

Appendix B

Hamiltonian for ion-electron Guiding Center dynamics

Start with full dynamics for ion and
GC dynamics for electron

$$\hat{H}_2 = \frac{\vec{p}_i^2 - \frac{e}{c} \vec{A}(\vec{r}_i)}{2m_i} + \frac{p_{ze}^2}{2m_e} - \frac{e^2}{\sqrt{(x_i - x_e)^2 + (y_i - \frac{p_{xe}}{m_e \Omega})^2 + (z_i - z_e)^2}}$$

use $\vec{A}(\vec{r}) = \frac{1}{2} \vec{B} \times \vec{r}$ with $\vec{B} = B \hat{z}$

make the canonical change of variables

$$\begin{aligned} & \left. \begin{aligned} z_{cm} &\equiv \frac{1}{M} (m_i z_i + m_e z_e) \\ z &\equiv z_i - z_e \\ q_x &\equiv x_i \\ q_y &\equiv y_i \\ q &\equiv \frac{p_{xe}}{m_e \Omega} - y_i \end{aligned} \right\} \\ & \left. \begin{aligned} p_{cm} &\equiv \frac{1}{M} (m_e p_{zi} - m_i p_{ze}) \\ p_z &\equiv \frac{1}{M} (m_e p_{zi} - m_i p_{ze}) \\ k_x &\equiv p_{ix} + p_{xe} - \frac{m_e \Omega}{2} y_i \\ k_y &\equiv p_{iy} + p_{ye} + \frac{m_e \Omega}{2} x_i \\ p &\equiv (x_i - x_e) m_e \Omega \end{aligned} \right\} \end{aligned}$$

constants

where $\Omega = \frac{eB}{m_e c}$ and $\vec{k} = m_e \Omega \hat{z} \times [(\vec{R}_{gc})_i - \vec{r}_e] = m_e \Omega \begin{pmatrix} -\underline{x} \\ \underline{y} \\ 0 \end{pmatrix}$

$$\underline{x} \equiv (R_{gc})_{ix} - x_e \quad \underline{y} \equiv (R_{gc})_{iy} - y_e$$

$$(\vec{R}_{gc})_i \equiv (\vec{r}_i)_1 + \frac{m_i c}{e} \frac{\vec{v}_i \times \vec{B}}{B^2}$$

the Hamiltonian can now be written as:

$$\hat{H}_2 = \frac{(k_x - m_e \Omega q)^2}{2m_i} + \frac{(k_y - p)^2}{2m_i} + \frac{p_{cm}^2}{2M} + \frac{p_z^2}{2\mu} - \frac{e^2}{\sqrt{q^2 + \left(\frac{p}{m_e \Omega}\right)^2 + z^2}}$$

where $M \equiv m_i + m_e \quad \frac{1}{\mu} = \frac{1}{m_i} + \frac{1}{m_e}$

Note that this transformation is canonical and q_x, q_y , and z_{cm} are cyclic. $\therefore k_x, k_y$ and P_{cm} are constants of the motion. Since the Hamiltonian is time independent, $\hat{H}_2$ is a fourth constant of the motion.

Now do the further change of variables:

$$\left\{ \begin{array}{ll} \bar{z}_{cm} = z_{cm} & \bar{P}_{cm} = P_{cm} \\ \bar{z} = z & \bar{p}_z = p_z \\ \psi' \equiv \tan^{-1} \left(\frac{q_y m_e \Omega}{p} \right) & \bar{p}_{\psi'} \equiv \frac{p^2 + (m_e \Omega q)^2}{2 m_e \Omega} \\ R \equiv \sqrt{k_x^2 + k_y^2} / m_e \Omega & \bar{p}_R \equiv \frac{-m_e \Omega}{\sqrt{k_x^2 + k_y^2}} (k_x q_x + k_y q_y) \\ \theta' \equiv \tan^{-1} \left(\frac{k_y}{k_x} \right) & \bar{p}_{\theta'} \equiv k_y q_x - k_x q_y \end{array} \right.$$

$$\text{where } q = \sqrt{\frac{2 \bar{p}_{\psi'}}{m_e \Omega}} \sin \psi', \quad p = \sqrt{2 \bar{p}_{\psi'} m_e \Omega} \cos \psi'$$

$$k_x = m_e \Omega R \cos \theta', \quad k_y = m_e \Omega R \sin \theta'$$

this leads to

$$H_2 \equiv \hat{H}_2 - \frac{P_{cm}^2}{2M} = \Omega_i \left[(m_e \Omega) \frac{R^2}{2} - R \sqrt{2 m_e \Omega \bar{p}_{\psi'}} \sin(\psi' - \theta') + \bar{p}_{\psi'} \right] + \frac{p_z^2}{2u} - \frac{c^2}{\sqrt{\frac{2 \bar{p}_{\psi'}}{m_e \Omega} + z^2}}$$

Now shift the two angles

$$\psi \equiv \psi' + \theta' - \frac{\pi}{2} \quad p_{\psi} = p_{\psi'}$$

$$\theta \equiv \theta' - \frac{\pi}{2} \quad p_{\theta} = p_{\theta'} - p_{\psi'}$$

$\vec{R}_{gc} = R(\sin\theta', -\cos\theta', 0)$
 $\vec{p} = \sqrt{\frac{2\rho\psi'}{me\Omega}} (\cos\psi', -\sin\psi', 0)$

$\Rightarrow$

$\vec{R} = \vec{R}_{gc} = (X, Y, 0)$
 $= R(\cos\theta, \sin\theta, 0)$
 $\vec{p} = (x_i - x_e, y_i - y_e, 0)$
 $= \sqrt{\frac{2\rho\psi}{me\Omega}} (\cos(\theta+\psi), \sin(\theta+\psi), 0)$

to give the Hamiltonian

$$H_2 = \Omega_i \left[me\Omega \frac{R^2}{2} - R \sqrt{2me\Omega \rho\psi} \cos\psi + \rho\psi \right] + \frac{\rho_z^2}{2\eta} - \frac{e^2}{\sqrt{\frac{2\rho\psi}{me\Omega} + z^2}}$$

of the variables $(R, \theta, z_{cm}, \psi, z; p_R, p_\theta, p_{cm}, p_\psi, p_z)$ where θ, z_{cm}, p_R and t are cyclic giving R, p_{cm}, p_θ as H_2 as constants of the motion. In fact the Hamiltonian is on phase space $(\psi, z; p_\psi, p_z)$ with R as a constant

$$H_2(R; \underbrace{\psi, \rho\psi}_{\text{ExB drift dynamics}}, z, p_z) = \Omega_i \left[me\Omega \frac{R^2}{2} - R \sqrt{2me\Omega \rho\psi} \cos\psi + \rho\psi \right] + \frac{\rho_z^2}{2\eta} - \frac{e^2}{\sqrt{\frac{2\rho\psi}{me\Omega} + z^2}}$$

Now to contact with the drifting ion and the fixed ion approximations consider the Hamiltonian in the following form and the two approximations

$$H_p(\vec{p}, \vec{R}; z, p_z) \equiv \frac{H_2}{me\Omega} = \frac{\Omega_i}{2} (\vec{R} - \vec{p})^2 + \frac{1}{me\Omega} \left(\frac{p_z^2}{2\eta} - \frac{e^2}{\sqrt{\rho^2 + z^2}} \right)$$

Guiding Center Drift Atom (drifting ion)

assume that $m_i \rightarrow \infty$ and $\vec{v}_i$ finite

$$\Rightarrow \begin{cases} \Omega_i \vec{R} \rightarrow \vec{v}_i \\ \Omega_i \rightarrow 0 \\ R \rightarrow \infty \end{cases}$$

$$\Rightarrow H_p \approx \underbrace{(\Omega_i \vec{R}) \cdot \vec{p}}_{= \vec{E}_0 / m_e \Omega_e} + \frac{1}{m_e \Omega} \left(\frac{p_z^2}{2\mu} - \frac{e^2}{\sqrt{\rho^2 + z^2}} \right)$$
$$\vec{E}_0 = (m_e \Omega_e) (\vec{v}_i \times \hat{z}) = \frac{e}{c} \vec{v}_i \times \vec{B}$$

That is it is the motion of an electron that is bouncing back and forth in the z direction due to the electrostatic force and $E \times B$ drifting in the direction $\perp$ to the magnetic field due to the constant $\vec{E}_0 \perp$ to $\vec{B}$ and the efield due to the electrostatic interaction.

Guiding Center Atom (stationary ion)

assume that $m_i \rightarrow \infty$ and $\vec{v}_i \rightarrow 0$

$$\Rightarrow H_p \approx \frac{1}{m_e \Omega} \left(\frac{p_z^2}{2\mu} - \frac{e^2}{\sqrt{\rho^2 + z^2}} \right) \quad \psi \text{ cyclic} \Rightarrow \begin{cases} p_\psi = \text{const} \\ \rho = \text{const} \end{cases}$$

That is it is the motion of an electron that is bouncing back and forth in the z direction due to the electrostatic force and $E \times B$ drifting at a constant $\perp$ radius due to the efield from the electrostatic interaction.

Finally for $v_{ce} \ll b$, so that the z motion is much faster than the $E \times B$ drift motion $\omega_z \gg \omega_{E \times B}$ we change coordinates:

$$(z, \rho_z, \psi, \rho_\psi) \rightarrow (\psi_z, J_z, \bar{\psi}, \bar{\rho}_\psi)$$

generated by $F(\bar{\psi}, \rho_\psi, z, J_z) = \bar{\psi} \rho_\psi + W(z, J_z, \rho_\psi)$

$$\rho_z = \frac{\partial F}{\partial z} = \frac{\partial W}{\partial z} \quad (\text{Eq. 1})$$

$$\psi_z = \frac{\partial F}{\partial J_z} = \frac{\partial W}{\partial J_z} \quad (\text{Eq. 2})$$

$$\psi = \frac{\partial F}{\partial p_\psi} = \bar{\psi} + \frac{\partial W}{\partial p_\psi} \quad (\text{Eq. 3})$$

$$\bar{p}_\psi = \frac{\partial F}{\partial \bar{\psi}} = p_\psi \quad (\text{Eq. 4})$$

