

Appendix F

Integration of Drift Equations

$$H(\rho, \theta) = \rho \sin \theta - H_1(J_z, \rho)$$

$$J_z \equiv \oint \rho_z dz$$

H_1, ρ constant

$$J_z H_1^{1/2} = (1 - H_1 \rho) g(H_1 \rho)$$

$$g(n) = 8\sqrt{2} \int_0^{\pi/2} \sqrt{\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n}} \cos^2 \beta d\beta$$

$$H_1 \equiv \frac{\rho_{1z}^2}{2} - \frac{1}{\sqrt{z^2 + \rho^2}}$$

$$A \equiv \frac{J_z^2}{\rho} \Rightarrow J_z H_1^{1/2} = A \sqrt{H_1 \rho}$$

$$A \sqrt{H_1 \rho} = (1 - H_1 \rho) g(H_1 \rho)$$

define y and n $\ni$ $n \equiv H_1 \rho$ $A \equiv g(1) \frac{1-y}{\sqrt{y}}$
quadratic for $y(A)$

$$\left(\frac{1-y}{\sqrt{y}}\right) g(1) \sqrt{n} = (1-n) g(n)$$

$$\left(\frac{1-y}{1-n}\right) \sqrt{\frac{n}{y}} - \frac{g(n)}{g(1)} = 0 \Rightarrow n(y)$$

as $n \rightarrow 1$ $y \rightarrow 1$

$$y = 1 - \delta_y$$

$$n = 1 - \delta_n$$

$$\frac{\delta_y}{\delta_n} \sqrt{\frac{1-\delta_y}{1-\delta_n}} - \left(1 + \frac{g'(1)}{g(1)} \delta_n\right) \approx 0$$

J_z scale by $m \bar{v} b \epsilon^{1/2}$

H, H_1 " $k_B T_e \epsilon^{1/2}$

ρ_z " $m \bar{v} \epsilon^{-1/2}$

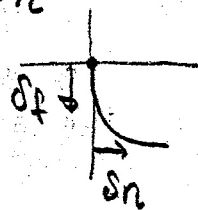
distance " $\bar{b} \epsilon^{-1}$

time $\frac{\bar{b}}{v} \frac{\bar{b}}{r_{ce}} \epsilon^{-3}$

$-\rho_x$ " $\frac{\bar{b}}{r_{ce}} m \bar{v} \epsilon^{-1}$

$$0 \approx \delta y - \delta n \Rightarrow \delta y = \delta n$$

to 1st order



$$\delta n = \delta \phi^2$$

$$\frac{d\delta n}{d\delta \phi} = 2\delta \phi = 2\sqrt{\delta n}$$

$$\frac{d\delta \phi}{d\delta n} = \frac{1}{2} \delta n^{-1/2}$$

as $n \rightarrow 0$ $y \rightarrow 0$

$$y = \delta y$$

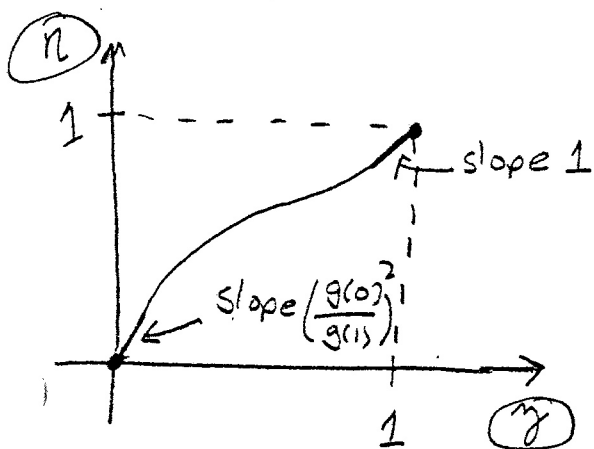
$$n = \delta n$$

$$\frac{1-\delta y}{1-\delta n} \sqrt{\frac{\delta n}{\delta y}} - \left(\frac{g(0)}{g(1)} + \underbrace{\frac{g'(0)}{g(1)} \delta n}_{\sim \delta n^{-1/2}} \right) \approx 0$$

$$\delta n^{1/2} - \delta y^{1/2} \frac{g(0)}{g(1)} \approx 0$$

order 1/2

$$\delta n = \left(\frac{g(0)}{g(1)} \right)^2 \delta y$$



now $H_1(J_z, \rho) = \frac{1}{\rho} n \left[y \left(\frac{J_z}{\rho} \right) \right]$

under the above scalings $p_x = y$ and the equations of motion are

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x}$$

make canonical change of coordinates

$$\tan \psi = \frac{y}{x}$$

$$\rho \psi = \frac{y^2 + x^2}{2} = \frac{p^2}{2}$$

$$H(\psi, p_\psi) = \sqrt{2} p_\psi^{1/2} \sin \psi - H_1(J_z, \sqrt{2} p_\psi^{1/2})$$

$$= \sqrt{2} p_\psi^{1/2} \sin \psi - \frac{1}{\sqrt{2}} p_\psi^{-1/2} n\left[\gamma\left(\frac{J_z}{p_\psi^{1/4} \sqrt{2}}\right)\right]$$

$$\frac{d\psi}{dt} = \frac{\partial H}{\partial p_\psi} = \frac{1}{\sqrt{2}} p_\psi^{-1/2} \sin \psi + 2 \frac{1}{\sqrt{2}} p_\psi^{-3/2} n\left[\gamma\left(\frac{J_z}{p_\psi^{1/4} \sqrt{2}}\right)\right]$$

$$- \frac{1}{\sqrt{2}} p_\psi^{-1/2} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \frac{J_z}{p_\psi^{5/4}} \left(\frac{-1}{4\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} p_\psi^{-1/2} \sin \psi + \frac{1}{2\sqrt{2}} p_\psi^{-3/2} \left[n\left(\gamma\left(\frac{J_z}{2^{1/4} p_\psi^{1/4}}\right)\right) \right.$$

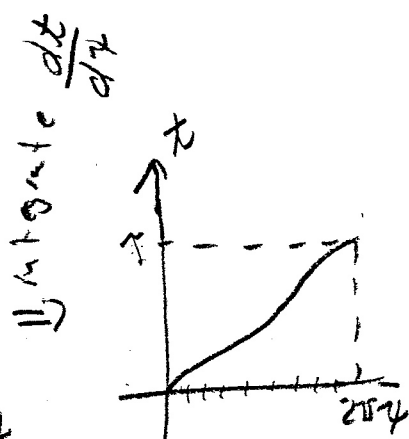
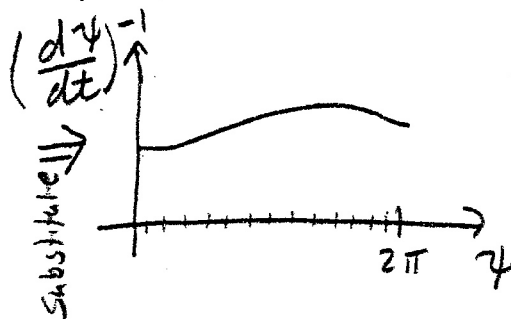
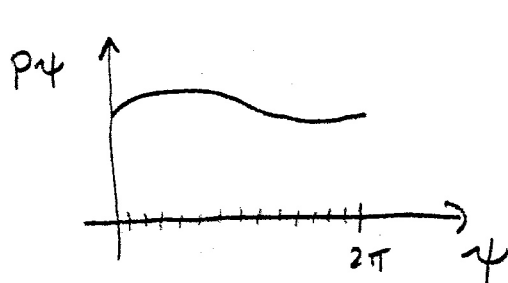
$$\left. + p_\psi^{-1/4} J_z 2^{-5/4} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \right]$$

$$= \frac{1}{p} \sin \psi + \frac{1}{p^3} \left[n\left(\gamma\left(\frac{J_z}{p}\right)\right) + \frac{J_z}{2p} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \right]$$

$$\frac{dp_\psi}{dt} = - \frac{\partial H}{\partial \psi} = \sqrt{2} p_\psi^{1/2} \cos \psi = p \cos \psi$$

$$J_D = \int_0^{2\pi} p_\psi(H, J_z; \psi) d\psi$$

↑ solve $H(J_z, p_\psi, \psi)$ for p_ψ



pick $\frac{\tau}{\tau}$ at random

Similarity Transformation

$$H(x, \frac{p_x}{m\bar{\omega}}, z, p_z) = eE \frac{p_x}{m\bar{\omega}} + \frac{p_z^2}{2m} - \frac{e^2}{\sqrt{z^2 + x^2 + \frac{p_x^2}{(m\bar{\omega})^2}}}$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} \quad \frac{dp_x}{dt} = -\frac{\partial H}{\partial x}$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \quad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}$$

$$\frac{H}{k_B T} = \left(\frac{E}{e/b^2} \right) \left(\frac{p_x}{m\bar{\omega}} \right) \left(\frac{r_{ce}}{b} \right) + \frac{(p_z/m\bar{\omega})^2}{2} - \frac{1}{\sqrt{\left(\frac{z}{b}\right)^2 + \left(\frac{x}{b}\right)^2 + \left(\frac{r_{ce}}{b}\right)^2 \bar{p}_x^2}}$$

$$\bar{H} = \varepsilon^2 \bar{p}_x \frac{r_{ce}}{b} + \frac{\bar{p}_z^2}{2} - \frac{1}{\sqrt{\bar{z}^2 + \bar{x}^2 + \left(\frac{r_{ce}}{b}\right)^2 \bar{p}_x^2}}$$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{\partial \bar{H}}{\partial \bar{p}_x} \quad \frac{d\bar{p}_x}{d\bar{t}} = -\frac{\partial \bar{H}}{\partial \bar{x}}$$

$$\frac{d\bar{z}}{d\bar{t}} = \frac{\partial \bar{H}}{\partial \bar{p}_z} \quad \frac{d\bar{p}_z}{d\bar{t}} = -\frac{\partial \bar{H}}{\partial \bar{z}}$$

$$\varepsilon^{-1} \bar{H} = \varepsilon \bar{p}_x \frac{r_{ce}}{b} + \frac{(\bar{p}_z \varepsilon^{-1/2})^2}{2} - \frac{1}{\sqrt{(\bar{z} \varepsilon)^2 + (\bar{x} \varepsilon)^2 + \left(\frac{r_{ce}}{b}\right)^2 (\bar{p}_x \varepsilon)^2}}$$

$$\bar{\bar{H}} = \bar{\bar{p}}_x + \frac{\bar{\bar{p}}_z^2}{2} - \frac{1}{\sqrt{\bar{\bar{z}}^2 + \bar{\bar{x}}^2 + \bar{\bar{p}}_x^2}}$$

$$\frac{1}{\frac{r_{ce}}{b} \varepsilon^3} \frac{d\bar{\bar{x}}}{d\bar{\bar{t}}} = \frac{\partial \bar{\bar{H}}}{\partial \bar{\bar{p}}_x} \quad \frac{1}{\frac{r_{ce}}{b} \varepsilon^3} \frac{d\bar{\bar{p}}_x}{d\bar{\bar{t}}} = -\frac{\partial \bar{\bar{H}}}{\partial \bar{\bar{x}}}$$

