

Appendix G Fluid & MHD

Start with Generalized Equation for $f_1(P, Q, t)$

$$\frac{\partial f_1}{\partial t} + \sum_i \tilde{H}_i + \langle \tilde{H}_{i2} \rangle_2, f_1 \rangle = 0 \quad \dot{P} = 0$$

where $\tilde{H}_i(P)$, that is $\partial \tilde{H}_i / \partial Q = 0$ and $\dot{Q} = \frac{\partial \tilde{H}_1}{\partial P} = \omega_Q$

integrate $\int dP$ with definition $n \equiv \int dP f_1$
 $\langle \omega_Q \rangle \equiv \bar{\omega}_Q \equiv \frac{1}{n} \int dP \omega_Q(P) f_1$

so that

$$\begin{aligned} 0 &= \frac{\partial n}{\partial t} + \int dP \left(\frac{\partial \tilde{H}_1}{\partial P} \frac{\partial f_1}{\partial Q} - \cancel{\frac{\partial \tilde{H}_1}{\partial Q} \frac{\partial f_1}{\partial P}} \right) + \int dP \langle \tilde{H}_{i2} \rangle, f_1 \rangle \\ &= \frac{\partial n}{\partial t} + \int dP \omega_Q(P) \frac{\partial f_1}{\partial Q} \\ &\quad \frac{\partial}{\partial Q} \left[\underbrace{\int dP \omega_Q(P) f_1(P, Q, t)}_{n \bar{\omega}_Q} \right] \end{aligned}$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial}{\partial Q} (n \bar{\omega}_Q) = 0 \text{ continuity equation}}$$

Operate on Ulsov equation with $\int \omega_Q dP$
 with the definition $\langle g \rangle \equiv \frac{1}{n} \int dP g f_1 = \bar{g}$

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} (n \bar{\omega}_Q) + \int dP \omega_Q(P) \omega_Q(P) \frac{\partial f_1}{\partial Q} + \int dP \omega_Q \langle \tilde{H}_{i2} \rangle, f_1 \rangle \\ &= \frac{\partial}{\partial t} (n \langle \omega_Q \rangle) + \frac{\partial}{\partial Q} (n \langle \omega_Q \omega_Q \rangle) + \int dP \omega_Q \langle \tilde{H}_{i2} \rangle, f_1 \rangle \end{aligned}$$

$$\Rightarrow \frac{d}{dt} (n \bar{\omega}_Q) + \frac{d}{dQ} (n \bar{\omega}_Q \omega_Q) = - \int dP \omega_Q \{ \langle H_{12} \rangle_2, f_1 \}$$

momentum equation
or
fluid force equation

if $\tilde{H}_1 = \frac{P_1^2}{2}$ and $\tilde{H}_{12} = V(Q_1 - Q_2)$

$$\omega_Q = P_1 = v \quad \frac{d}{dQ} = \nabla \cdot \quad - \int dP \omega_Q \{ \langle \tilde{H}_{12} \rangle, f_1 \}$$

$$= - \frac{\partial V}{\partial Q} n = \langle a_{12} \rangle n$$

so $\frac{dn}{dt} + \nabla \cdot (n \bar{v}) = 0$

$$\frac{d(n \bar{v})}{dt} + \nabla \cdot (n \bar{v} \bar{v}) = - \frac{\partial V}{\partial Q} n = \langle a_{12} \rangle n$$

$\uparrow$ pressure/m
 if isotropic $T \equiv m \bar{v}^2$
 $\bar{v} \bar{v} = \bar{v}^2 \overset{\leftrightarrow}{I}$

$= \frac{q}{m} n (\vec{E} + \frac{\bar{v}}{c} \times \vec{B})$
 for magnetized plasma

MHD

What we wish to do is find a form of the "fluid" equations which would be valid if $\omega_Q \ll n \bar{v} b^2$, that is for low frequency. In this case we need to start with the Collisional Vlasov Equation based on the VFP Equation (6.1).

$$\frac{\partial f_1}{\partial t} + \sum_i \{ \tilde{H}_1 + \langle \tilde{H}_{12} \rangle_2, f_1 \} = C[f_1]$$

where $C[f_1]$ can be a Boltzmann, Master or Fokker-Planck collision operator, as shown in Eqs. (), (), and (). We will assume the most general, Boltzmann form, for the collision operator.

