

Appendix C

Pull back of Variational Flux

$$\Phi = \int_S \rho(x) i_v \text{vol}^{2n}$$

where $S = \{x: E_0 = E_1(x)\}$

$$\omega = dp_i \wedge dq^i \quad \omega^n = \underbrace{\omega \wedge \dots \wedge \omega}_{n\text{-times}}$$

$$\omega^n = (-1)^{\frac{n(n+1)}{2}} n! \underbrace{dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n}_{\substack{\text{permutations of } dp_i \wedge dq_i \\ \sum_{i=1}^n i \text{ from ordering}}} \quad \text{pullback } \varphi$$

$$= (-1)^{n(n+1)/2} n! \text{vol}^{2n}$$

$$\Rightarrow \text{vol}^{2n} = \frac{1}{n!} (-1)^{n(n+1)/2} \omega^n = A(n) \omega^n$$

$$\Phi = A(n) \int_S \rho(x) i_v \omega^n$$

$$= A \int_S \rho n (i_v \omega) \wedge \omega^{n-1}$$

$$= A(n) \int_S \rho n (-dH) \wedge \omega^{n-1}$$

$$= [-n A(n)] \int_S \rho(x) dH \wedge \omega^{n-1}$$

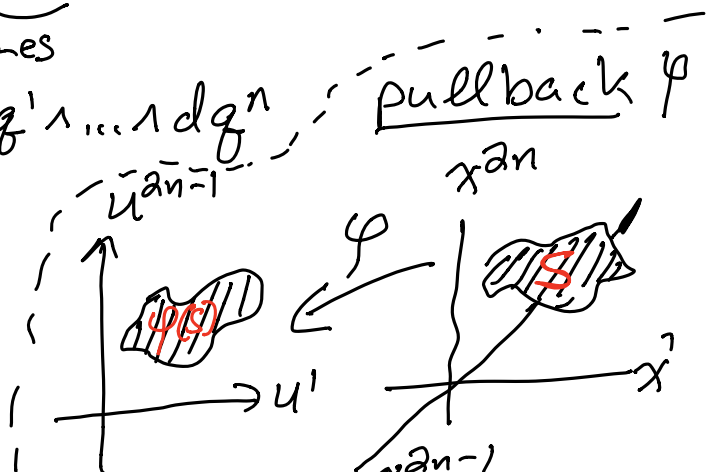
$$= \tilde{A} \int_S \rho(x) dH \wedge \omega^{n-1}$$

$$= \tilde{A} \int_{\varphi(S)} \rho(x) \varphi^*(dH \wedge \omega^{n-1})$$

$$= \tilde{A} \int_{\varphi(S)} \rho \varphi^*(dH \wedge \omega^{n-1}) \left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{2n-1}} \right) du^1 \wedge \dots \wedge du^{2n-1}$$

$$= \tilde{A} \int_{\varphi(S)} \rho dH \wedge \omega^{n-1} \left(\varphi^* \frac{\partial}{\partial u^1}, \dots, \varphi^* \frac{\partial}{\partial u^{2n-1}} \right) du^1 \wedge \dots \wedge du^{2n-1}$$

$$= \tilde{A} \int_{\varphi(S)} \rho dH \wedge \omega^{n-1} \left(\frac{\partial \vec{x}}{\partial u^1}, \dots, \frac{\partial \vec{x}}{\partial u^{2n-1}} \right) du^1 \wedge \dots \wedge du^{2n-1}$$



$$\varphi: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n-1}$$

$$x \rightarrow u$$

$$\text{let } (u^1, \dots, u^{2n-1})$$

$$= (p_2, \dots, p_n, q^1, \dots, q^n)$$

$$p_1 = p, (E_1, p_2, \dots, p_n, q^1, \dots, q^n)$$

$$\tilde{A} = -n A(n) = \frac{-1}{(n-1)!} (-1)^{n(n+1)/2}$$

$$i_v \omega^n = (i_v \omega) \wedge \omega^{n-1} + \omega \wedge (i_v \omega^{n-1})$$

$$= (i_v \omega) \wedge \omega^{n-1} + \omega \wedge (i_v \omega) \wedge \omega^{n-2}$$

$$+ \omega^2 \wedge i_v \omega^{n-2}$$

$$= 2(i_v \omega) \wedge \omega^{n-1} + \omega^2 \wedge i_v \omega^{n-2}$$

$$= 3(i_v \omega) \wedge \omega^{n-1} + \omega^3 \wedge i_v \omega^{n-3}$$

$$= \dots = (n-1)(i_v \omega) \wedge \omega^{n-1} + \omega^{n-1} \wedge i_v \omega^{n-n+1}$$

$$= n(i_v \omega) \wedge \omega^{n-1}$$

$$= \tilde{A} \int_{\varphi(S)} \rho \left(\frac{\partial H}{\partial x^i} dx^i \right) \wedge \overbrace{(dp_i \wedge dq^i) \wedge \dots \wedge (dp_i \wedge dq^i)}^{(n-1) \text{ times}} \\ \left(\frac{\partial x^i}{\partial u^1} \frac{\partial}{\partial x^i}, \dots, \frac{\partial x^i}{\partial u^{2n+1}} \frac{\partial}{\partial x^i} \right) \\ \text{choose } + \frac{\partial}{\partial p_i} \text{ for } \vec{N} \quad du^1 \wedge \dots \wedge du^{2n-1}$$