$$\frac{1}{\varepsilon^{3/2}} \frac{d\bar{\bar{z}}}{d\bar{\bar{t}}} = \frac{\partial \bar{\bar{H}}}{\partial \bar{\bar{p}}_z} \quad \frac{1}{\varepsilon^{3/2}} \frac{d\bar{\bar{p}}_z}{d\bar{\bar{t}}} = -\frac{\partial \bar{\bar{H}}}{\partial \bar{\bar{z}}}$$

$$\omega_D \equiv \frac{\bar{\omega}}{b} \frac{r_{ce}}{b} \varepsilon^3$$

$$\omega_z = \varepsilon^{3/2} \frac{\bar{\omega}}{b}$$

$$t_D = \omega_D t \quad t_z = \omega_z t$$

$$\frac{d\bar{x}}{dt_0} = \frac{\partial \bar{H}}{\partial \bar{p}_x}$$

$$\frac{d\bar{p}_x}{dt_0} = - \frac{\partial \bar{H}}{\partial \bar{x}}$$

$$\frac{d\bar{z}}{dt_z} = \frac{\partial \bar{H}}{\partial \bar{p}_z}$$

$$\frac{d\bar{p}_z}{dt_z} = - \frac{\partial \bar{H}}{\partial \bar{z}}$$

$$\bar{H} \equiv \frac{H}{\epsilon k_B T}$$

$$\bar{x} \equiv \frac{x}{\epsilon^{-1} b}$$

$$\bar{p}_x \equiv \frac{p_x}{\epsilon^{-1} b} = \frac{p_x}{\epsilon^{-1} m \bar{v} b}$$

$$\bar{z} \equiv \frac{z}{\epsilon^{-1} b}$$

$$\bar{p}_z \equiv \frac{p_z}{\epsilon^{1/2} m \bar{v}}$$

$$= \frac{p_x}{\epsilon^{-1} m \Omega b}$$

$$t_0 \equiv \omega_0 t = \frac{\bar{v}}{b} \frac{r_{ce}}{b} \epsilon^3 t$$

$$\frac{\omega_0}{\omega_z} = \frac{r_{ce}}{b} \epsilon^{3/2}$$

$$t_z \equiv \omega_z t = \frac{\bar{v}}{b} \epsilon^{3/2} t$$

$$\bar{H} = \bar{p}_x + \frac{\bar{p}_z^2}{2} - \frac{1}{\sqrt{\bar{z}^2 + \bar{x}^2 + \bar{p}_x^2}}$$

Action Calculation

$$\varepsilon = -\frac{\bar{v}^2}{2} + \frac{1}{\sqrt{\bar{z}^2 + \rho^2}}$$

$$\bar{v}^2 = 2 \left[\frac{1}{\sqrt{\bar{z}^2 + \rho^2}} - \varepsilon \right]$$

$$J_z = \oint m \dot{\bar{z}} d\bar{z}$$

$$\bar{J}_z = \frac{J_z}{m \bar{v} b} = \oint \bar{v} d\bar{z} = \sqrt{2} \oint \sqrt{\frac{1}{\sqrt{\bar{z}^2 + \rho^2}} - \varepsilon} d\bar{z}$$

$$= 4\sqrt{2} \int_{\bar{z}=0}^{\sqrt{\frac{1}{\varepsilon^2} - \rho^2}} \sqrt{\frac{1}{\sqrt{\bar{z}^2 + \rho^2}} - \varepsilon} d\bar{z}$$

$$x \equiv \varepsilon \bar{z}$$

$$n \equiv \varepsilon \rho$$

$$= 4\sqrt{2} \varepsilon^{\frac{1}{2}} \int_{\bar{z}=0}^{\frac{1}{\varepsilon} \sqrt{1-n^2}} \sqrt{\frac{1}{\sqrt{(\varepsilon \bar{z})^2 + n^2}} - 1} d\bar{z}$$

$$= 4\sqrt{2} \varepsilon^{-1/2} \int_{x=0}^{\sqrt{1-n^2}} \sqrt{\frac{1}{\sqrt{x^2 + n^2}} - 1} dx$$

$$y^2 \equiv x^2 + n^2$$

$$y dy = x dx$$

$$= 4\sqrt{2} \varepsilon^{-1/2} \int_{y=n}^1 \sqrt{\frac{1}{y} - 1} \frac{y}{\sqrt{y^2 - n^2}} dy$$

$$= 4\sqrt{2} \varepsilon^{-1/2} \int_{y=n}^1 \sqrt{\frac{(1-y)y}{(y-n)(y+n)}} dy$$

$$\xi \equiv \frac{y-n}{1-n}$$

$$d\xi = dy \frac{1}{(1-n)}$$

$$= 4\sqrt{2} \varepsilon^{-1/2} (1-n) \int_{\xi=0}^1 \sqrt{\frac{1-\xi}{\xi} \frac{(1-n)\xi+n}{(1-n)\xi+2n}} d\xi$$

$$(1-n)\xi + n = y$$

$$1-y = (1-n)(1-\xi)$$

$$= 4\sqrt{2} \varepsilon^{-1/2} (1-n) \int_{\beta=0}^{\pi/2} \frac{\cos \beta}{\sin \beta} \sqrt{\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n}} 2 \sin \beta \cos \beta d\beta$$

$$\xi \equiv \sin^2 \beta$$

$$d\xi = 2 \sin \beta \cos \beta d\beta$$

$$= 8\sqrt{2} \varepsilon^{-1/2} (1-n) \int_{\beta=0}^{\pi/2} \sqrt{\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n}} \cos^2 \beta d\beta$$

$$= \varepsilon^{-1/2} (1-n) g(n)$$

$$\frac{1}{2} \frac{\pi}{2} \frac{2}{\sqrt{2}}$$

$$g(n) \equiv 8\sqrt{2} \int_{\beta=0}^{\pi/2} \sqrt{\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n}} \cos^2 \beta d\beta$$

$$g'(n) = \frac{dg}{dn} = 8\sqrt{2} \int_{\beta=0}^{\pi/2} \left\{ \frac{d}{dn} \left[\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n} \right]^{1/2} \right\} \cos^2 \beta d\beta$$

$$\frac{1}{2} \left[\frac{(1-n)\sin^2 \beta + 2n}{(1-n)\sin^2 \beta + n} \right]^{1/2} \frac{[(1-n)\sin^2 \beta + 2n](1-\sin^2 \beta) - [(1-n)\sin^2 \beta + n](2-\sin^2 \beta)}{[(1-n)\sin^2 \beta + 2n]^2}$$

$$\frac{-n \sin^2 \beta - (1-n)\sin^2 \beta}{2 [(1-n)\sin^2 \beta + n]^{1/2} [(1-n)\sin^2 \beta + 2n]^{3/2}}$$

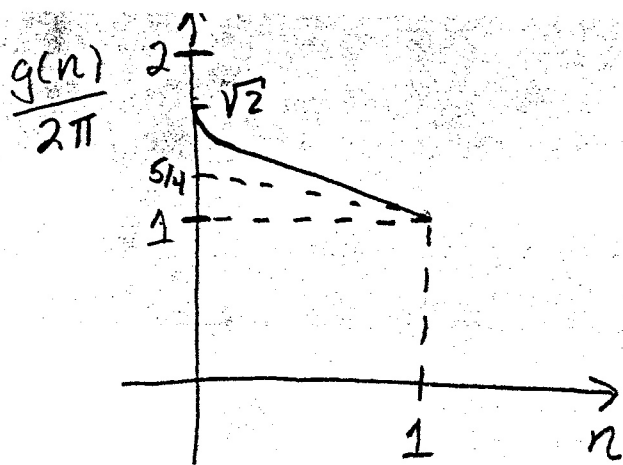
$$\frac{-\sin^2 \beta}{2 [(1-n)\sin^2 \beta + n]^{1/2} [(1-n)\sin^2 \beta + 2n]^{3/2}}$$

$$= 4\sqrt{2} \int_{\beta=0}^{\pi/2} \frac{\sin^2 \beta \cos^2 \beta d\beta}{[(1-n)\sin^2 \beta + n]^{1/2} [(1-n)\sin^2 \beta + 2n]^{3/2}}$$

$$g'(n) \xrightarrow{n \rightarrow 0} O(n^{-1/2}) \quad g(0) = 2\sqrt{2}\pi$$

$$g'(1) = -\frac{1}{16}$$

$$g(1) = 2\pi$$



$$\bar{J}_z = \varepsilon^{-1/2} (1-n) g(n)$$

$$\varepsilon = H_1$$

$$n = H_1 \rho$$

$$H(\rho, \theta) = \rho \sin \theta - H_1(\rho, \bar{J}_z)$$

$$\text{where } H_1^{1/2} \bar{J}_z = (1 - H_1 \rho) g(H_1 \rho)$$

$$\text{defines } H_1(\rho, \bar{J}_z)$$

$$\begin{aligned}
\frac{dn}{dA} &= \frac{n}{\sqrt{n} g'(n) (1-n) - \sqrt{n} g(n) - A/2} \\
&= \frac{n}{\sqrt{n} g'(n) (1-n) - \sqrt{n} g(n) - \frac{1-n}{2\sqrt{n}} g(n)} \\
&= \frac{-n^{3/2}}{n g(n) + \frac{1}{2}(1-n) g(n) - n(1-n) g'(n)} \\
&= \frac{-n^{3/2}}{\frac{1}{2}(1+n) g(n) - n(1-n) g'(n)}
\end{aligned}$$

define $g(y) \equiv g(1)$ $g'(y) = 0$
 by

$$\frac{dy}{dA} = \frac{-y^{3/2}}{\frac{1}{2}(1+y) g(1)}$$

$$\frac{dn}{dy} = - \frac{dn/dA}{dy/dA} = \left(\frac{n}{y}\right)^{3/2} \frac{\frac{1}{2}(1+y) g(1)}{\frac{1}{2}(1+n) g(n) - n(1-n) g'(n)}$$

$$\text{if } n=1 \text{ then } y=1 \Rightarrow \left. \frac{dn}{dy} \right|_{n=y=1} = 1$$

$$\text{if } n \rightarrow 0 \text{ then } y \rightarrow 0 \Rightarrow \left. \frac{dn}{dy} \right|_{n=y=0} = \left(\frac{dn}{dy}\right)^{3/2} \frac{g(1)}{g(0)}$$

$$\frac{dn}{dy} = \left(\frac{dn}{dy}\right)^{3/2} \frac{g(1)/2}{\frac{g(0)}{2} - n(1-0) n^{-1/2}} = \left(\frac{dn}{dy}\right)^{3/2} \frac{g(1)}{g(0)}$$

$$\Rightarrow \left. \frac{dn}{dy} \right| = \left[\frac{g(0)}{g(1)} \right]^2 = 2$$