We can start by, ^{integrating the collisional Vlasov equation over P} $\int dP$, to get the continuity equation as before

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial Q} (n \bar{\omega}_Q) = 0$$

Since $\int dP C[f_1] = 0$.

Next operate on the Collisional Vlasov Equation with $\int \omega_Q dP$ to get

$$\frac{\partial}{\partial t} (n \bar{\omega}_Q) = - \underbrace{\frac{\partial}{\partial Q} (n \bar{\omega}_Q \omega_Q)}_{\substack{\text{thermal} \\ \text{pressure} \\ \text{force}}} - \underbrace{\int dP \omega_Q \{ \langle \tilde{H}_{12} \rangle_2, f_1 \}}_{\substack{\text{two-particle} \\ \text{interaction} \\ \text{force}}} + \underbrace{\int dP \omega_Q C[f_1]}_{\text{collisional force}}$$

the Momentum equation. We now assume that we have a two-fluid model where the fluids are of species $S = \{+, -\} = \{i, e\}$.
 Now, proceed to the one fluid model, also known as Magnetohydrodynamics (MHD).
 Define,

for ions & electrons

$$\text{mass density} \equiv \rho_M \equiv m_+ n_+ + m_- n_- \approx m_+ n_+$$

$$\text{charge density} \equiv \rho_c \equiv q_+ n_+ + q_- n_-$$

$$\text{center of mass velocity} \equiv \mathbf{v}_M \equiv \frac{1}{\rho_M} (m_+ n_+ \mathbf{v}_+ + m_- n_- \mathbf{v}_-)$$

$$\text{current density} \equiv \mathbf{J}_c = q_+ n_+ \mathbf{v}_+ + q_- n_- \mathbf{v}_-$$

Multiply the ion continuity equation by m_+ , and multiply the electron continuity equation by m_- , then add together to get

$$\boxed{\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial Q} (\rho_M \mathbf{v}_M) = 0,}$$

the ^{generalized} mass conservation equation.

Next multiply the ion continuity equation by q_+ , and multiply the electron continuity equation by q_- , then add together to get

$$\boxed{\frac{\partial \rho_c}{\partial t} + \frac{\partial}{\partial Q} (\mathbf{J}_c) = 0,}$$

the ^{generalized} charge conservation equation.

Now multiply the ion momentum equation by m_+ , and multiply the electron momentum equation by m_- , then add together to get

$$\left[\frac{d}{dt} (\rho_m \Omega_m) + \frac{d}{dQ} (P_t) = F_{ti} \right]$$

the generalized force equation, where $K_+ = \int dP m_+ \omega_+ \mathcal{C}[f_{1+}] = - \int dP m_- \omega_- \mathcal{C}[f_{1-}] = K_-$
 $P_t \equiv m_+ n_+ \overline{\omega_+ \omega_+} + m_- n_- \overline{\omega_- \omega_-} = \text{total pressure tensor} = K_i$
 $= P_+ + P_-$

$$F_{ti} \equiv - \int dP m_+ \omega_+ \{ \langle H_{12} \rangle_2, f_{1+} \} + m_- \omega_- \{ \langle H_{12} \rangle_2, f_{1-} \}$$

$$= F_t(\rho_c, J_c) = \text{total two-particle interaction force}$$

$$F_{tp} = - \frac{d}{dQ} (P_t) = \text{total thermal pressure force}$$

Finally, multiply the ion momentum equation by q_+ and the electron momentum equation by q_- , then add together to get

$$\left[\frac{dJ_c}{dt} + \frac{d}{dQ} (P_{qt}) = F_{qti} + F_{qtp} \right]$$

the generalized Ohm's law, where

$$\frac{q_+}{m_+} \rho_+ + \frac{q_-}{m_-} \rho_- = \rho_{qt} \equiv q_+ n_+ \overline{\omega_+ \omega_+} + q_- n_- \overline{\omega_- \omega_-} = \text{total charge pressure tensor}$$