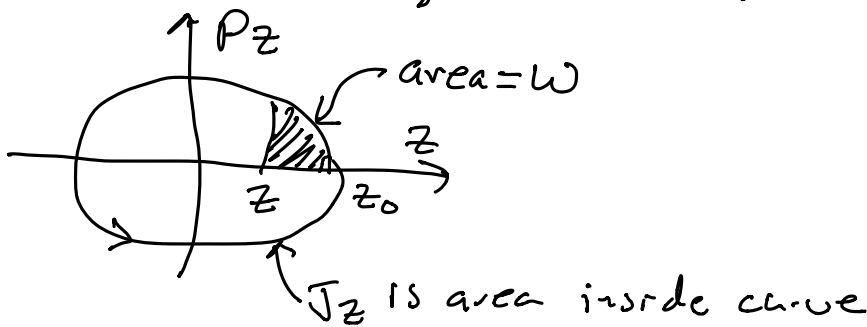
$$W(z, J_z, p_\psi) = \int_{z_0}^z \rho_z(E(J_z, p_\psi), p_\psi, z') dz'$$

by Eq. 1

where

$$E \equiv \frac{p_z^2}{2m} - \frac{e^2}{\sqrt{\frac{2p_\psi}{m_e \Omega} + z^2}} \Rightarrow \rho_z(E, p_\psi, z)$$

$$J_z \equiv \oint \rho_z(E, p_\psi, z) dz \Rightarrow J_z(E, p_\psi) \Rightarrow E(J_z, p_\psi)$$



by Eq. 2

$$\begin{aligned} \psi_z = \frac{\partial W}{\partial J_z} &= \int_{z_0}^z \frac{\partial \rho_z}{\partial J_z} dz' = \int_{z_0}^z \frac{\partial \rho_z}{\partial E} \frac{\partial E}{\partial J_z} dz' \\ &= \frac{\int_{z_0}^z \frac{dz'}{\psi_z}}{\oint \frac{dz'}{\psi_z}} = \frac{t}{T} = \frac{\omega_z t}{2\pi} \end{aligned}$$

by Eq. 3

$$\begin{aligned} \psi - \bar{\psi} &= \frac{\partial W}{\partial p_\psi} = \int_{z_0}^z \frac{\partial \rho_z}{\partial p_\psi} dz' \\ &= \left(\frac{r_{ce}}{b} \right) \int_{z_0/b}^{z/b} \frac{\partial (\rho_z / m_e \bar{\psi})}{\partial (\rho_\psi / m_e \Omega b^2)} \frac{dz'}{b} \end{aligned}$$

$\mathcal{O}(1)$

$$\text{So } \psi = \bar{\psi} + \underbrace{\left(\frac{r_{ce}}{b}\right) f(\psi_z, \bar{\psi}_z, \bar{\rho}_\psi)}_{\mathcal{O}(1)}$$

by Eq. 4

$$\rho_\psi = \bar{\rho}_\psi$$

$$\frac{r_{ce}}{b} \sim \frac{\omega_{ExB}}{\omega_z} \sim \frac{\omega_\psi}{\omega_z}$$

So

$$\underbrace{\frac{r_{ce}}{b}}_{\mathcal{O}(1)} \omega_z$$

$$H_2(R; \psi_z, \bar{\psi}_z, \bar{\psi}, \bar{\rho}_\psi) = \Omega_i \left[m_e \Omega_e \frac{R^2}{2} - R \sqrt{2 m_e \Omega_e \bar{\rho}_\psi} \cos \left[\bar{\psi} + \frac{\omega_\psi}{\omega_z} f(\psi_z, \bar{\psi}_z, \bar{\rho}_\psi) + \bar{\rho}_\psi \right] \right] + H_1(\bar{\psi}_z, \bar{\rho}_\psi)$$

Here we have removed the ψ_z dependence from H_1 and put it as a term of order r_{ce}/b . Another way of looking at this is that H_1 has been averaged over z .

Now let's look at the character of the orbits of this Hamiltonian, neglecting the ω_ψ/ω_z term, which is an integral Hamiltonian in the drift coordinate, $\bar{\psi}$.

$$H_2(R, \bar{\psi}_z; \bar{\psi}, \bar{\rho}_\psi) = \Omega_i \left[m_e \Omega_e \frac{R^2}{2} - R \sqrt{2 m_e \Omega_e \bar{\rho}_\psi} \cos \bar{\psi} + \bar{\rho}_\psi \right] + H_1(\bar{\psi}_z, \bar{\rho}_\psi)$$

Let's write this as (divide by $\Omega_i m_e \Omega_e$)

$$H_p(\vec{R}, \bar{\psi}_z; \vec{\rho}) = \frac{1}{2} (\vec{\rho} - \vec{R})^2 + a \bar{H}_1(\bar{\psi}_z, \rho)$$

$$\rho^2 = \frac{2 \rho_\psi}{m_e \Omega_e}$$

$$\vec{\rho} \cdot \vec{R} = \rho R \cos \bar{\psi}$$

where

$$\frac{1}{a} \xrightarrow[\rho^2 \text{ term}]{m_i \rightarrow \infty} 0$$

$\vec{\rho}, \vec{R}$ normed by b

$$\vec{\rho} \equiv (\bar{x}, \bar{y}, 0)$$

H_1 normed by $k_B T$

$\bar{\psi}_z$ normed by $m v_e b$

$$\frac{1}{a} |\vec{R}| = \frac{v_i}{v_e} \frac{b}{r_{ce}} = \frac{v_i/b}{\omega_{ExB}}$$

$$\vec{\rho} \equiv (\bar{x}, \bar{y}, 0)$$

$$a \equiv \left(\frac{r_{ce}}{b}\right)^2 \frac{m_i}{m_e} = \frac{\left(\frac{r_{ce}}{b}\right) \left(\frac{v_e}{b}\right)}{\Omega_i} = \frac{\omega_{ExB}}{\Omega_i}$$

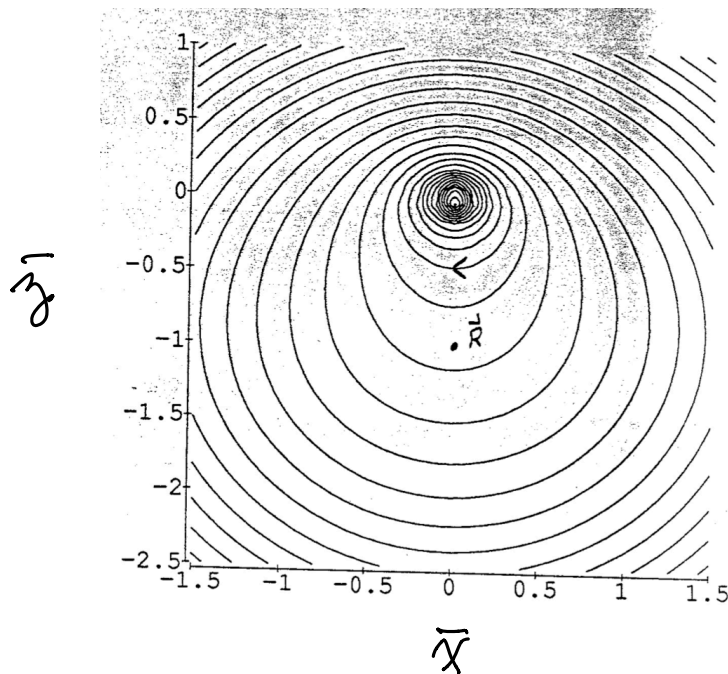
$$\frac{1}{a} |\vec{R}| \xrightarrow[\vec{\rho} \cdot \vec{R} \text{ term}]{v_i \rightarrow 0} 0$$

if $v_{i\perp}$ finite $R \rightarrow \infty$

Details can be found at

http://www.qitech.biz/papers_refereed/phys_plasmas_guiding_center_04.pdf

There are two characteristic phase diagrams for this Hamiltonian one with one o-point for $\frac{1}{a}, R/a \lesssim 1$ where the ion cyclotron motion is a minor perturbation to the $E \times B$ drift motion, yielding Guiding Center Atom.



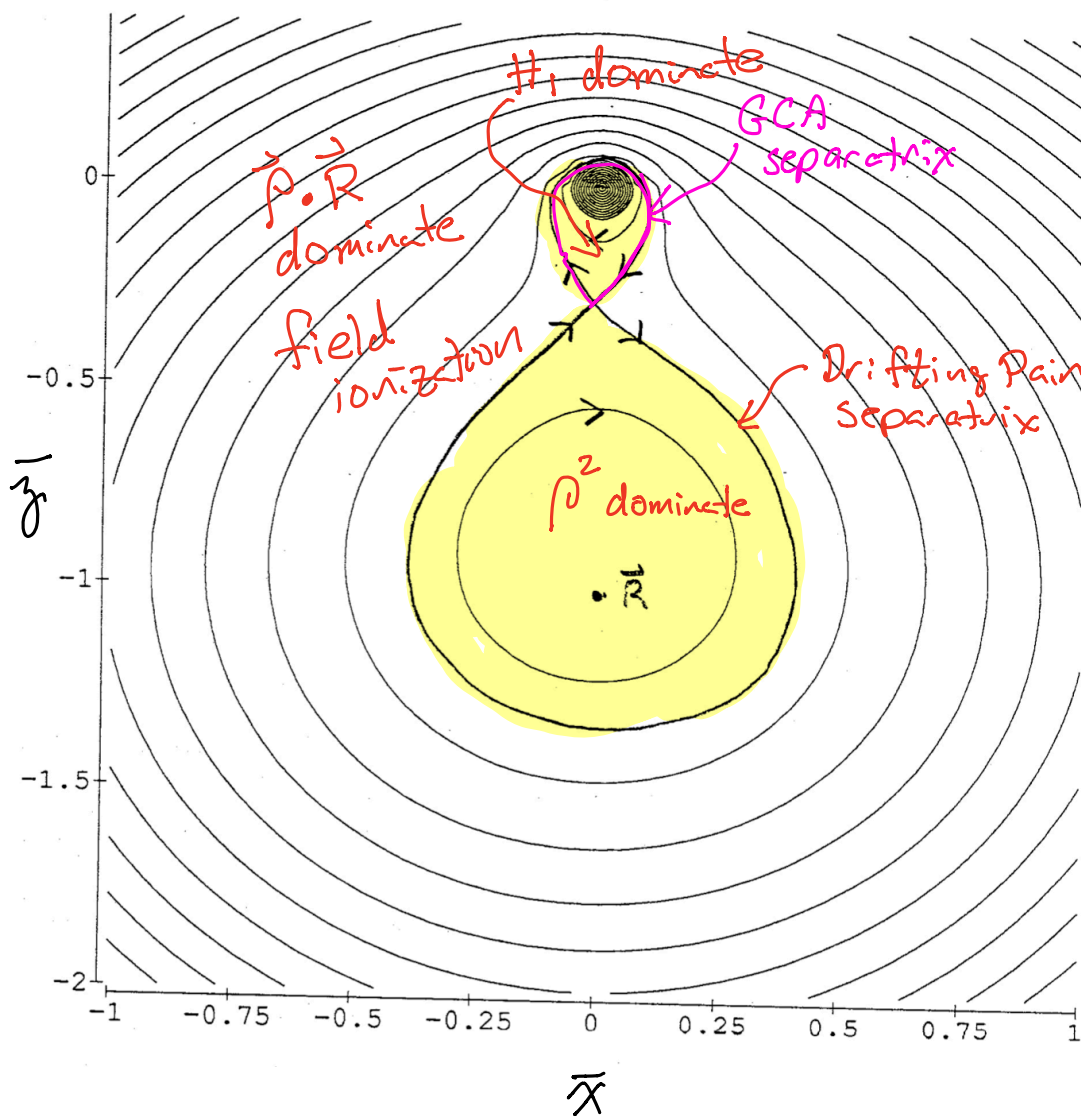
$$\vec{R} = -\hat{y}$$

$$\frac{J_z}{2\pi} = 0.4$$

$$a = 1$$

one o-point

And the other with two o-points and one x-point for $\frac{1}{a}$ or $R/a \gg 1$ where there are two islands of stability, one where the electron $E \times B$ drifts around the ion, and another where the ion and electron maintain their separation both undergoing their separate cyclotron orbits. This second island of stability we call a drifting pair. The guiding center atom (the first island of stability) will be "field ionized" if it is transported by 3-body collisions outside the separatrix.