$$A \sqrt{H, p} = (1 - H, p) g(H, p)$$

$$n \equiv H, p \quad A^2 \equiv \frac{J_z^2}{p} \quad H, = \frac{1}{p} n$$

$$A \sqrt{n} = (1 - n) g(n)$$

$$F(A, n) = 0 = (1 - n) g(n) - A \sqrt{n} \Rightarrow n(A)$$

$$\frac{dn}{dA} = - \frac{\partial F / \partial A}{\partial F / \partial n}$$

$$= - \frac{-\sqrt{n}}{-g(n) + (1 - n) g'(n) - A \frac{1}{2\sqrt{n}}}$$

$$= \frac{n}{\sqrt{n} g'(n) (1 - n) - \sqrt{n} g(n) - A/2}$$

if $g(n) = 1$ (what used in simplification)

$$\frac{dn}{dA} = \frac{\sqrt{n} - n}{\sqrt{n} + A/2}$$

$$n(A) = \left[\frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 + 1} \right]^2$$

$$\frac{\partial H(p, \theta = -\frac{\pi}{2})}{\partial p} = -1 + \frac{1}{p^2} \left[n(A) + \frac{A}{2} \frac{dn}{dA} \right]$$

$$\frac{\partial H(p_s, \theta = -\frac{\pi}{2})}{\partial p} = 0$$

$$\Rightarrow p_s(J_z)$$

$$\frac{d\psi}{dt} = \frac{1}{p} \sin \psi + \frac{1}{p^3} \left[n(A) + \frac{A}{2} \frac{dn}{dA} \right]$$

$$H_3(J_z) = H(J_z; p_s(J_z), -\frac{\pi}{2})$$

$$H(J_z; p, \theta) = p \sin \theta - \frac{1}{p} n(A) \Rightarrow p(H, J_z; \theta)$$

$$J_n = \int_0^{2\pi} p^2 d\theta$$

$$H(\rho, \theta) = \rho \sin \theta - \frac{1}{\rho} n \left[\gamma \left(\frac{J_z}{\rho} \right) \right]$$

$$\frac{\partial H(\rho, \theta = -\frac{\pi}{2})}{\partial \rho} = -1 - \left\{ -\frac{1}{\rho^2} n \left[\gamma \left(\frac{J_z}{\rho} \right) \right] + \frac{1}{\rho} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \left(\frac{-J_z}{2\rho^{3/2}} \right) \right\}$$

$$= -1 + \frac{1}{\rho^2} \left\{ n \left[\gamma \left(\frac{J_z}{\rho} \right) \right] + \underbrace{\left(\frac{J_z}{2\rho} \right)}_{\frac{A}{2}} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \right\}$$

$$= -1 + \frac{1}{\rho^2} \left\{ n \left[\gamma \left(\frac{J_z}{\rho} \right) \right] + \frac{A}{2} \frac{dn}{d\gamma} \frac{d\gamma}{dA} \right\}$$

if $n = \gamma$ (approximation made before)

$$\underset{n=\gamma}{\approx} -1 + \frac{1}{\rho^2} \left\{ \gamma(A) + \frac{A}{2} \frac{d\gamma}{dA} \right\}$$

$$A^2 \equiv \frac{J_z^2}{\rho} \quad \frac{dA}{d\rho} = -\frac{A}{2\rho}$$

$$0 = \frac{\partial H(\rho_s, \theta = -\frac{\pi}{2})}{\partial \rho} \quad \text{gives } \rho_s$$

$$H_s = H(\rho_s, \theta = -\frac{\pi}{2})$$

$$\gamma + \frac{A}{g(1)} \sqrt{\gamma} - 1 = 0 \quad \Rightarrow \quad \sqrt{\gamma} = \frac{-\frac{A}{g(1)} \pm \sqrt{\left(\frac{A}{g(1)}\right)^2 + 4}}{2}$$

$$\gamma = \left[\frac{A}{2g(1)} - \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1} \right]^2$$

$$\frac{d\gamma}{dA} = \left[\frac{A}{2g(1)} - \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1} \right] \left[\frac{1}{2g(1)} - \frac{\frac{A}{2(2g(1))^2}}{2\sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1}} \right]$$

$$= \left[\frac{A}{2g(1)} - \sqrt{\frac{A^2}{4g(1)^2} - \frac{A}{2g(1)}} \right]$$

$$= - \frac{\left[\frac{A}{2g(1)} - \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1} \right]^2}{2g(1) \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1}} = - \frac{\gamma(A)}{2g(1) \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1}}$$

$$H_p = -1 + \frac{\gamma}{\rho z} \left[1 - \frac{A \gamma}{4g(1) \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1}} \right]$$

$$= -1 + \frac{\gamma}{\rho z} \left\{ 1 - \left(\frac{A}{2g(1)}\right) \frac{\gamma/2}{\sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1}} \right\}$$

$$\gamma = 17.1 \rho$$

$$A^2 = \frac{J z^2}{\rho}$$

Outline of Numerical Strategy

(I) Graph $n(y)$ by solving $\frac{1-y}{1-n} \sqrt{\frac{n}{y}} - \frac{g(n)}{g(1)} = 0$

Graph $\frac{dn(y)}{dy}$ by substituting into

$$\frac{dn}{dy} = \left(\frac{n}{y}\right)^{3/2} \frac{(1+y)g(1)/2}{(1+n)g(n)/2 - n(1-n)g'(n)}$$

(II) Given $A = \frac{\sqrt{J_z}}{\sqrt{\rho}}$ one can find y by

$$y(A) = \left[\frac{A}{2g(1)} - \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1} \right]^2$$

and then find $n(y)$ and dn/dy by interpolation of tables found in (I)

This will lead to following quantities

$$H(J_z; \rho, \psi) = \rho \sin \psi - \frac{1}{\rho} n$$

$$\frac{d\psi}{dt} = \frac{1}{\rho} \sin \psi + \frac{1}{\rho^2} \left[n + \frac{A}{2} \frac{dn}{dy} \frac{dy}{dA} \right]$$

$$\frac{\partial H}{\partial \rho} = -1 + \frac{1}{\rho^2} \left[n + \frac{A}{2} \frac{dn}{dy} \frac{dy}{dA} \right]$$

where $\frac{dy}{dA} = \frac{-2 y^{3/2}}{(1+y)g(1)}$

Can now solve $H(J_z; \rho, \psi)$ to find $\rho(H, J_z; \psi)$

then have $\left[\frac{d\psi(H, J_z; \psi)}{dt} \right]^{-1} = \frac{dt(H, J_z; \psi)}{d\psi}$

$$\kappa(\psi) = \int_0^\psi \frac{d\psi}{dt} d\psi \Rightarrow \text{action angle mapping}$$

Also can find $\rho_s(J_z)$ by condition $\frac{\partial H(J_z; \rho_s, -\frac{\pi}{2})}{\partial \rho} = 0$

then $H_s(J_z) = H(J_z; \rho_s(J_z), -\frac{\pi}{2})$

$$J_D = \int_0^{2\pi} \frac{\rho^2}{2} d\psi \quad \text{if needed}$$

(III) At end of collision we have $(\rho, \bar{z}, \bar{\rho}_z)$
 now form $n = \left[\frac{1}{\sqrt{\bar{z}^2 + \rho^2}} - \frac{\bar{\rho}_z^2}{2} \right] \rho - 11$

Invert graph in (I) to get $y(n)$
 then find $\frac{dn(y(n))}{dy}$ by interpolation

Finally $A \bar{J}_z^{-1} = \sqrt{\rho} A = \sqrt{\rho} \frac{1-n}{\sqrt{n}} g(n)$

Continue as before $= \sqrt{\rho} \frac{1-y}{\sqrt{y}} g(1)$

$\frac{\partial H}{\partial \rho}$ calculation

$$\frac{J_2}{\rho} \equiv A \equiv \frac{1-n}{\sqrt{n}} g(n) \quad \frac{dA}{dn} = -n^{-3/2} \left[\frac{1}{2}(n+1)g(n) - n(1-n)g'(n) \right]$$

$$\Rightarrow \frac{dn}{dA} = \frac{-n^{3/2}}{\frac{1}{2}(n+1)g(n) - n(1-n)g'(n)}$$

$$f(n) = n + \frac{A}{2} \frac{dn}{dA} = n + \frac{(1-n)}{2\sqrt{n}} g(n) \frac{-n^{3/2}}{\frac{1}{2}(n+1)g(n) - n(1-n)g'(n)}$$

$$= \frac{\frac{1}{2}n(n+1)g(n) - n^2(1-n)g'(n) - n(1-n)g(n)}{\frac{1}{2}(n+1)g(n) - n(1-n)g'(n)}$$

$$= \frac{n^2 g(n) - n^2(1-n)g'(n)}{\frac{1}{2}(n+1)g(n) - n(1-n)g'(n)}$$

$$= n^2 \frac{g(n) - (1-n)g'(n)}{\frac{1}{2}(n+1)g(n) - n(1-n)g'(n)}$$

know $g(n) \xrightarrow{n \rightarrow 0} \sqrt{2}$

$$g'(n) \xrightarrow{n \rightarrow 0} -Bn^{-1/2} \quad (B \approx 0.54)$$

$$g(n) \xrightarrow{n \rightarrow 1} 1$$

$$g'(n) \xrightarrow{n \rightarrow 1} -\frac{1}{16}$$

$$f(n) \xrightarrow{n \rightarrow 0} \sqrt{2} B n^{3/2}$$

$$f(n) \xrightarrow{n \rightarrow 1} 1$$

$$\frac{1}{\sqrt{2}} \approx 0.707$$

$$\frac{\partial H}{\partial \rho} = \sin \theta + \frac{1}{\rho^2} f(n) = \sin \theta + (1-n)^4 g^4(n) n^{-2} f(n) \frac{1}{J_z^4}$$

$$\frac{\partial H}{\partial \rho} \xrightarrow{n \rightarrow 1} \sin \theta + \frac{1}{\rho^2}$$

$$\frac{\partial H}{\partial \rho} \xrightarrow{n \rightarrow 0} \sin \theta + \frac{B 4 \sqrt{2}}{J_z^4} n^{-1/2} \rightarrow \sin \theta + \frac{4}{J_z^3} \frac{B}{\sqrt{\rho}}$$

$$\text{for } n \rightarrow 0 \quad n \xrightarrow{n \rightarrow 0} 2\gamma$$

$$\gamma \xrightarrow{\gamma \rightarrow 0} \frac{1}{A^2} = \frac{\rho}{J_z^2} \Rightarrow n \xrightarrow{n \rightarrow 0} \frac{2\rho}{J_z^2}$$

$$\text{for } \frac{\partial H(\theta = -\frac{\pi}{2})}{\partial \rho} = 0 \Rightarrow \rho_s \xrightarrow{J_z \gg 1} \frac{16}{J_z^6}$$

$$\rho_s \xrightarrow{J_z \ll 1} 1$$

Old function $g(n) \equiv 1 \quad g'(n) = 0$

$$f(n) = n^2 \left(\frac{2}{n+1} \right)$$

$$\frac{\partial H}{\partial \rho} \xrightarrow{n \rightarrow 1} \sin \theta + \frac{1}{\rho^2}$$

$$\frac{\partial H}{\partial \rho} \xrightarrow{n \rightarrow 0} \sin \theta + \frac{2}{J_z^4} (1-5n) = \sin \theta + \frac{2}{J_z^4} - \frac{10}{J_z^6} \rho$$

$$n \xrightarrow{n \rightarrow 0} \frac{1}{A^2} = \frac{\rho}{J_z^2}$$

$$\text{for } \frac{dH}{dp}(\theta = -\frac{\pi}{2}) = 0 \Rightarrow$$

$$p_s \xrightarrow{J_z \ll 1} 1$$

$$(J_z)_{\max} = 2^{1/4}$$

$$p_s \xrightarrow{\Delta J \ll 1} \frac{2^{1/4}}{5} \underbrace{(2^{1/4} - J_z)}_{\Delta J}$$