$$F_{qtp} \equiv - \frac{d}{dQ} (P_{qt}) = \text{total charge thermal force pressure}$$

$$F_{qti} = - \int dP q_+ \omega_+ \{ \langle H_{12} \rangle_2, f_{1+} \} + q_- \omega_- \{ \langle H_{12} \rangle_2, f_{1-} \}$$

$$= \text{total charge two-particle interaction force}$$

$$= F_{qt}(\vec{J}_c, \Omega_m, \rho_m)$$

$$= \int dP q_+ \omega_+ \mathcal{C}[f_{1+}] + \int dP q_- \omega_- \mathcal{C}[f_{1-}]$$

$$= \frac{q_+}{m_+} \int dP m_+ \omega_+ \mathcal{C}[f_{1+}] + \frac{q_-}{m_-} \int dP m_- \omega_- \mathcal{C}[f_{1-}]$$

$$= \frac{q_+}{m_+} K_+ - \frac{q_-}{m_-} K_+ = \left(\frac{q_+}{m_+} - \frac{q_-}{m_-} \right) K_+ \quad K_i$$

$$\text{total charge collisional force} = \frac{q_+ m_- - q_- m_+}{m_+ m_-} K_+ = \frac{q_+ m_- - q_- m_+}{m_+ m_-} \int dP m_+ \omega_+ \mathcal{C}[f_{1+}]$$

As for the fluid equations, let's reduce the generalized expression to the traditional expressions. Remember that

$$\begin{aligned} \frac{d}{dt} &= \nabla \cdot \\ \rho_m &= \text{mathbf{(\rho)}} = \vec{\rho} \\ \vec{J}_c &= \text{mathbf{(\vec{J})}} = \vec{J} \end{aligned} \quad \begin{aligned} m_+ &= m_i \\ m_- &= m_e \\ q_+ &= e \\ q_- &= -e \end{aligned}$$

So that the mass conservation equation is

$$\boxed{\frac{d\rho_m}{dt} + \nabla \cdot (\rho_m \vec{V}) = 0,}$$

the charge conservation equation is

$$\boxed{\frac{d\rho_c}{dt} + \nabla \cdot \vec{J} = 0,}$$

The force equation is

$$\boxed{\rho_m \frac{d\vec{V}}{dt} = -\nabla p + \rho_c \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}}$$

and Ohm's Law is

$$\frac{m_e m_i}{\rho_m e^2} \frac{d\vec{J}}{dt} = \frac{m_i}{2\rho_m e} \nabla p + \vec{E} + \frac{1}{c} \vec{V} \times \vec{B} - \frac{m_i}{\rho_m e c} \vec{J} \times \vec{B} - \frac{\vec{J}}{\sigma},$$

where

$$\rho_{\pm} = \rho = \rho_i + \rho_e$$

$$F_{\pm i} = \rho_c \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}$$

$$\rho_{q\pm} = \frac{e}{2m} \rho$$

$$F_{q\pm i} = \frac{\rho_m e^2}{m_e m_i} \left(\vec{E} + \frac{1}{c} \vec{V} \times \vec{B} \right) - \frac{e}{m_e} \vec{J} \times \vec{B}$$

$$F_{\text{etc}} = - \frac{\rho_m e^2}{m e m_i \sigma} \vec{J} \quad \sigma \equiv \text{electrical conductivity}$$

For very low frequencies the plasma becomes quasineutral so that $\rho_e = 0$, leading to the ideal resistive MHD equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{V}) = 0$$

$$\nabla \cdot \vec{J} = 0$$

$$\rho_m \frac{d\vec{V}}{dt} = \frac{\vec{J} \times \vec{B}}{c} - \nabla p$$

$$\vec{E} + \frac{\vec{V} \times \vec{B}}{c} = \frac{\vec{J}}{\sigma}$$

(mass continuity)
equation

(charge continuity)
equation

(force equation)

(Ohm's Law
or
charge momentum
equation)