$$\vec{R} = -\hat{y}$$

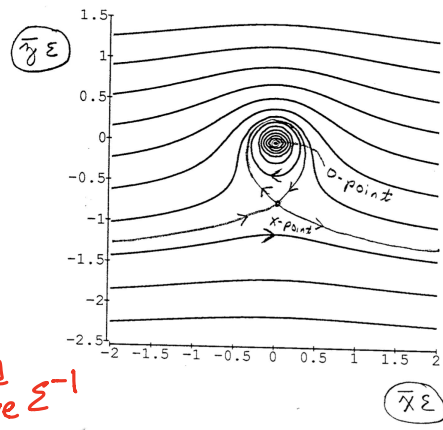
$$\frac{\sqrt{J_z}}{2\pi} = 0.4$$

$$a = 0.15$$

For the case of an infinitely massive drifting ion one loses the second O-point and the drifting pair as the ion cyclotron radius goes to infinity

$$1/a \rightarrow 0 \quad (m_i \rightarrow \infty)$$

$$R/a \gg 1 \quad v_i \text{ finite}$$



$$\frac{\sqrt{J_z}}{2\pi} \varepsilon^{1/2} = 0.4$$

$$\vec{E} = E \hat{y}$$

$$\rho_x \text{ scaled by } \frac{b}{r_{ce}} \varepsilon^{-1}$$

$$H \text{ scaled by } \varepsilon$$

$$\rho_x, \rho, x, y \text{ scaled by } \varepsilon^{-1}$$

$$\rho_z \text{ scaled by } \varepsilon^{1/2}$$

$$J_z \text{ scaled by } \varepsilon^{-1/2}$$

$$w_z, 1/x_z \text{ scaled by } \varepsilon^{3/2}$$

$$w_0, 1/x_0 \text{ scaled by } \frac{r_{ce}}{b} \varepsilon^3$$

$$\rho_s \varepsilon = 0.702819$$

$$\bar{H}_s \varepsilon^{-1} = -1.60216$$

$$\vec{B} = B \hat{z}$$

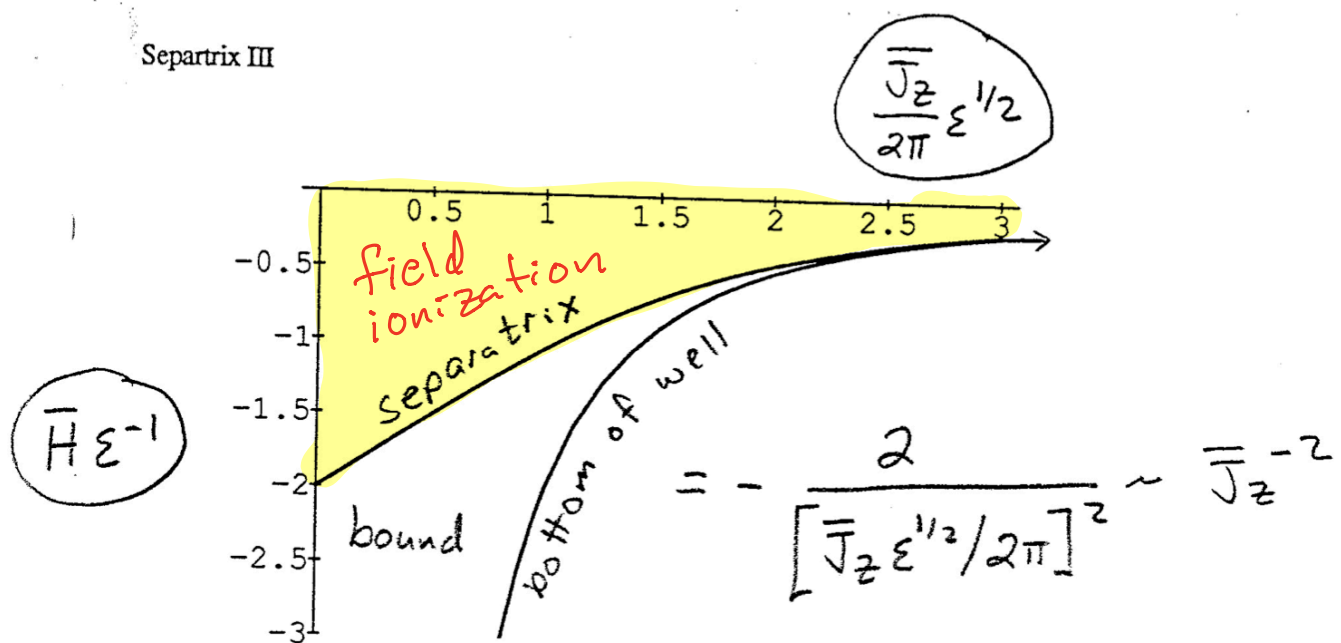
$$\vec{v}_\perp = -\vec{v}_\perp \hat{x}$$

$$\varepsilon^2 = \frac{E}{e/b^2}$$

$$\text{Self similar solution} \Rightarrow \left(\frac{\vec{v}_\perp}{\vec{v}_e} \right) \left(\frac{b}{r_{ce}} \right) = |\vec{R}| \frac{1}{a}$$

phase_space.mov

This leads to the following area where the atom will be field ionized.



So there are two parameters which determine whether or not there is field ionization

The first is

$$a = \left(\frac{m_i}{m_e} \right) \left(\frac{r_{ce}}{b} \right)^2 \ll 1 \quad \text{upper limit on } m_i$$

and the second is

$$\epsilon^2 = \left(\frac{v_i}{v_e} \right) \left(\frac{b}{r_{ce}} \right) \gg 1 \quad \text{lower limit on } v_i$$

$$\text{or } \frac{T_i}{T_e} \gg a^2 \quad \text{lower limit on } T_i$$

note $a \ll 1$

for field ionization.

A better way of understanding the character of the atoms is to separate the Hamiltonian into three terms

$$H_p = \underbrace{\Omega_i}_{\omega_D} \rho^2 + v_i \rho + b^2 \left(\frac{r_{ce}}{b} \right) \left(\frac{v_e}{b} \right) \bar{H}_1(\bar{J}_z, \rho/b)$$

Now since $\bar{H}_i \sim (\rho/b)^{-1}$, the frequency of each of the terms are:

Ω_i
①
Drifting Pair
(DP)



$\frac{v_i}{\rho}$
②
Guiding Center
Drifting Atom
(GCDA)
field ionization



$$\omega_D = \left(\frac{b}{\rho}\right)^3 \left(\frac{\bar{v}}{b}\right) \left(\frac{r_{ce}}{b}\right)$$

③
Guiding center
Atom
(GCA)



$$\frac{③}{①} = \frac{\omega_D}{\Omega_i} \gg 1 \Rightarrow \bar{\rho} = \frac{\rho}{b} \ll \bar{r}_{s1} \equiv \left[\left(\frac{m_e}{m_i} \right)^{1/2} \frac{b}{r_{ce}} \right]^{-2/3}$$

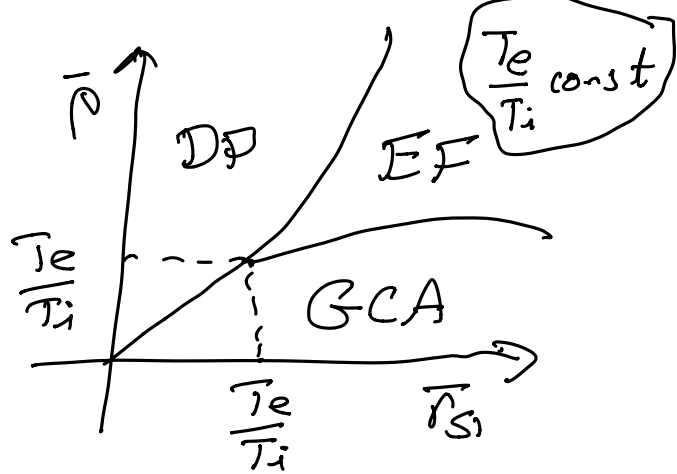
$$\frac{③}{②} = \frac{\omega_D}{v_i/\rho} \gg 1 \Rightarrow \bar{\rho} \ll \bar{r}_{s2} \equiv \left(\frac{v_i}{v_e} \frac{b}{r_{ce}} \right)^{-1/2}$$

if $\bar{\rho} \ll \bar{r}_{s1}, \bar{r}_{s2}$ ③ $\gg$ ① and ②
 $\Rightarrow$ Guiding Center Atom (GCA)

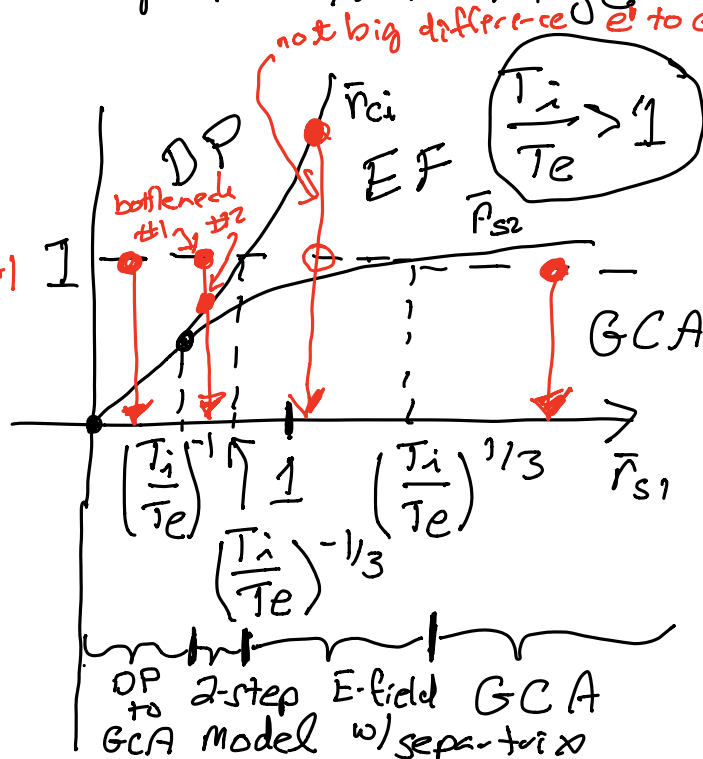
$$\frac{②}{①} = \frac{v_i/\rho}{\Omega_i} \gg 1 \Rightarrow \bar{\rho} \ll \bar{r}_{ci} \equiv \frac{v_i}{b\Omega_i} = \frac{r_{ci}}{b}$$