$f(n) \gamma^{-3/2}$	γ
1.65484	10^{-2}
1.96034	10^{-3}
2.09013	10^{-4}
2.15736	10^{-6}

$$4B = 2.15736$$

numerical results

$$\frac{dH}{dp} \xrightarrow{n \rightarrow 0} \sin \theta + \frac{4B}{Jz^3} \frac{1}{p}$$

$$H \xrightarrow{n \rightarrow 0} p \sin \theta + \underbrace{\frac{8B}{Jz^3} \sqrt{p}} - \frac{2}{Jz^2} + \frac{1}{p} \frac{8B}{Jz^3} p^{3/2}$$

$$= p \sin \theta + \frac{1}{p} \left[\underbrace{8B \left(\frac{p}{Jz^2} \right)^{3/2} - \frac{2p}{Jz^2}}_{\underbrace{8B \left(\frac{1}{A^2} \right)^{3/2} - \frac{2}{A^2}}_n} \right]$$

$$B \approx 0.54$$

$$n = 2\gamma - D\gamma^{3/2}$$

$$D = 8B$$

$$\gamma = \frac{1}{2}n + \delta_n \quad \text{sub into}$$

$$\frac{-n + 2\gamma}{\gamma^{3/2}} = +D$$

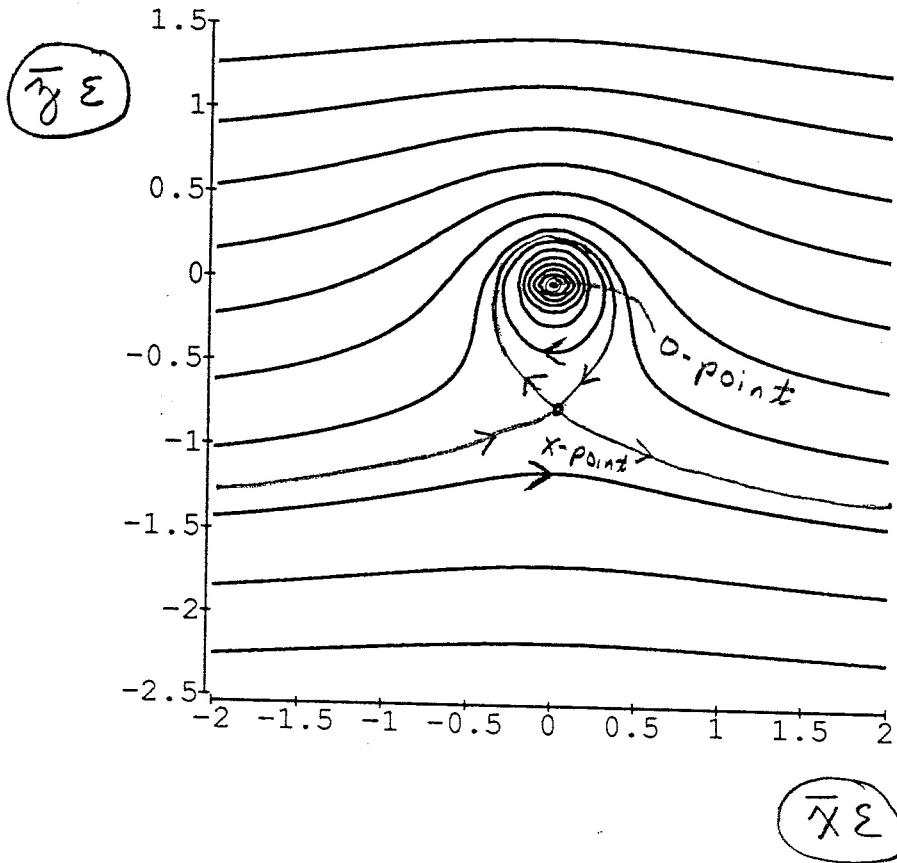
$$n \approx 2\left(\frac{1}{2}n + \delta_n\right) - D\left(\frac{1}{2}n + \delta_n\right)^{3/2}$$

$$\approx n + 2\delta_n - \frac{Dn^{3/2}}{2^{3/2}}$$

$$\delta_n = \frac{+D}{2^{5/2}} n^{3/2}$$

$$\gamma = \frac{1}{2}n + \frac{D}{2^{5/2}} n^{3/2}$$

$$4.32 \approx D$$



$$\frac{\bar{J}_z}{2\pi} \epsilon^{1/2} = 0.4$$

↑ $\vec{E} = E \hat{z}$

$$\rho_s \epsilon = 0.702819$$

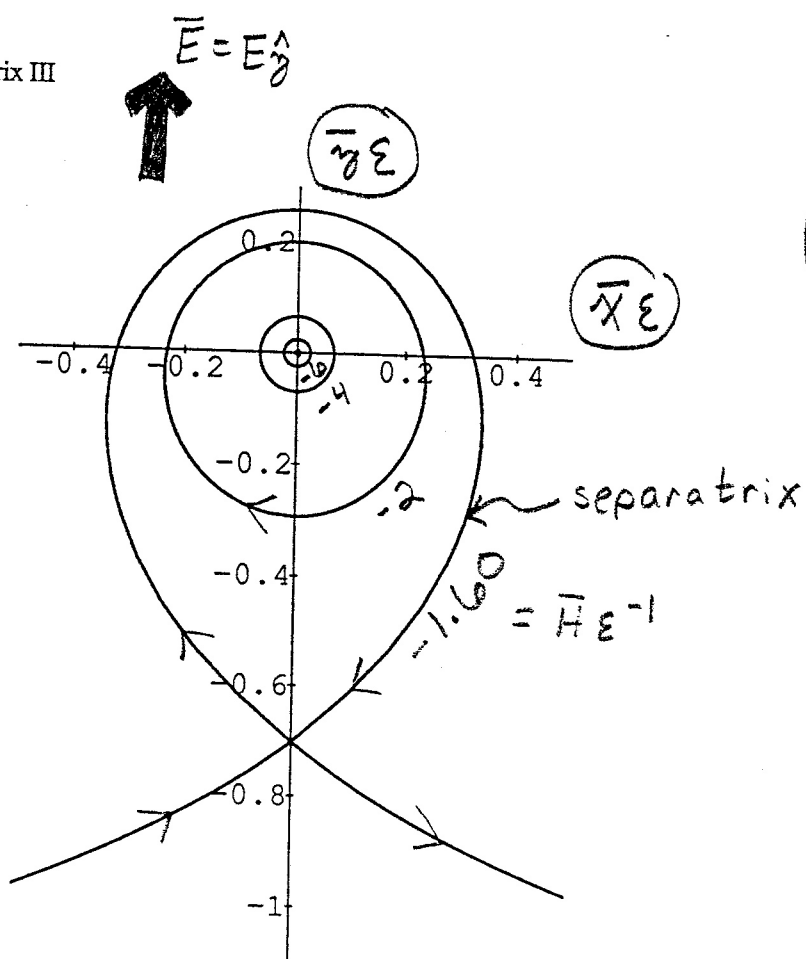
$$\bar{H}_s \epsilon^{-1} = -1.60216$$

$$\vec{B} = B \hat{z}$$

$$\vec{v}_i = -\bar{v}_i \hat{x}$$

$$\epsilon^2 = \frac{E}{e \bar{b}^2}$$

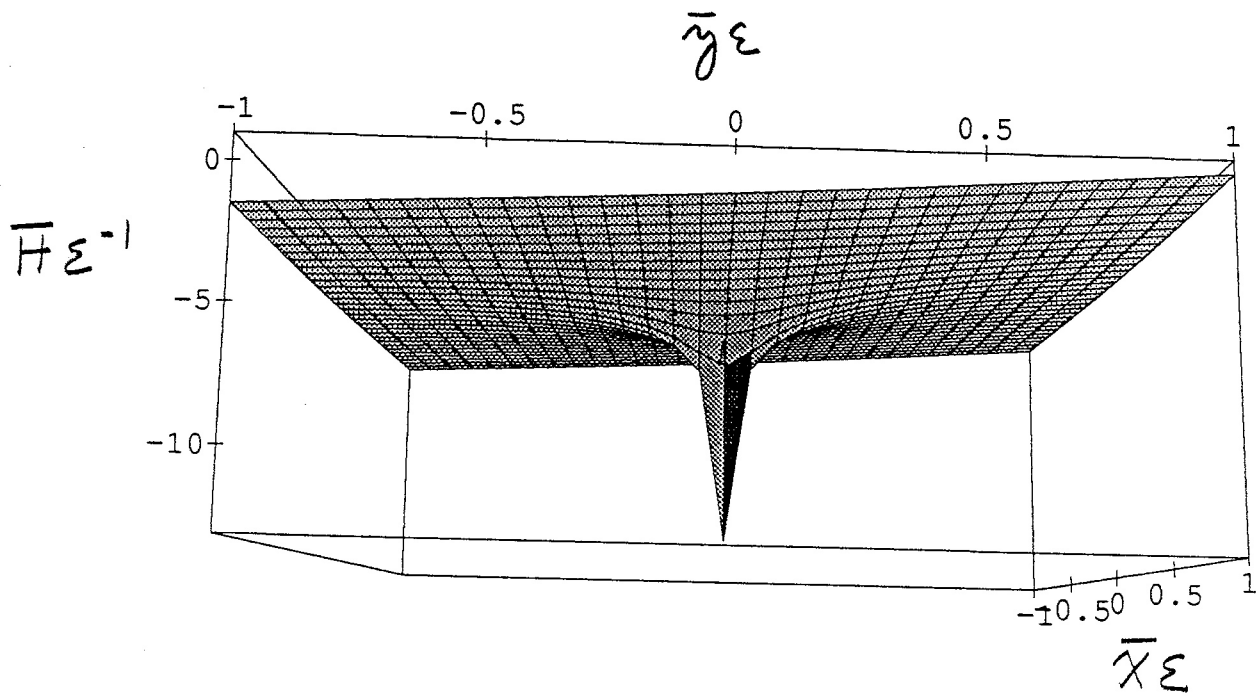
$$= \left(\frac{\bar{v}_i}{\bar{v}_e} \right) \left(\frac{b}{r_{ce}} \right)$$