$$= \tilde{A} \int_{\mathbb{R}^{2n-1}} \rho (-)^n (n-1)! (dp_1 \wedge dq^1 \wedge \dots \wedge dp_i \wedge dq^i \wedge \dots \wedge (dp_i \wedge dq^i) \\ \wedge \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \right) \left(\frac{\partial p_1}{\partial q^1} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^1}, \dots, \frac{\partial p_1}{\partial q^n} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^n}, \right. \\ \left. \frac{\partial p_1}{\partial p_2} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \dots, \frac{\partial p_1}{\partial p_n} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_n} \right) dq^1 \wedge \dots \wedge dq^n dp_2 \wedge \dots \wedge dp_n$$

$$= (-)^n (n-1)! \tilde{A} \int_{\mathbb{R}^{2n-1}} \rho \left(\frac{\partial H}{\partial p_i} dp_1 \wedge dq^1 \wedge \dots \wedge \widehat{dq^i} \wedge \dots \wedge dp_n \wedge dq^n \right. \\ \left. + \frac{\partial H}{\partial q^i} dp_1 \wedge dq^1 \wedge \dots \wedge \widehat{dp_i} \wedge \dots \wedge dp_n \wedge dq^n \right)$$

$$\left(\frac{\partial p_1}{\partial q^1} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^1}, \dots, \frac{\partial p_1}{\partial q^n} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^n}, \frac{\partial p_1}{\partial p_2} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \dots, \frac{\partial p_1}{\partial p_n} \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_n} \right) \\ dq^1 \wedge \dots \wedge dq^n dp_2 \wedge \dots \wedge dp_n$$

$$= (-)^n (n-1)! \tilde{A} \int_{\mathbb{R}^{2n-1}} \rho \left(\frac{\partial H}{\partial p_i} \frac{\partial p_1}{\partial q^i} - \frac{\partial H}{\partial q^2} \frac{\partial p_1}{\partial p_2} - \dots - \frac{\partial H}{\partial q^n} \frac{\partial p_1}{\partial p_n} \right) (-)^{n(n-1)/2} \\ dq^1 \wedge \dots \wedge dq^n dp_2 \wedge \dots \wedge dp_n$$

Using  $E_0 = E_1(p_1, q_1, \dots, p_n, q_n)$

$$\Rightarrow p_1 = p_1(E_0, q^1, \dots, p_n, q^n)$$

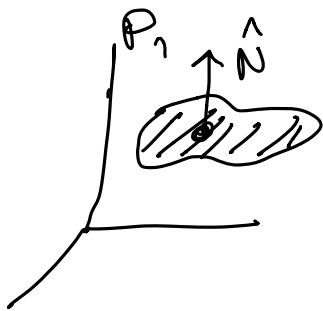
$$\frac{dE_0}{dq^i} = 0 = \frac{\partial E_1}{\partial p_1} \frac{\partial p_1}{\partial q^i} + \frac{\partial E_1}{\partial q^i} \Rightarrow \frac{\partial p_1}{\partial q^i} = - \frac{\partial E_1 / \partial q^i}{\partial E_1 / \partial p_1}$$

$$\frac{dE_0}{dp_i} = 0 \Rightarrow \frac{\partial p_1}{\partial p_i} = - \frac{\partial E_1 / \partial p_i}{\partial E_1 / \partial p_1}$$

$$= (-)^n (n-1)! (-)^{n(n-1)/2} \tilde{A} \int_{\mathbb{R}^{2n-1}} \rho (-) \frac{1}{\partial E_1 / \partial p_1} \left(\frac{\partial H}{\partial p_i} \frac{\partial E_1}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial E_1}{\partial p_i} \right)$$

$$\begin{aligned}
 & dq^1 \dots dq^n dp_2 \dots dp_n \\
 &= (-)^n (n-1)! (-)^{n(n-1)/2} \frac{-1}{(n-1)!} (-)^{n(n+1)/2} \int_{\mathbb{R}^{2n-1}} \rho \frac{\partial \rho_1}{\partial E_1} \left(\frac{\partial H}{\partial p_i} \frac{\partial E_1}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial E_1}{\partial p_i} \right) dq^1 \dots dq^n dp_2 \dots dp_n \\
 &= - \int_{\mathbb{R}^{2n-1}} \rho \{H, E_1\} \frac{\partial \rho_1}{\partial E_1} dq^1 \dots dq^n dp_2 \dots dp_n \\
 &= \int_{\mathbb{R}^{2n-1}} \rho \{E_1, H\} \frac{\partial \rho_1}{\partial E_1} dq^1 \dots dq^n dp_2 \dots dp_n
 \end{aligned}$$

$$\bar{\Phi} = \int_{\mathbb{R}^{2n-1}} \rho(p, q) \underbrace{\{E_1, H\}}_{\frac{dE_1}{dt}} \frac{\partial \rho_1}{\partial E_1} dq^1 \dots dq^n dp_2 \dots dp_n$$



where

$$\rho_1 = \rho_1(E_s, q^1, p_2, q^2, \dots, p_n, q^n)$$

or

$$E_s = E_1(p_1, q^1, \dots, p_n, q^n)$$

defines surface