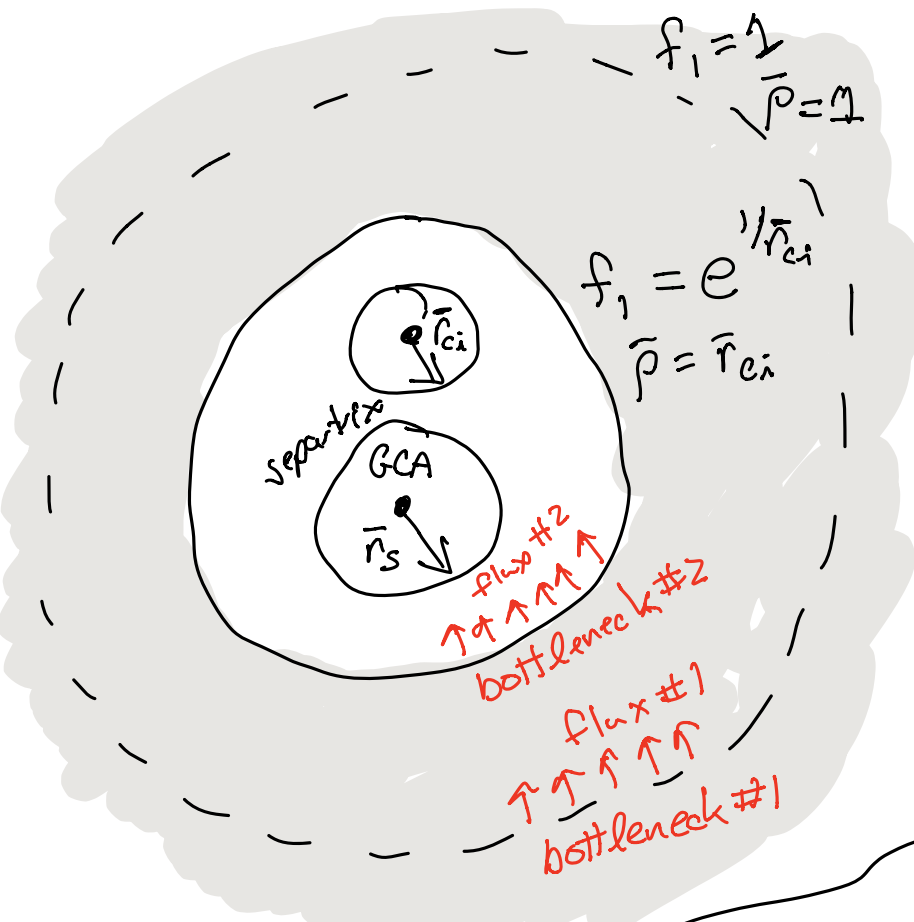
if not GCA but $\bar{\rho} \ll \bar{r}_{ci}$ ② $\gg$ ①
 $\Rightarrow$ Constant E-field (EF)

if not GCA and $\bar{\rho} \gg \bar{r}_{ci}$ ① $\gg$ ②
 $\Rightarrow$ Drifting Pair (DP)



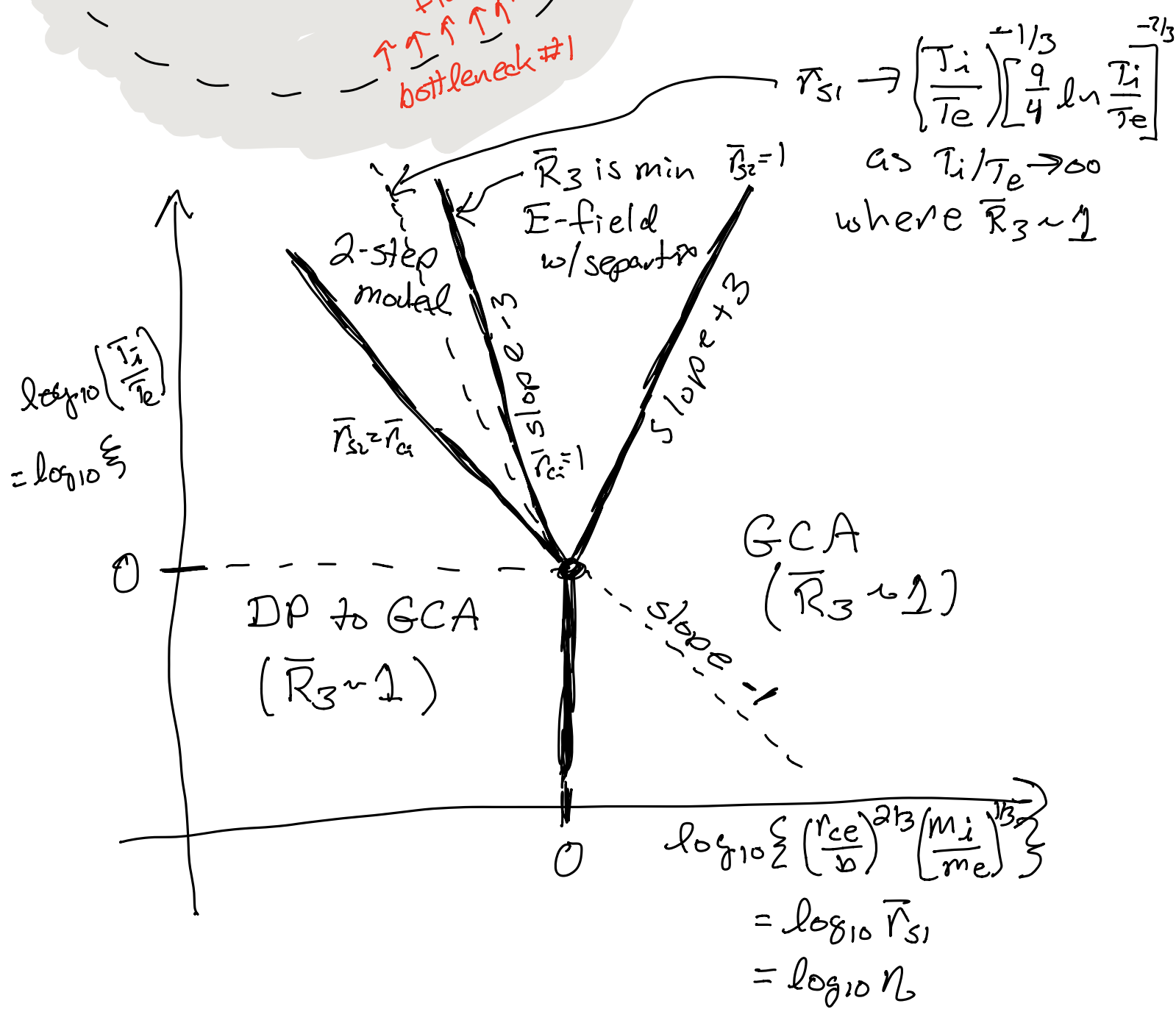
$$\begin{aligned} \bar{r}_{ci} &= \frac{r_{ci}}{b} = \frac{v_i}{v_e} \frac{m_i}{m_e} \frac{r_{ce}}{b} = \frac{v_e}{v_i} \frac{r_{ce}}{b} \frac{T_i}{T_e} = \bar{r}_{s2}^2 \left(\frac{T_i}{T_e} \right) \\ &= \left(\frac{T_i}{T_e} \right)^{1/2} \left(\frac{m_i}{m_e} \right)^{1/2} \frac{r_{ce}}{b} = \left(\frac{T_i}{T_e} \right)^{1/2} r_{s1}^{3/2} = \left(\frac{T_i}{T_e} \right)^{1/2} n^{3/2} \end{aligned}$$

[illegible]



2-step (bottleneck) model

will fill up down to $E = -\frac{e^2}{r_{ci}}$ if flux #2 is less than flux #1



$$\bar{R}_3 = \frac{R_3}{n^2 \tilde{\sigma} b^5} \sim \min \left[1, e^{1/\bar{r}_{ci}} \bar{r}_{s2}^{9/2} \right]$$

$$\bar{r}_{ci} = \xi^{1/2} n^{3/2}$$

$$\bar{r}_{s2} = \xi^{-1/4} n^{3/4}$$

$$\xi^{1/2} n^{3/2} = \xi^{-1/4} n^{3/4} \Rightarrow n = \xi^{-1}$$

bottleneck #1

GCA

DP to GCA

2-step model

(bottleneck #1)