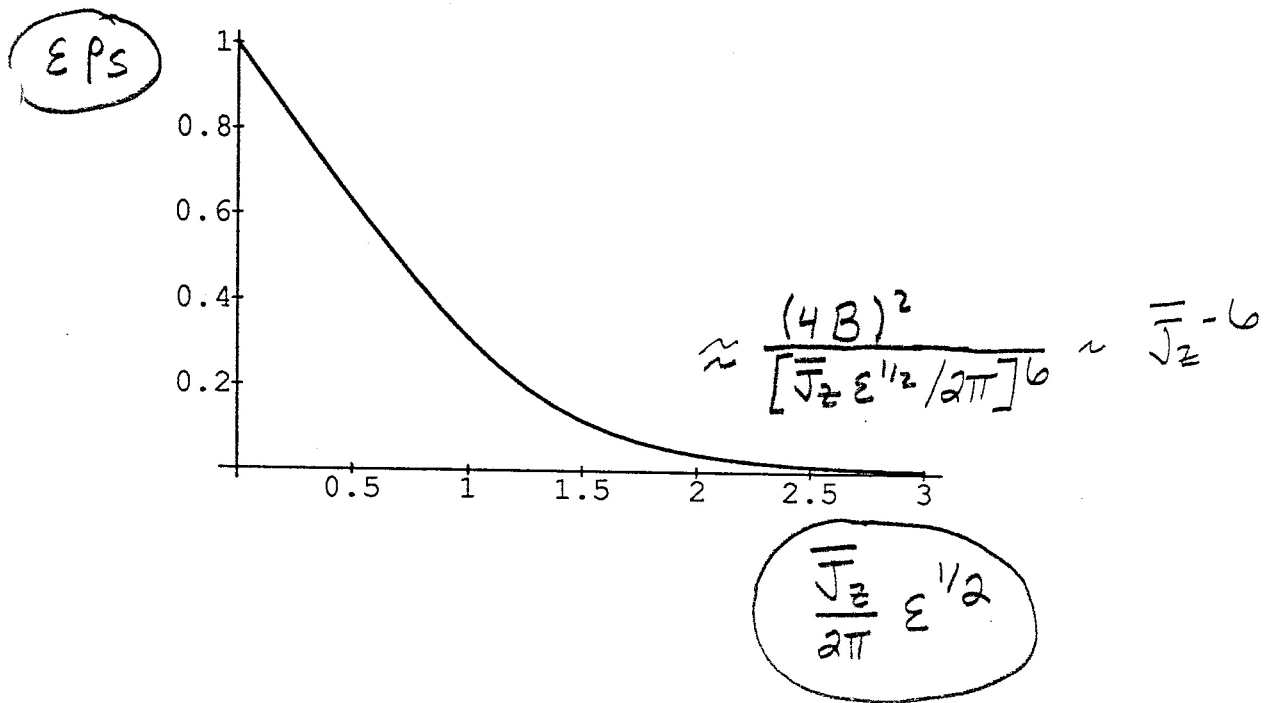
$$\frac{\bar{J}_z}{2\pi} \epsilon^{1/2} = 0.4$$

center point is

$$\bar{H} \epsilon^{-1} = -12.5$$



$$\frac{\bar{J}_2 \varepsilon^{1/2}}{2\pi} = 0.4$$

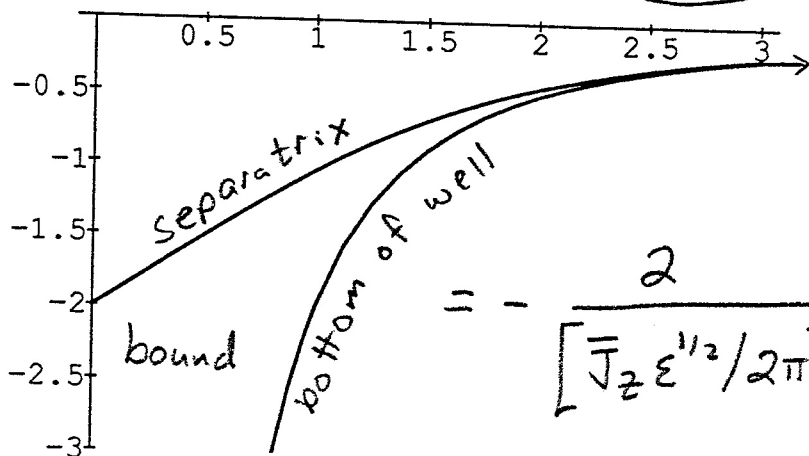


$$B \approx 0.53934(1)$$

$$B = \lim_{n \rightarrow 0} \left[\frac{n^{1/2} g'(n)}{2\pi} \right]$$

$$g(n) = 8\sqrt{2} \int_0^{\pi/2} \sqrt{\frac{(1-n) \sin^2 \beta + n}{(1-n) \sin^2 \beta + 2n}} \cos^2 \beta \, d\beta$$

$$\bar{H} \varepsilon^{-1}$$



$$\frac{\bar{J}_z}{2\pi} \varepsilon^{1/2}$$

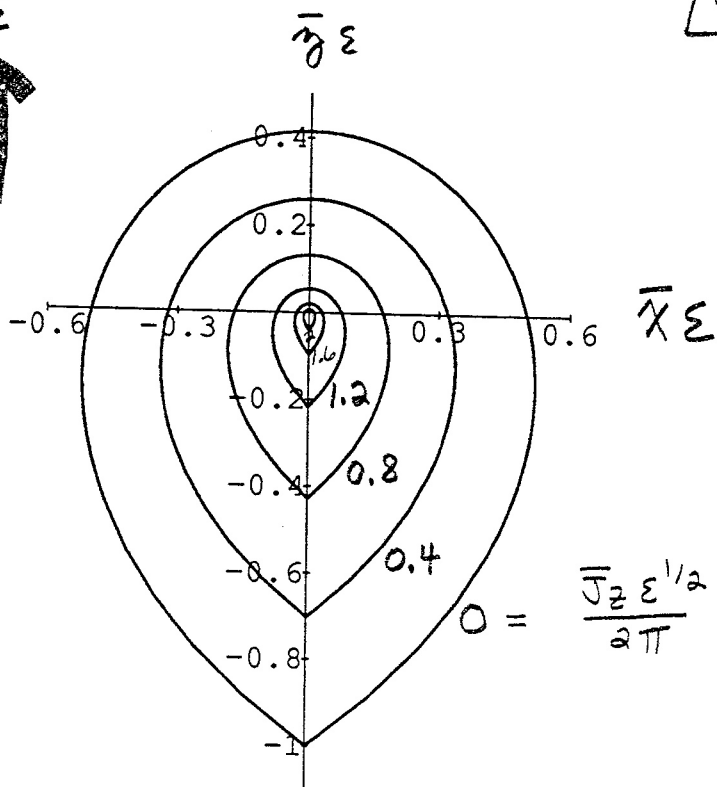
$$= - \frac{2}{\left[\bar{J}_z \varepsilon^{1/2} / 2\pi \right]^2} \sim \bar{J}_z^{-2}$$

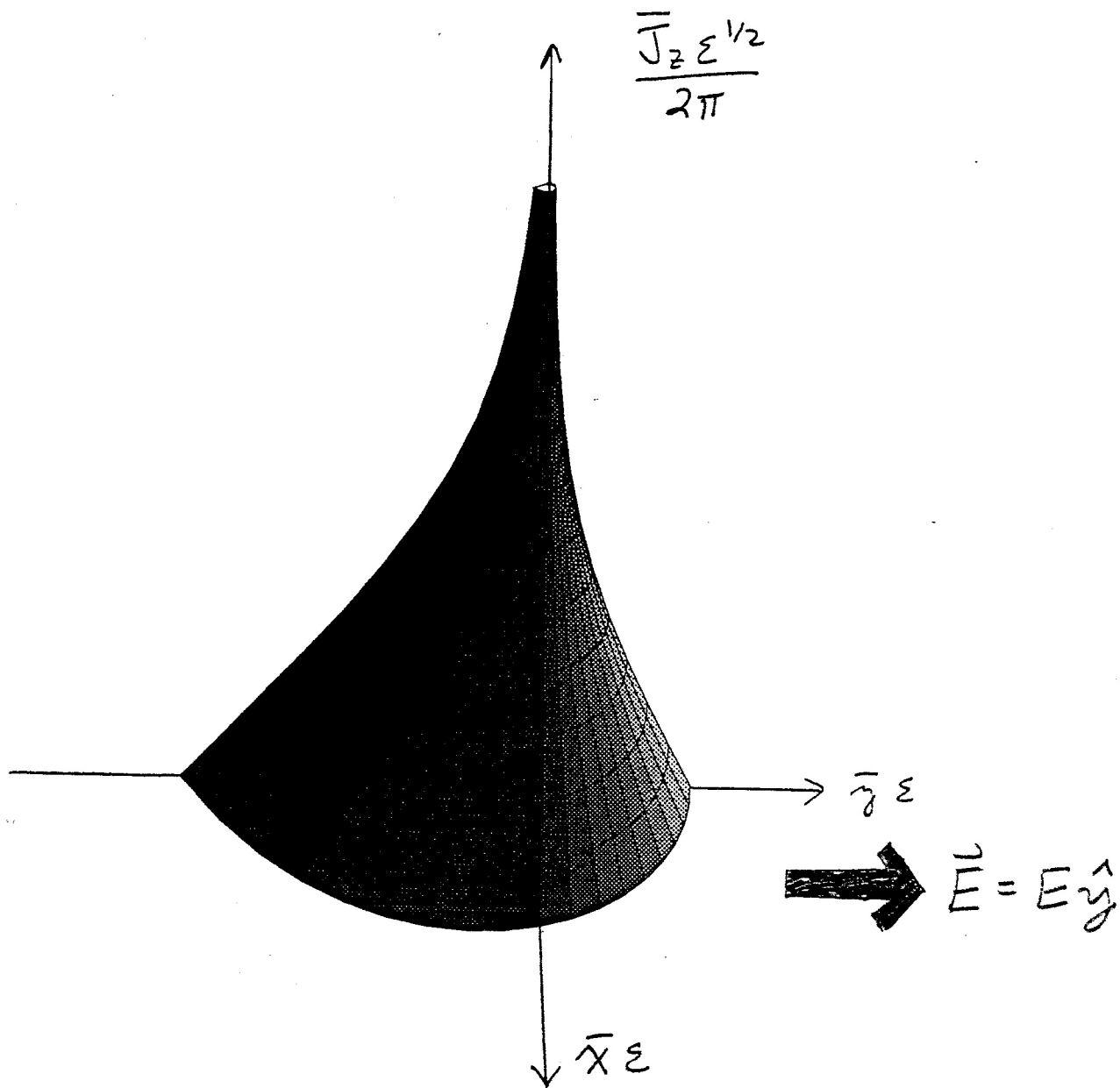
$$(\bar{H}_s - \bar{H}_b) \varepsilon^{-1} \xrightarrow{\frac{\bar{J}_z}{2\pi} \varepsilon^{1/2} \rightarrow \infty} \frac{(4B)^2}{\left[\bar{J}_z \varepsilon^{1/2} / 2\pi \right]^6} = \varepsilon \rho_s \sim \bar{J}_z^{-6}$$

