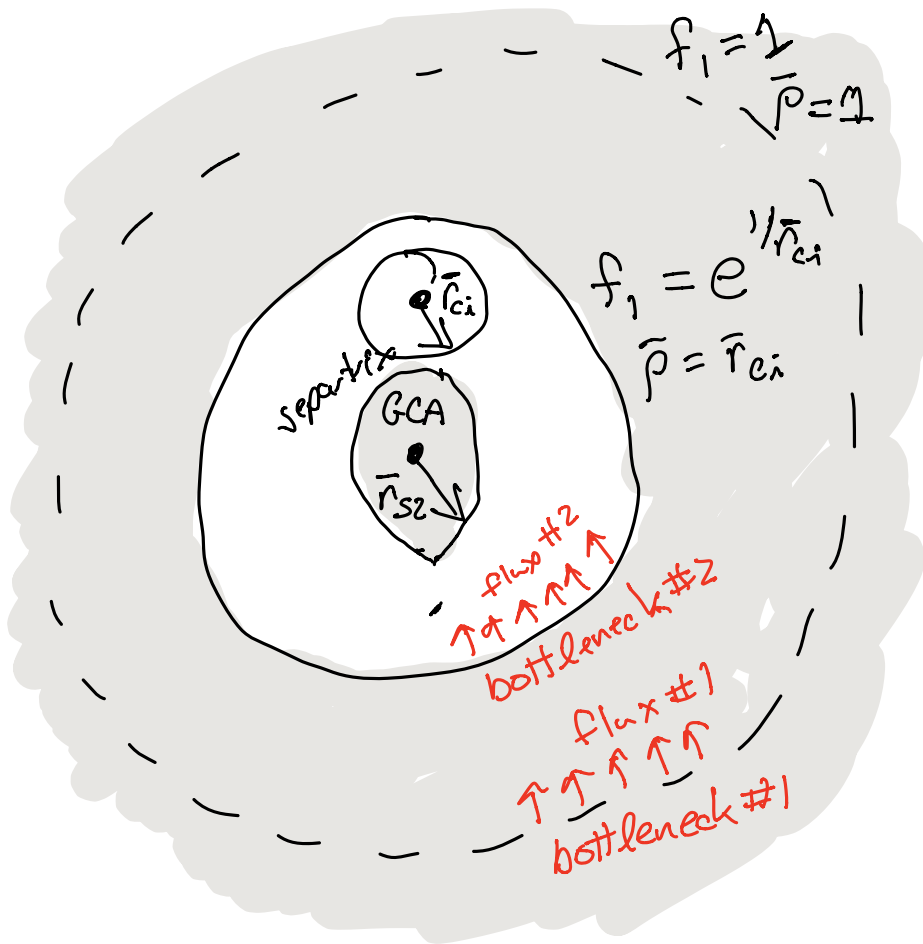
Another way of looking at this is

$$e^{1/r_{ci} - r_{sz}^{a/2}} = 1 \text{ or } n = \xi^{-1/3} \left[\frac{a}{4} \ln \xi \right]^{2/3}$$

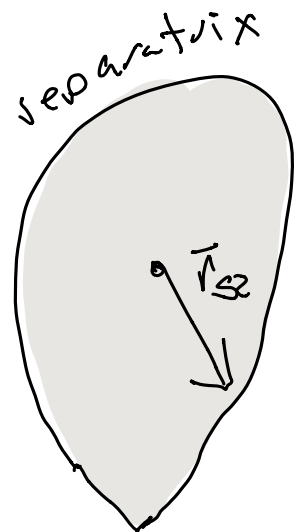


where

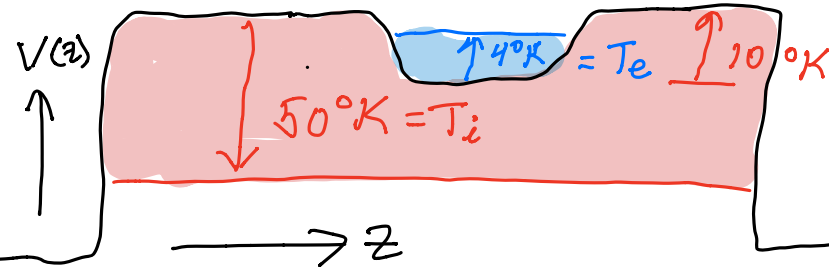
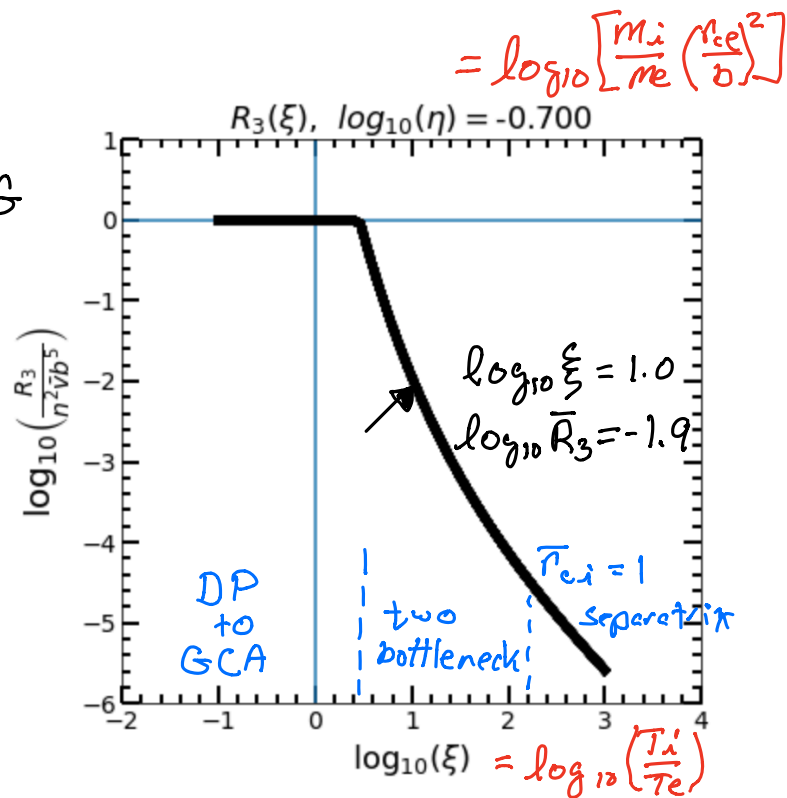
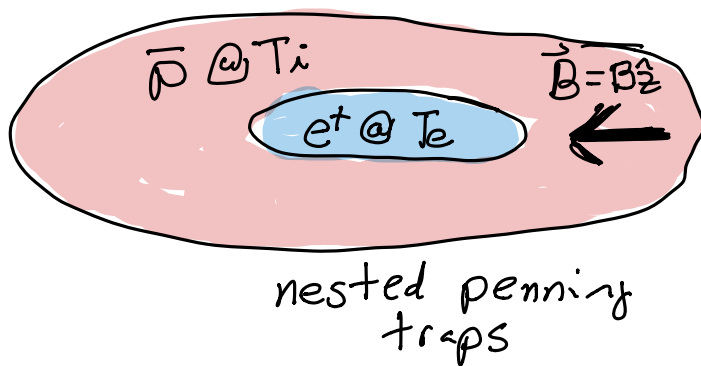
two bottleneck



separatrix



for $T_e = 4^\circ K$ and $B = 40 kG$
 $\log_{10} \eta = -0.7$



What is the R_3 for forming $H @ CERN$?