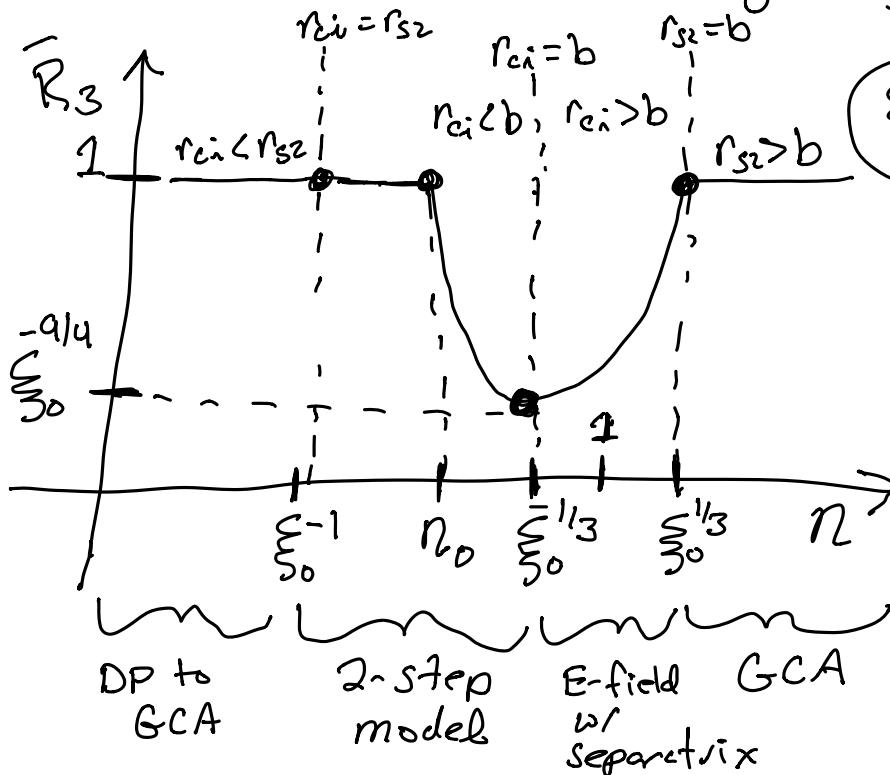
minimum

2-step model

(bottleneck #2)

minimum

Now look at a slice through $\bar{R}_3$ for $\xi > 1$



$$n^{3/2} = \xi^{-1/2}$$

$$n = \xi^{-1/3}$$

$$1 = \xi^{1/4}$$

$$\xi_0^{1/4} = n^{3/4}$$

$$n = \xi_0^{1/3}$$

$$1 = e^{\xi_0^{-1/2} n^{-3/2}} \left(\xi_0^{-1/4} n^{3/4} \right)^{9/2} \text{ as } \xi_0 \rightarrow \infty$$

$$0 = \xi_0^{-1/2} n^{-3/2} + \frac{9}{2} \ln \left(\xi_0^{-1/4} n^{3/4} \right)$$

$$\frac{9}{8} \ln \xi_0 = \xi_0^{-1/2} n^{-3/2} - \frac{1}{4} \ln \xi_0 + \frac{3}{4} \ln n$$

$$n^{-3/2} = \xi_0^{1/2} \frac{9}{8} \ln \xi_0$$

$$n_0 = \xi_0^{-1/3} \left[\frac{9}{8} \ln \xi_0 \right]^{-2/3}$$

Expression for rate @ r_{s2} separatrix is

Rate $\sim$ (Flux) (Probability near separatrix)

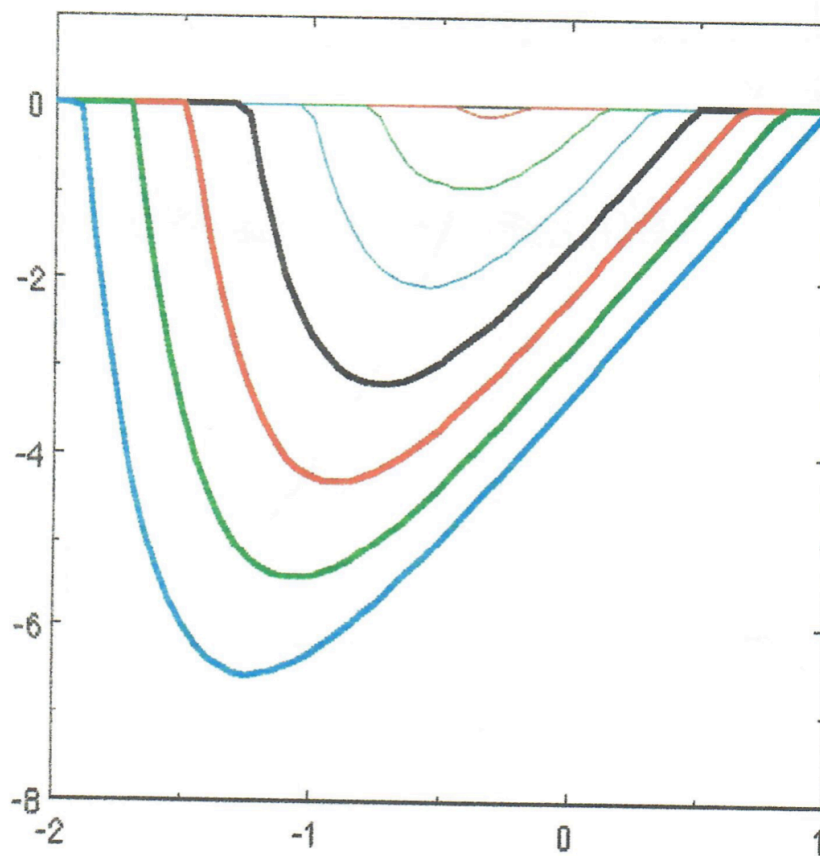
$$n_e v_e r_{s2}^2$$

$$\frac{n_e}{m_e^2 v_e \Sigma_e} \underbrace{(m_e \Sigma_e r_{s2}^2)}_{\Delta \Sigma_D} \underbrace{(m_e v_e \bar{r}_{s2}^{-1/2} r_{s2})}_{\Delta \rho_z \Delta z}$$

$$\underbrace{\Delta \Sigma_D}_{\Delta \rho_z \Delta z}$$

$$\text{Rate} \sim n_e^2 v_e r_{s2}^{9/2} b^{1/2} = n_e^2 v_e b^5 \bar{r}_{s2}^{-9/2}$$

$$\log_{10} \left[\frac{R_3}{n_e^2 b^5 v_e} \right] = \log \bar{R}_3$$

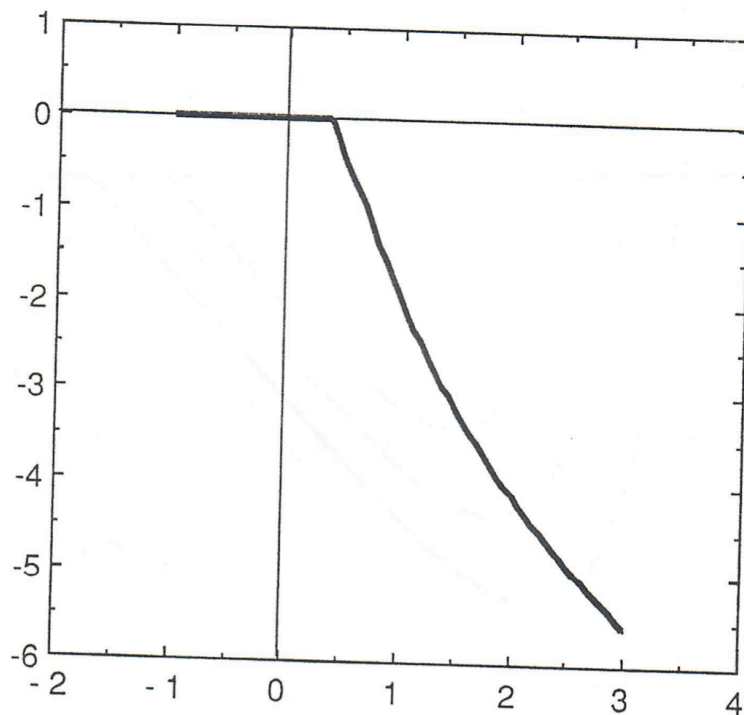


$$\log_{10} \left[\frac{T_{iL}}{T_e} \right] = \log_{10} \xi$$

- -0.50
- 0.00
- 0.50
- 1.00
- 1.50
- 2.00
- 2.50
- 3.00

$$\log_{10} \left\{ \left[\left(\frac{r_{ce}}{b} \right)^2 \frac{m_i}{m_e} \right]^{1/3} \right\} = \log \eta = \log \bar{r}_{s1}$$

$$\log_{10} \left[\frac{R_3}{n_e^2 b^5 v_e} \right] = \log \bar{R}_3$$



$$\log_{10} \left[\frac{T_{i\perp}}{T_e} \right] = \log \xi$$

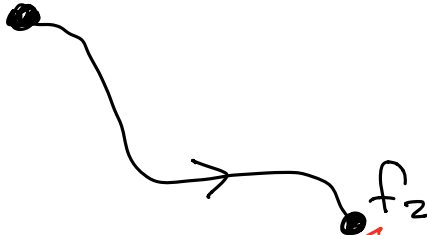
for $T_e = 4^\circ\text{K}$ and $B = 40 \text{ kG}$, or

$$\log_{10} \left\{ \left[\left(\frac{r_{ce}}{b} \right)^2 \frac{m_i}{m_e} \right]^{1/3} \right\} = -0.7 = \log \eta = \log \bar{r}_{s1}$$

Now for a more careful look at the flux calculation

$$\frac{df_2}{dt} \approx 0$$

$f_{2\infty}$



separatrix

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e^2}{r_1} - \frac{e^2}{r_2} + \frac{e^2}{r_{12}} - eE y_1 - eE y_2$$

$$f_{200} = \frac{n^2}{(m^2 \bar{v} \Omega)^2} e^{-\frac{1}{k_B T} \left(\frac{p_{100}^2}{2m} + \frac{p_{200}^2}{2m} \right)}$$

$$= \frac{n^2}{(m^2 \bar{v} \Omega)^2} e^{-\frac{1}{k_B T} \left[\frac{p_1^2 + p_2^2}{2m} - \frac{e^2}{r_1} - \frac{e^2}{r_2} + \frac{e^2}{r_{12}} + eE(\bar{y}_{100} - \bar{y}_1 + \bar{y}_{200} - \bar{y}_2) \right]}$$

$$= f_2(\underbrace{\bar{x}_1, \bar{p}_{x1}, \bar{x}_1, \bar{p}_{x1}}_{\bar{X}_1}, \underbrace{\bar{x}_2, \bar{p}_{x2}, \bar{x}_2, \bar{p}_{x2}}_{\bar{X}_2})$$

Must have both particles initially unbound
 $\Rightarrow K_{\infty} \geq 0$

$$\left(\frac{m^2 \bar{v} \Omega}{n} \right)^2 f_2 = e^{-\bar{K}_{\infty}}$$

$$\bar{K}_{\infty} = \frac{\bar{p}_1^2}{2} + \frac{\bar{p}_2^2}{2} - \frac{1}{\bar{r}_1} - \frac{1}{\bar{r}_2} + \frac{1}{\bar{r}_{12}} + \varepsilon^2(\bar{y}_{100} - \bar{y}_1 + \bar{y}_{200} - \bar{y}_2)$$

scale distance $\sim b$

momentum $\sim m \bar{v}_e$

action $\sim m \bar{v}_e b$

$$\varepsilon^2 \equiv \frac{E}{e^2/b} = \frac{v_i}{v_e} \frac{b}{r_{ce}} \equiv \bar{r}_{s2}^{-2} \quad \varepsilon = \frac{1}{\bar{r}_{s2}}$$