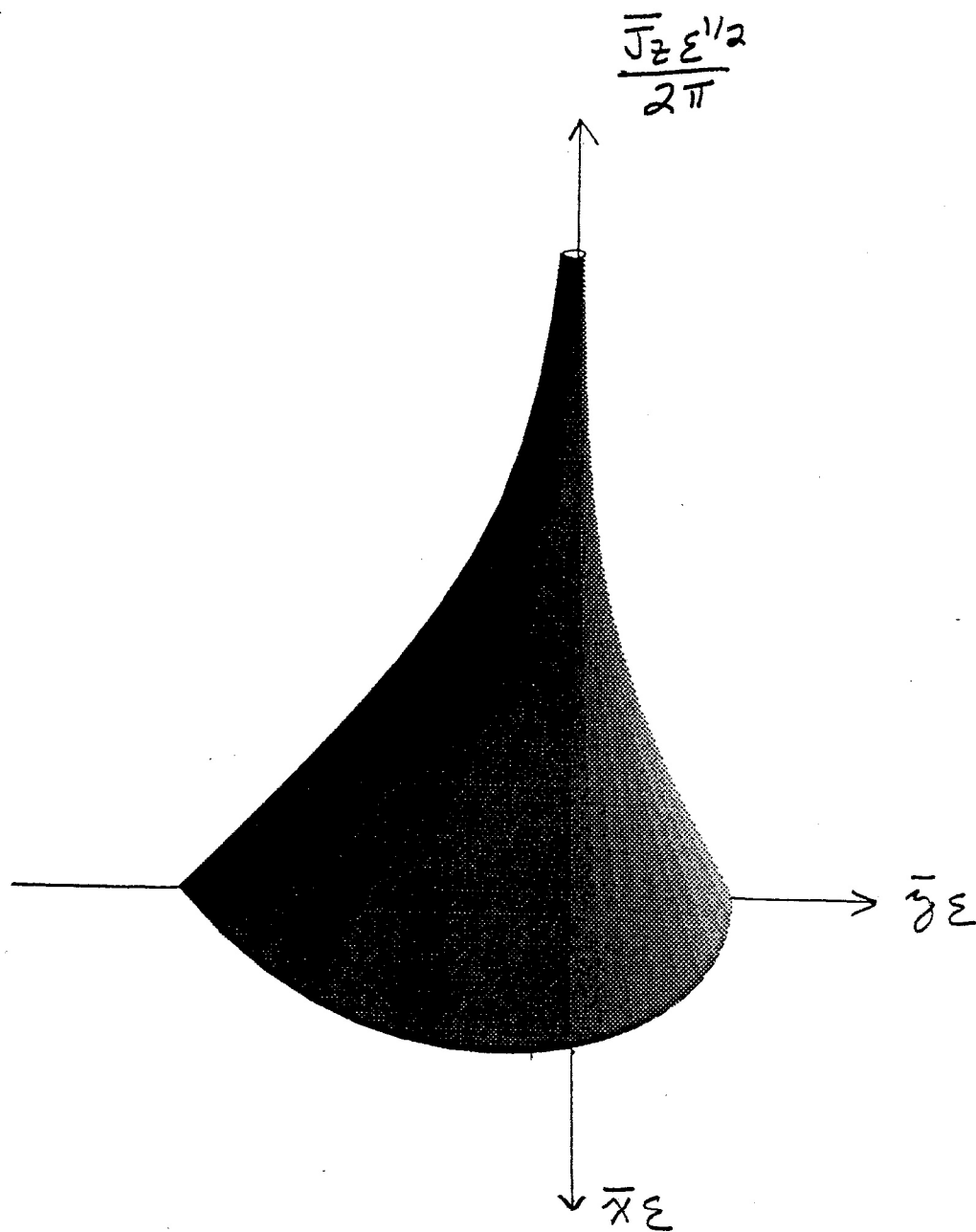
Separatrix III

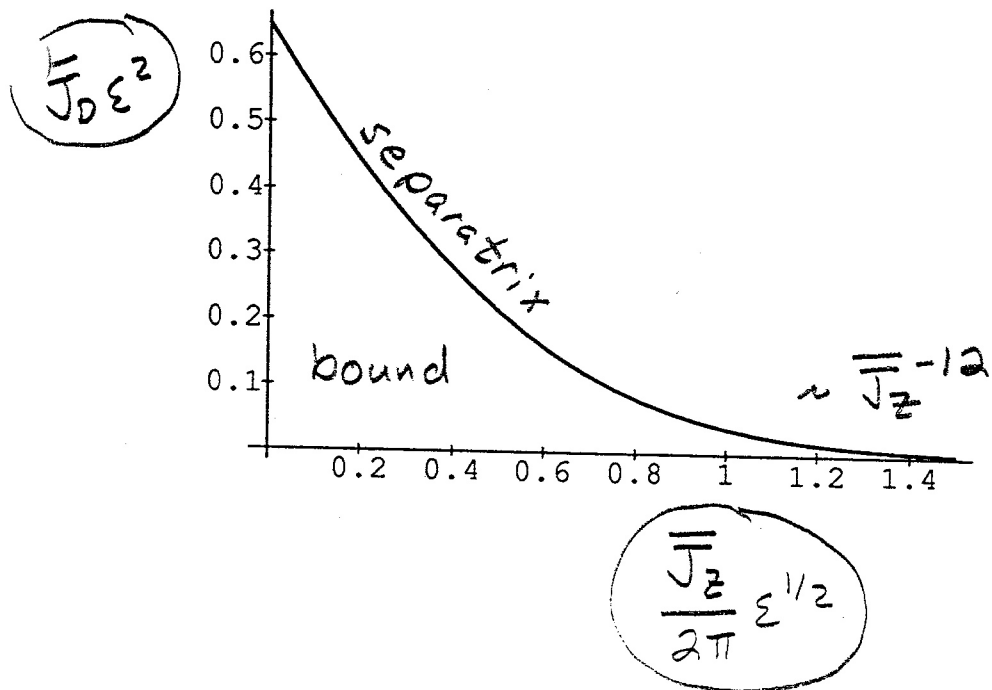


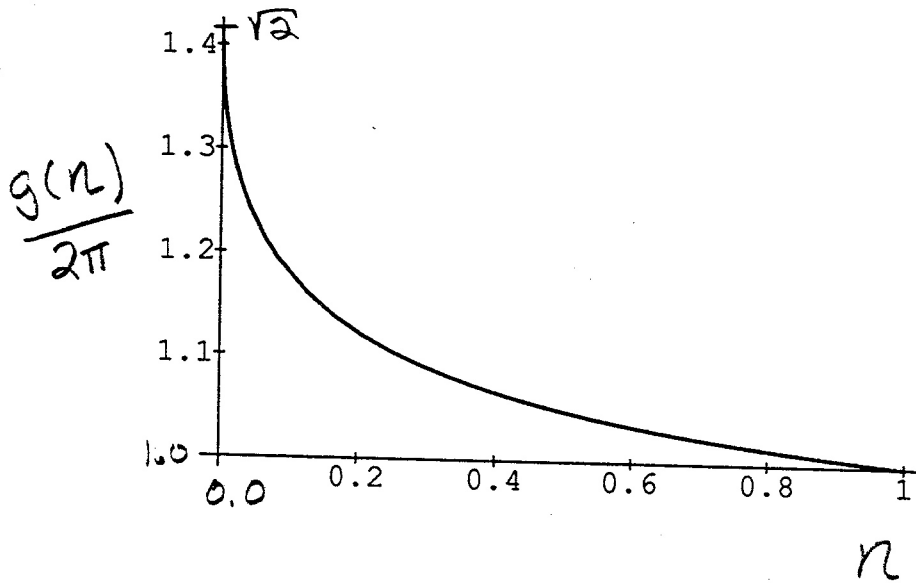
Separatrix Surface











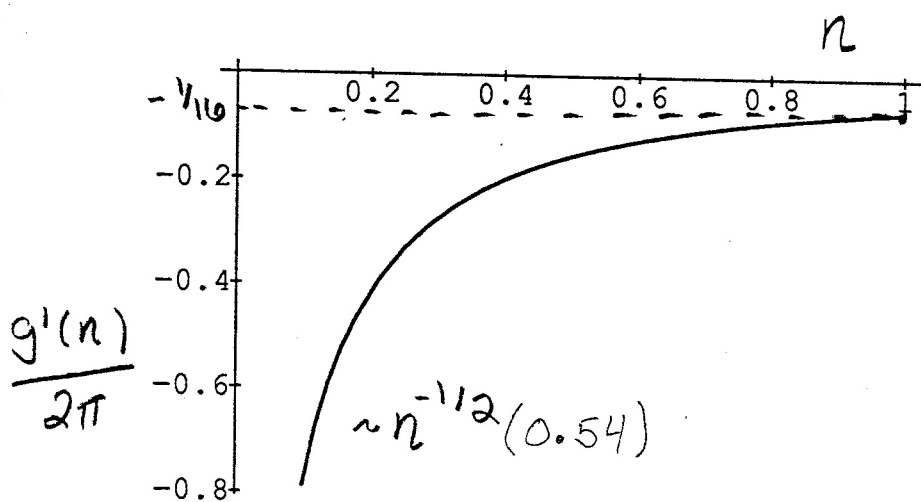
$$\bar{J}_z = \bar{H}_1^{-1/2} (1 - \bar{H}_1 \rho) g(\bar{H}_1 \rho)$$

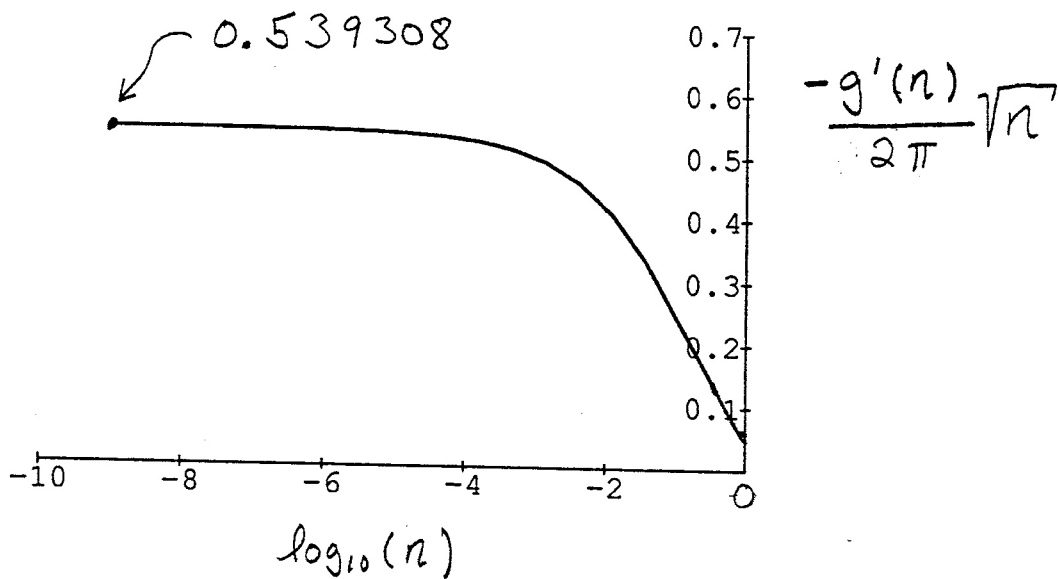
$$n \equiv \bar{H}_1 \rho$$

$$g(n) \equiv 8\sqrt{2} \int_0^{\pi/2} \sqrt{\frac{(1-n)\sin^2 \beta + n}{(1-n)\sin^2 \beta + 2n}} \cos^2 \beta \, d\beta$$

$$\bar{H} = \varepsilon^2 \bar{y} - \bar{H}_1(\bar{J}_z, \rho)$$

$$\text{or } \bar{H} \varepsilon^{-1} = (\varepsilon \bar{y}) - \varepsilon^{-1} \bar{H}_1(\bar{J}_z \varepsilon^{1/2}, \varepsilon \rho)$$