Know that at separatrix: $\bar{r}_1 \sim \mathcal{O}(\varepsilon^{-1})$
 $\bar{p}_1 \sim \mathcal{O}(\varepsilon^{1/2})$
 $\bar{y}_2 \sim \mathcal{O}(\varepsilon^{-1/2})$

$$\bar{K}_{\infty} = \bar{K}_1 + \bar{K}_2$$

$$\bar{K}_1 \equiv \frac{\bar{p}_1^2}{2} - \frac{1}{\bar{r}_1} - \varepsilon^2 \bar{y}_1$$

$$\bar{K}_2 \equiv \frac{\bar{p}_2^2}{2} - \frac{1}{\bar{r}_2} + \frac{1}{\bar{r}_{12}} + \varepsilon^2(\bar{y}_{200} - \bar{y}_2 + \bar{y}_{100})$$

remember $\bar{y}_{100}(\bar{X}_1, \bar{X}_2)$ $\bar{y}_{200}(\bar{X}_1, \bar{X}_2)$

At separatrix $\bar{K}_1 \sim \mathcal{O}(\varepsilon) \Rightarrow \bar{K}_2 \sim \mathcal{O}(\varepsilon)$

$$\bar{K}_2 = \bar{K}_2(\bar{X}, \bar{p}_2, \bar{r}_{12}, \theta, \varphi)$$

given $\bar{p}_2 \leq \mathcal{O}(\varepsilon^{1/2})$
 $\theta \leq \mathcal{O}(1)$
 $\varphi \leq \mathcal{O}(1)$

One can find $(\bar{r}_{12})_0 \sim \mathcal{O}(\varepsilon^{-1}) \rightarrow (\bar{K}_2)_0 = -\bar{K}$,

$$\bar{K}_2 \approx (\bar{K}_2)_0 + \left(\frac{\partial \bar{K}_2}{\partial \bar{r}_{12}} \right)_0 \delta \bar{r}_{12}$$

$$\left(\frac{\partial \bar{K}_2}{\partial \bar{r}_{12}} \right)_0 \sim \mathcal{O} \left[\frac{(\bar{K}_2)_0}{(\bar{r}_{12})_0} \right] = \mathcal{O} \left(\frac{\varepsilon}{\varepsilon^{-1}} \right) = \mathcal{O}(\varepsilon^2)$$

$$\bar{K}_\infty = \bar{K}_1 + \bar{K}_2 \approx \frac{\partial \bar{K}_2}{\partial \bar{r}_{12}} \delta \bar{r}_{12} = \mathcal{O}(\varepsilon^2) \delta \bar{r}_{12} \lesssim \mathcal{O}(1)$$

$$\Rightarrow \delta \bar{r}_{12} \sim \mathcal{O}(\varepsilon^{-2})$$

for significant contribution

$$\frac{1}{n^2 b^5 v_e} \frac{dP_z}{dt} = \int \underbrace{d\bar{x}, d\bar{y}, d\bar{z}}_{\varepsilon^{-3}} \underbrace{d\bar{p}_2}_{\varepsilon^{1/2}} \underbrace{\bar{r}_{12}^2 d\bar{r}_{12}}_{\varepsilon^{-2}} d(\cos\theta) d\varphi \left(\frac{\cos\theta}{\bar{r}_{12}^2} \right) e^{-\bar{K}_\infty}$$

$$\sim \mathcal{O}[\varepsilon^{-3} \varepsilon^{1/2} \varepsilon^{-2}]$$

$$\sim \mathcal{O}[\varepsilon^{-9/2}] \sim \mathcal{O}[\bar{r}_{s2}^{9/2}]$$

$$\frac{1}{n^2 b^5 v_e} \frac{dP_D}{dt} = \left(\frac{r_{ce}}{b} \right) \int \underbrace{d\bar{J}_z}_{\varepsilon^{-1/2}} \underbrace{d\psi_z}_{\varepsilon^{1/2}} \underbrace{d\bar{p}_2}_{\varepsilon^{1/2}} \underbrace{\bar{r}_{12}^2 d\bar{r}_{12}}_{\varepsilon^{-2}} d(\cos\theta) \underbrace{d\varphi}_{\varepsilon^{-1}} e^{-\bar{K}_\infty} \left(\frac{\sin\theta \cos\varphi}{\bar{r}_{12}^2} d\bar{x}_1 + \frac{\sin\theta \sin\varphi}{\bar{r}_{12}^2} d\bar{y}_1 \right)$$

$$\sim \mathcal{O} \left[\frac{r_{ce}}{b} \varepsilon^{-1/2} \varepsilon^{1/2} \varepsilon^{-2} \varepsilon^{-1} \right]$$

$$\sim \mathcal{O} \left[\frac{r_{ce}}{b} \varepsilon^{-3} \right]$$

$$\sim \mathcal{O} \left[\frac{r_{ce}}{b} \varepsilon^{3/2} \varepsilon^{-9/2} \right]$$

$$\frac{\omega_P}{\omega_Z} = \frac{r_{ce}}{b} \frac{1}{\bar{r}_{s2}^{3/2}}$$

Define $\bar{t} = n_e^2 b^5 v_e t$

$$\frac{dP}{d\bar{t}} = \underbrace{\frac{dP_z}{d\bar{t}}}_{\mathcal{O}[\bar{r}_{s2}^{9/2}]} + \underbrace{\frac{dP_\perp}{d\bar{t}}}_{\mathcal{O}\left[\frac{\omega_D}{\omega_B} \bar{r}_{s2}^{9/2}\right]}$$

$\bar{r}_{s2}^3 = \bar{r}_{s2}^{6/2}$

Details of the calculation of 1D phase space trajectories

$$H_2(R, J_z; \bar{\psi}, \bar{p}_\psi) = \Omega_i \left[m_e \Omega \frac{R^2}{2} - R \sqrt{2 m_e \Omega} \bar{p}_\psi \cos \bar{\psi} + \bar{p}_\psi \right] + H_1(J_z, \bar{p}_\psi)$$

Let's write this as

$$H_p(J_z, \vec{R}, a; \vec{p}) = \frac{1}{2} (\vec{p} - \vec{R})^2 + a H_1(\bar{J}_z, \rho) \quad \text{where } \vec{R}, \vec{p} \text{ are 2D in } (x, y) \text{ plane}$$

where

$$\vec{p}, \vec{R} \text{ normed by } b \quad \vec{p} \equiv (\bar{x}, \bar{y}, 0)$$

$$H_1 \text{ normed by } k_B T$$

$$\bar{J}_z \text{ normed by } m v_e b$$

$$\vec{p} \equiv (\bar{x}, \bar{y}, 0)$$

$$a \equiv \left(\frac{r_{ce}}{b}\right)^2 \frac{m_i}{m_e} = \frac{\left(\frac{r_{ce}}{b}\right) \left(\frac{v_e}{b}\right)}{\Omega_i} = \frac{\omega_{E \times B}}{\Omega_i} = \eta^3$$

$$\frac{1}{a} |\vec{R}| = \frac{v_i}{v_e} \frac{b}{r_{ce}} = \frac{v_i / b}{\omega_{E \times B}} = \bar{r}_{s2}^{-2} \varepsilon^2 = \bar{r}_{s1}^{-3}$$

$$\frac{1}{a}, R/a \ll 1 \text{ for GCA}$$

$$\frac{1}{a} = \bar{r}_{s1}^3$$

$$\frac{R}{a} = \bar{r}_{s2}^{-2}$$

We will examine for $\frac{1}{a} \ll 1, R/a \gg 1$ so that there

is one separatrix (one x-point) so that we can write
and one o-point

$$\bar{H}_p(\vec{R}, \bar{J}_z, a; \vec{\rho}) = \frac{1}{2} \vec{R} \cdot \vec{\rho} + \bar{H}_1(\bar{J}_z, \rho)$$

We now can scale a and $|\vec{R}|$ out of the problem by a similarity transformation where the scaling parameter is $\varepsilon = \sqrt{\frac{1}{a} |\vec{R}|}$ when we scale

$$\begin{aligned} H & \text{ by } \varepsilon & (\text{keV units}) \\ \vec{\rho} & \text{ by } \varepsilon^{-1} & (\bar{b} \text{ units}) \\ \bar{J}_z & \text{ by } \varepsilon^{-1/2} & (m_e \bar{v} \bar{b} \text{ units}) \\ \rho_z & \text{ by } \varepsilon^{1/2} & (m \bar{v} \text{ units}) \\ \rho_x & \text{ by } \frac{b}{r_{ce}} \varepsilon^{-1} & (m \bar{v} \text{ units}) \\ \omega_z, 1/t_z & \text{ by } \varepsilon^{3/2} & (\bar{v}/b \text{ units}) \\ \omega_D, 1/t_D & \text{ by } \frac{r_{ce}}{b} \varepsilon^3 & \text{so that } \frac{\omega_D}{\omega_z} \sim \frac{r_{ce}}{b} \varepsilon^{3/2} \\ J_D & \text{ by } (b/r_{ce}) \varepsilon^{-2} & (m_e \bar{v} \bar{b} \text{ units}) \end{aligned}$$

We are left with

$$\bar{H}_p(\bar{J}_z; \vec{\rho}, \theta) = \vec{\rho} \sin \theta + \bar{H}_1(\bar{J}_z, \vec{\rho})$$

for convenience we now drop the double bar and will reintroduce the scale factors in the end.

$$H(J_z; \rho, \theta) = \rho \sin \theta + H_1(J_z, \rho)$$

Do note that the expression for H_1 that we calculate can be used in the more general expression:

$$H_p(\vec{R}, a, \bar{J}_z; \vec{\rho}) = \frac{1}{2} (\vec{\rho} - \vec{R})^2 + a \bar{H}_1(\bar{J}_z, \vec{\rho})$$

Details of the calculation can be found in App. F.

Let's first turn our attention to calculating $H_1(\bar{J}_z, \rho)$ in $H(\rho, \theta) = \rho \sin \theta - H_1(\bar{J}_z, \rho)$

$$\bar{J}_z \equiv \int \rho_z dz \quad H_1 \equiv \frac{\rho_z^2}{2} - \frac{1}{\sqrt{z^2 + \rho^2}}$$

$H_1, \rho = \text{constant}$

can be shown that

$$\bar{J}_z H_1^{1/2} = (1 - H_1, \rho) g(H_1, \rho) \quad \text{where}$$

$$g(\eta) = 8\sqrt{2} \int_0^{\pi/2} \sqrt{\frac{(1-\eta)\sin^2 \beta + \eta}{(1-\eta)\sin^2 \beta + 2\eta}} \cos^2 \beta d\beta \quad (0 \leq \eta \leq 1)$$

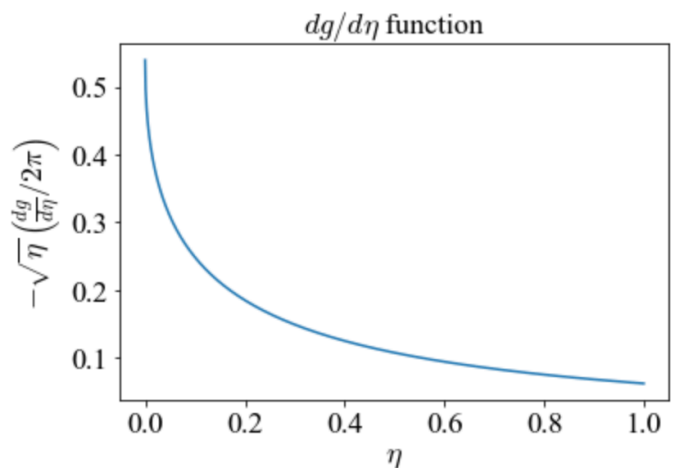
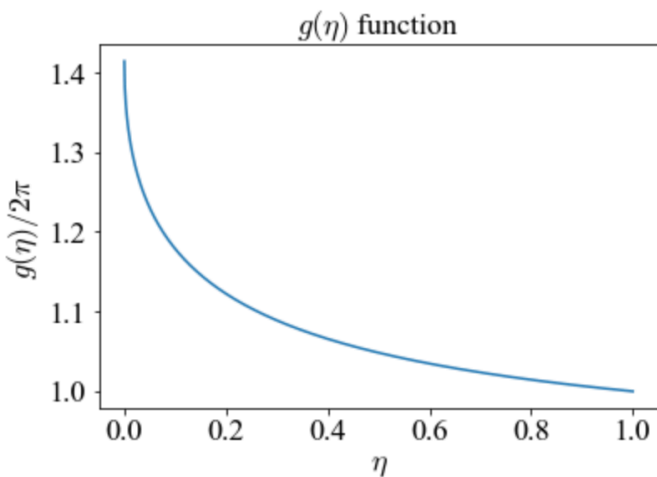
$$g'(\eta) = -4/\sqrt{2} \int_0^{\pi/2} \frac{\sin^2 \beta \cos^2 \beta d\beta}{[(1-\eta)\sin^2 \beta + \eta]^{1/2} [(1-\eta)\sin^2 \beta + 2\eta]^{3/2}}$$

$$g(0) = 2\sqrt{2}\pi$$

$$g(1) = 2\pi$$

$$g'(\eta) \xrightarrow{\eta \rightarrow 0} 2\pi B \eta^{-1/2} \quad B \approx 0.539$$

$$g(1) = -(1/16)(2\pi)$$



Now let us define $A \equiv \bar{J}_z^2 / \rho$ and $\eta \equiv H_1, \rho$
 $\Rightarrow A\sqrt{\eta} = (1 - \eta) g(\eta)$

Let's also y such that $A = g(1) \frac{1-y}{\sqrt{y}}$

which is a quadratic for $y(A)$

$$\eta(A) = \left[\frac{A}{2g(1)} - \sqrt{\left(\frac{A}{2g(1)}\right)^2 + 1} \right]^2 \quad \begin{matrix} 0 \leq \eta \leq 1 \\ 0 \leq \gamma \leq 1 \\ 0 \leq A \end{matrix}$$

and

$$\frac{d\eta}{dA} = \frac{-2\gamma^{3/2}}{(1+\gamma)g(1)}$$

To solve for $\eta(\gamma)$

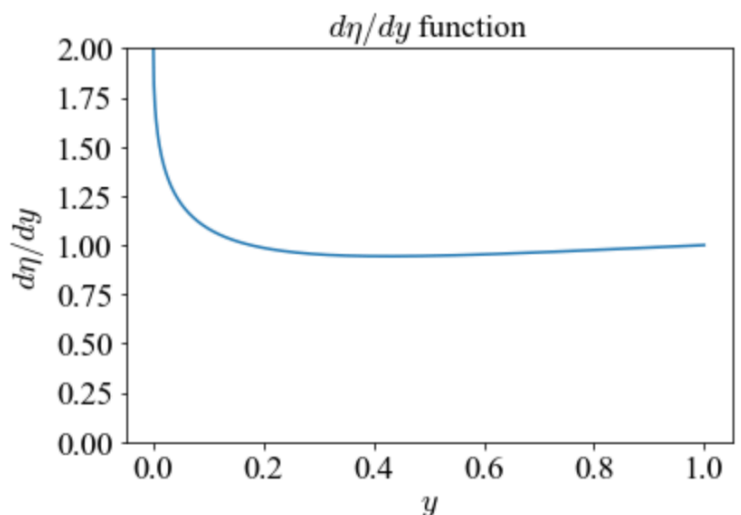
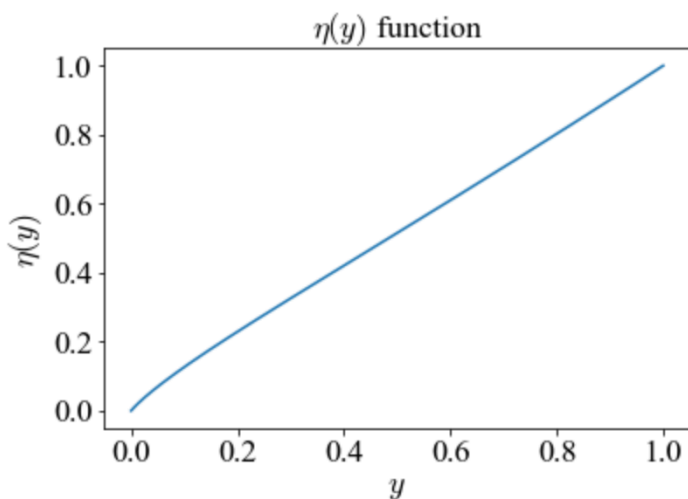
$$\left(\frac{1-\gamma}{1+\gamma}\right) g(1) \sqrt{\eta} = (1-\eta) g(\eta)$$

$$\left(\frac{1-\gamma}{1-\eta}\right) \sqrt{\frac{\eta}{\gamma}} - \frac{g(\eta)}{g(1)} = 0 \Rightarrow \eta(\gamma)$$

Now

$$\frac{d\eta}{d\gamma} = \left(\frac{\eta}{\gamma}\right)^{3/2} \frac{(1+\gamma)g(1)/2}{(1+\eta)g(\eta)/2 - \eta(1-\eta)g'(\eta)}$$

$$\begin{aligned} \eta(0) &= 0 \\ \eta(1) &= 1 \end{aligned} \quad \begin{aligned} \eta'(0) &= [g(0)/g(1)]^2 = 2 \\ \eta'(1) &= 1 \end{aligned}$$



Putting this together we have

$$H_1(\mathcal{I}_z, \rho) = \frac{1}{\rho} \eta\left(\gamma\left(\frac{\mathcal{I}_z}{\rho}\right)\right)$$

So that

$$H(J_z; \rho, \psi) = \rho \sin \psi - \frac{1}{\rho} n\left(y\left(\frac{J_z^2}{\rho}\right)\right)$$

$$\frac{\partial H}{\partial \rho} = \sin \psi + \frac{1}{\rho^2} \left[n + \frac{A}{2} \frac{dn}{dy} \frac{dy}{dA} \right]$$

$$\frac{d\psi}{dt} = \frac{1}{\rho} \sin \psi + \frac{1}{\rho^3} \left[n + \frac{A}{2} \frac{dn}{dy} \frac{dy}{dA} \right]$$

Now at separatrix

$$\frac{\partial H(J_z; \rho_s, -\frac{\pi}{2})}{\partial \rho} = 0 \Rightarrow \begin{cases} \rho_s(J_z) \\ H_s(J_z) = H(J_z; \rho_s(J_z), -\frac{\pi}{2}) \end{cases}$$

Can now solve

$$H - H(J_z; \rho, \psi) = 0 \Rightarrow \rho(J_z, H; \psi)$$

For the ExB drift action J_D we can calculate

$$J_D(J_z, H) = \int_0^{2\pi} \frac{1}{2} \rho^2(J_z, H; \psi) d\psi$$

Note that $\frac{d\psi}{dt}(J_z; \rho(J_z, H; \psi), \psi) = \frac{d\psi}{dt}(J_z, H; \psi)$

So $t(\psi) = \int_0^\psi \left[\frac{d\psi}{dt} \right]^{-1} d\psi$

$$\Rightarrow \begin{cases} \psi(t), \rho(t) = \rho(J_z, H; \psi(t)) \\ J_D = \int_0^{2\pi} \frac{\rho^2}{2} d\psi \Rightarrow H(J_z, J_D) \\ \omega_D = 2\pi / t(2\pi) = \frac{\partial H(J_z, J_D)}{\partial J_D} \\ \text{action-angle mapping} \end{cases}$$

First let us take a look at an example phase space contour plot for the full Hamiltonian