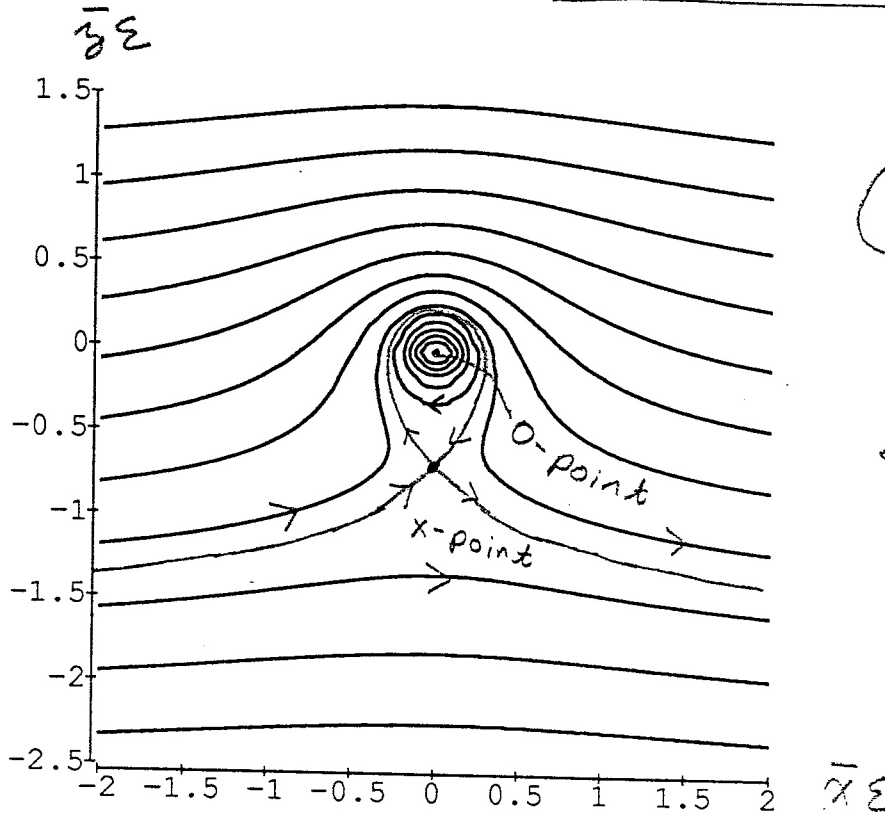


$$\begin{array}{rcl} \frac{n}{n_1 = 10^{-9}} & \frac{-\sqrt{n} g'(n)/2\pi}{0.539308} & \\ n_2 = 10^{-10} & 0.539338 & \end{array} \Bigg\rangle \Delta = 3 \times 10^{-5}$$

$$\sqrt{n_1} - \sqrt{n_2} = 2 \times 10^{-5}$$

$$g'(n) \xrightarrow{n \rightarrow 0} \left[\begin{array}{c} 0.53934(1) \\ 2\pi n^{-1/2} \end{array} \right]$$

$$g(n) = 1 \text{ Simplification}$$



$$\bar{J}_z \varepsilon^{1/2} = 0.4$$

$$\vec{E} = E \hat{y}$$

$$\bar{\rho}_s \varepsilon = 0.6893$$

$$\bar{H}_s \varepsilon^{-1} = -1.5895$$

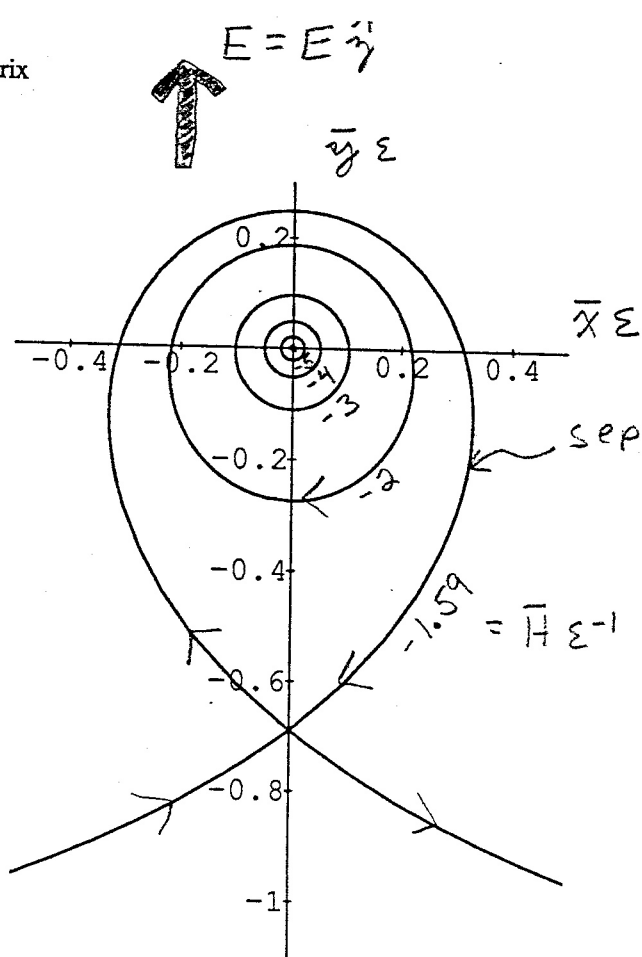
$$\vec{B} = B \hat{z}$$

$$\vec{v}_x = -\bar{v}_x \hat{x}$$

$$\varepsilon^2 = \frac{E}{e/\bar{b}^2}$$

$$= \left(\frac{\bar{v}_x}{\bar{v}_e} \right) \left(\frac{\bar{b}}{r_{ce}} \right)$$

Separatrix



$$\bar{J}_2 \varepsilon^{1/2} = 0.4$$

center point is

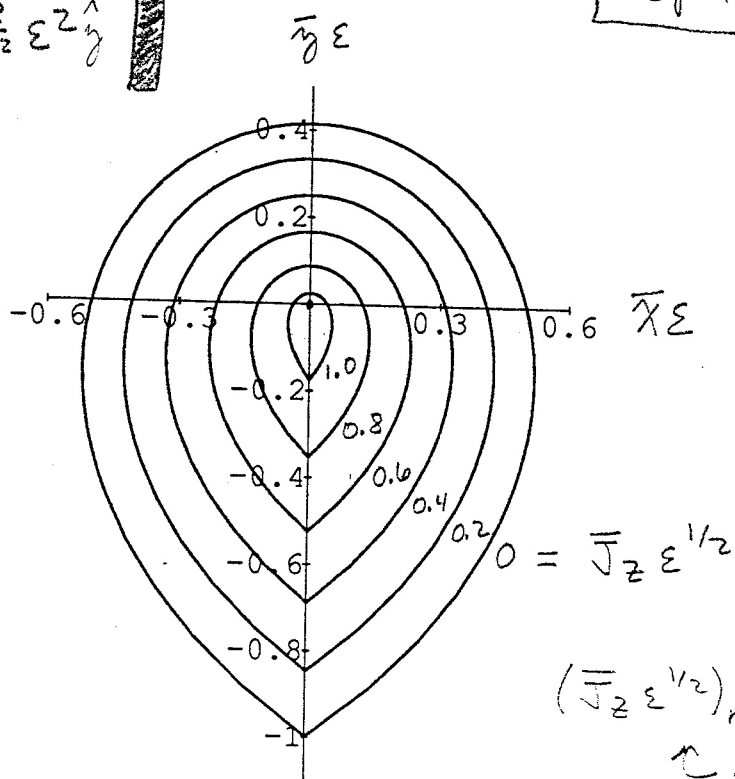
$$\bar{H} \varepsilon^{-1} = -6.25$$

Separatrix

$$\bar{E} = \frac{e}{\sigma_{1/2}} \varepsilon^2 \hat{y}$$

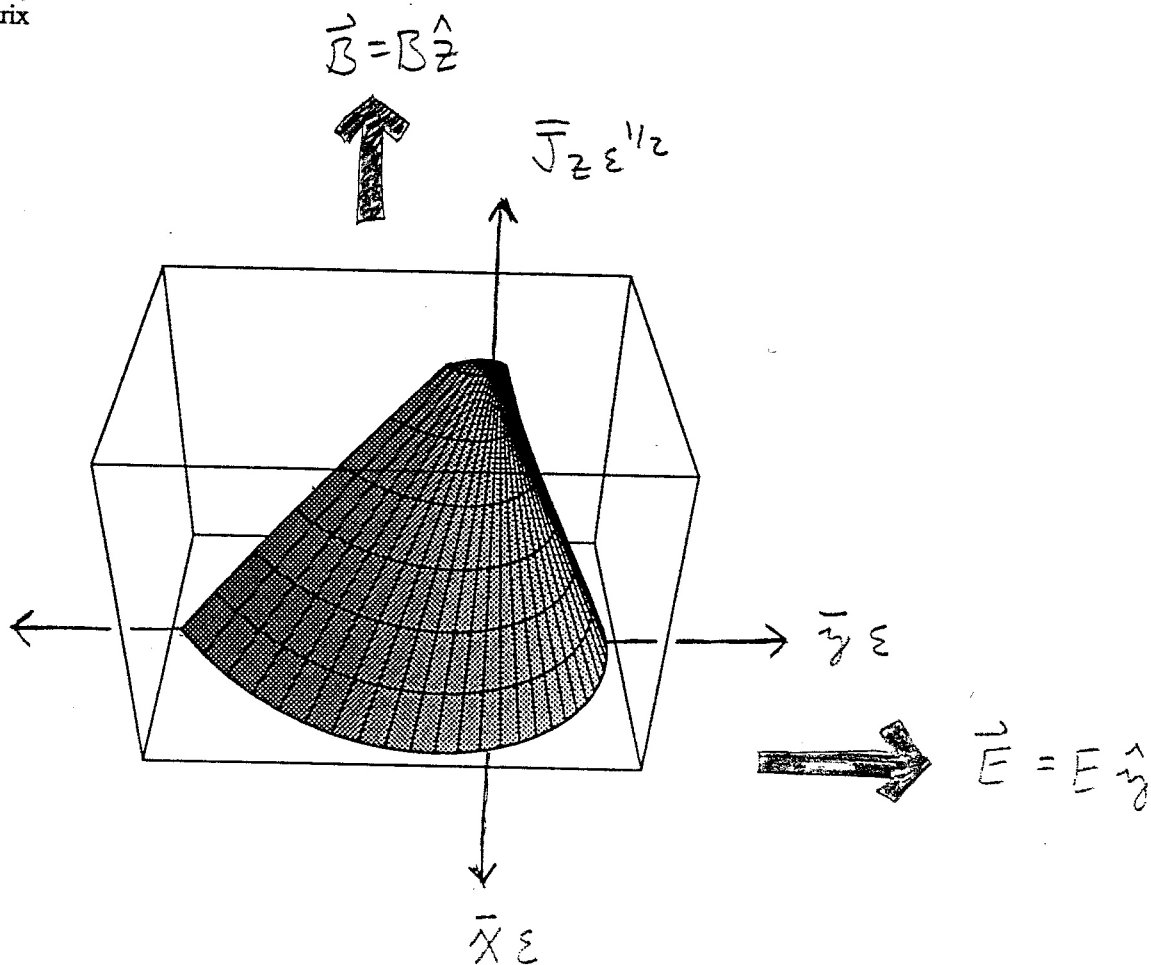


Separatrix Surface



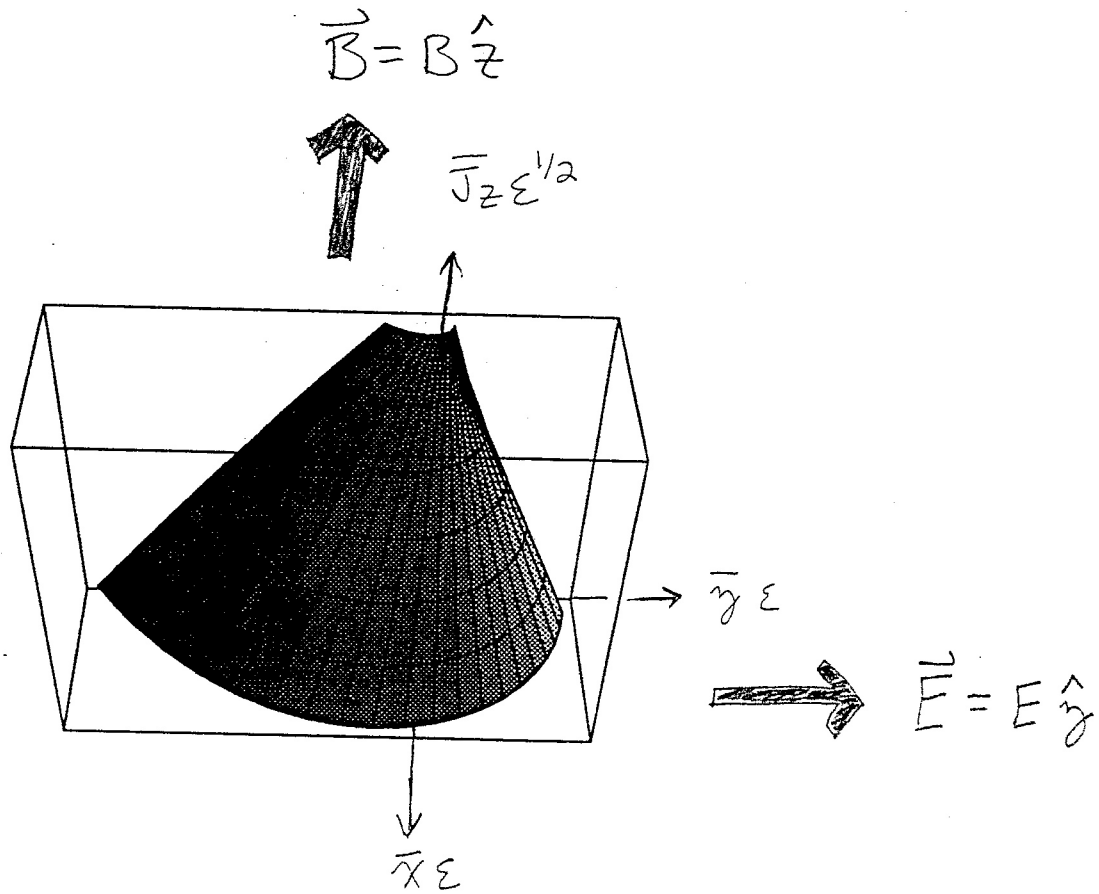
$$(\bar{y}_z \varepsilon^{1/2})_{\max} = 2^{1/4}$$

↑ separatrix is point at (0,0)



viewpoint = $(2, 0, \frac{3}{2})$
 Lightsource = $(4, 0, 4) + 0.9$
 Ambient = 0

Separatrix



$\{2, 0, 2\} = \text{viewpoint}$

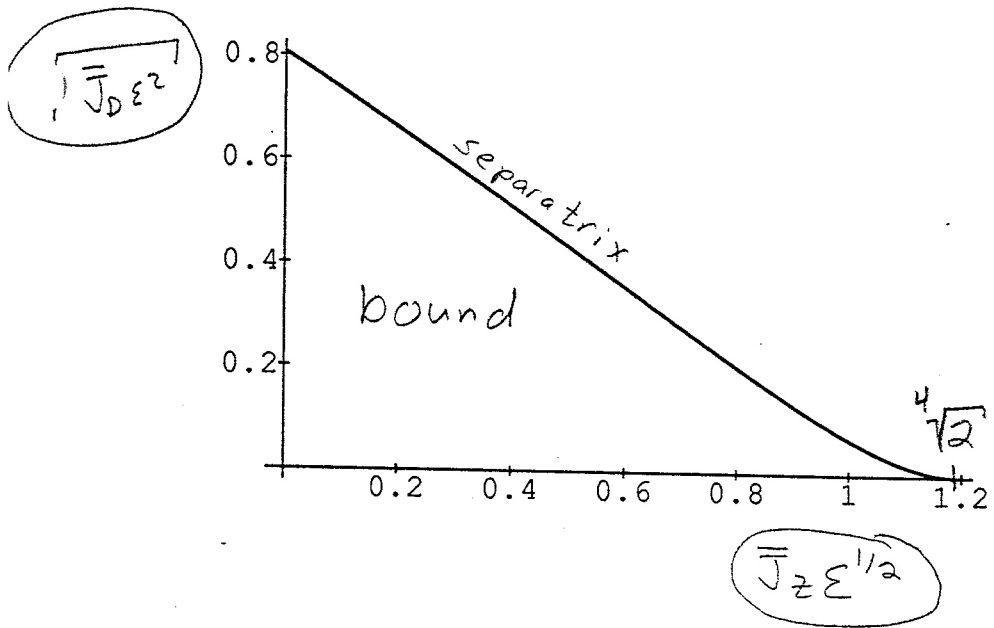
$\{1.5, 1.5, 0.5\} = \text{light}$

Ambient = GrayLevel [0.05]

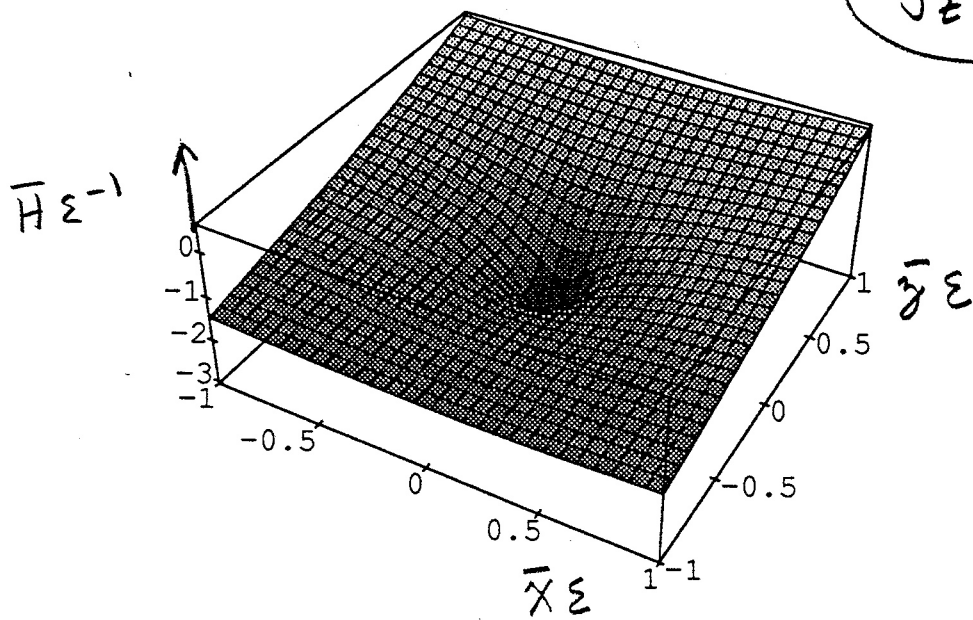
$$\frac{\pi}{20} = du$$

$$\frac{2^{1/4}}{7} = dv$$

Separatrix



H plot

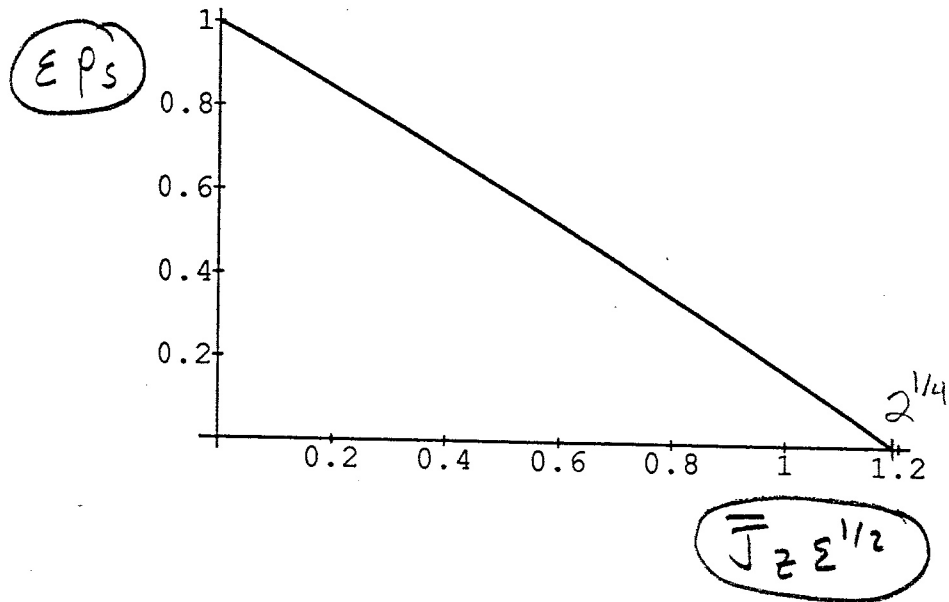


$$\bar{J}_2 \epsilon^{1/2} = \frac{1}{2}$$

$$\frac{1}{\epsilon}$$

$$n \frac{1}{n} = 1$$

Separatrix



Separatrix

$$\bar{H} \varepsilon^{-1}$$

$$\bar{J}_2 \varepsilon^{1/2}$$