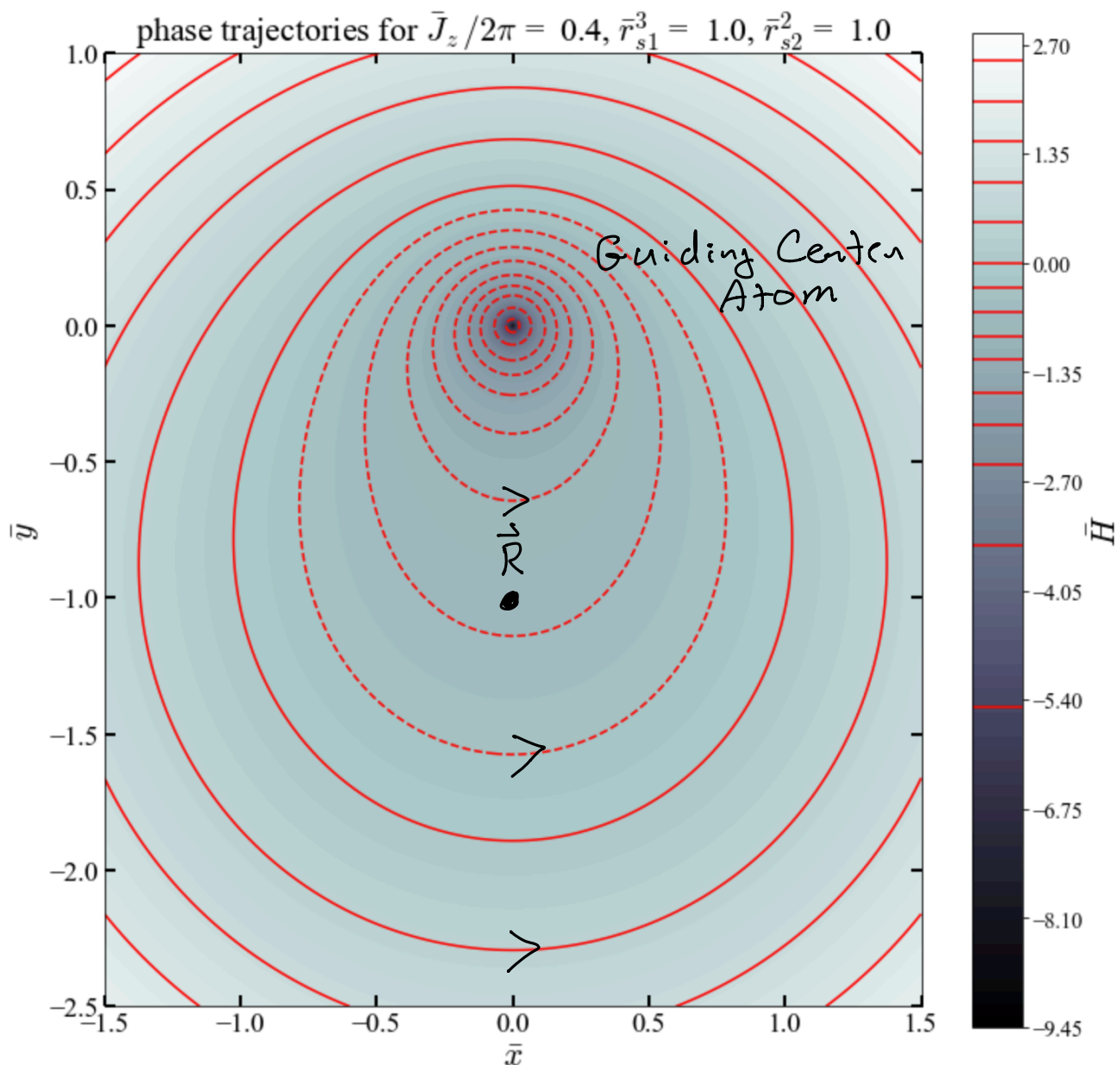
$$\bar{H}(\bar{J}_z, \vec{R}, a; \vec{p}) = \frac{1}{2} (\vec{p} - \vec{R})^2 + a \bar{H}_1(\bar{J}_z, \rho)$$

$$a \equiv \left(\frac{r_{ce}}{b}\right)^2 \frac{m_i}{m_e} = \frac{\omega_{E \times B}}{\omega_{ci}} = \bar{r}_{s1}^{-3}$$

$$\frac{1}{a} |\vec{R}| = \frac{v_i}{v_e} \frac{b}{r_{ce}} = \frac{v_i / b}{\omega_{E \times B}} = \bar{r}_{s2}^{-2} \equiv \varepsilon^2$$

$$\begin{aligned} 1/a &= \bar{r}_{s1}^{-3} \quad (\rho^2) \\ R/a &= \bar{r}_{s2}^{-2} \quad (\rho R) \end{aligned}$$

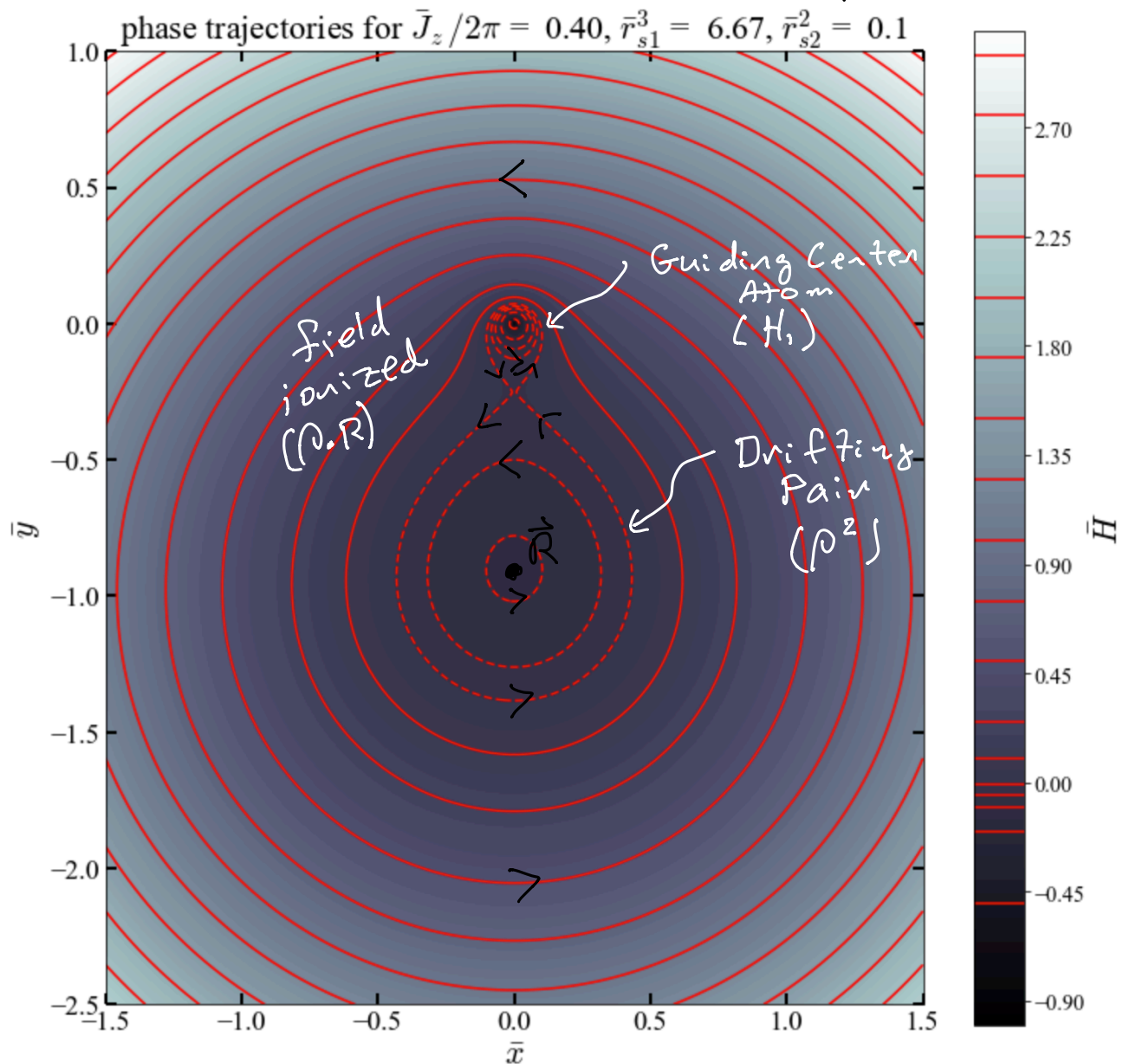
$$\boxed{1/a=1 \quad R/a=1} \quad a=1 \quad \vec{R} = -\hat{y} \quad (\bar{H}_1 \text{ is dominate, } G \ll A)$$



Now let's take a look at $a \lesssim 1$ so that the $(\vec{p} - \vec{R})^2$ interacts with the $\bar{H}_1$ term so that there is a bifurcation between one 0-point to two 0-points and an x-point

$$a = 0.15 \quad \vec{R} = -\hat{y}$$

$$1/a = 6.67, R/a = 10$$

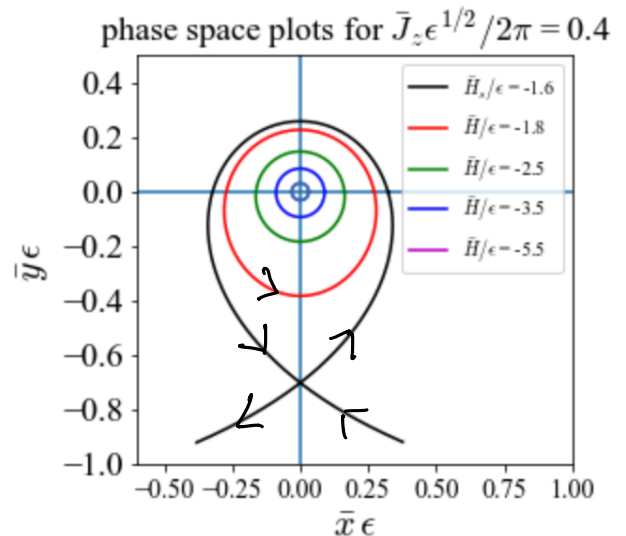
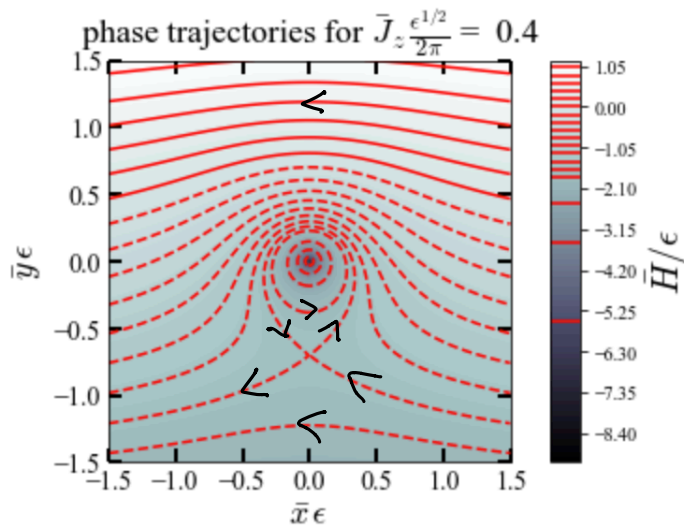


We now have three regions. The area inside the separatrix around the origin where $\bar{H}_1$ dominates and we have a GCA. The scale of this separatrix is $\bar{r}_{s1} = \epsilon^{-1}$. There is a second separatrix around $\vec{R}$ where we have a drifting pair $\begin{matrix} \uparrow \text{drift} \\ \text{e}^- \end{matrix}$ $\begin{matrix} \uparrow \text{drift} \\ \text{ion} \end{matrix}$. Finally we have the area outside the separatrix where the electron is field ionized and moves

and moves away from ion until it reaches the ion cyclotron radius.
 Now let's zoom in on the separatrix at the origin letting $m_i \rightarrow \infty$, taking the ion electron radius to infinity, $\vec{R} \rightarrow \infty$, but keep $v_{i\perp}$ finite so that $\Omega_i \vec{R} \rightarrow \vec{v}_{i\perp}$, taking the outer curvature to 0. One is left with the scaled Hamiltonian

$$\bar{H}_\rho(\bar{J}_z; \bar{\rho}, \theta) = \bar{\rho} \sin \theta + \bar{H}_1(\bar{J}_z, \bar{\rho})$$

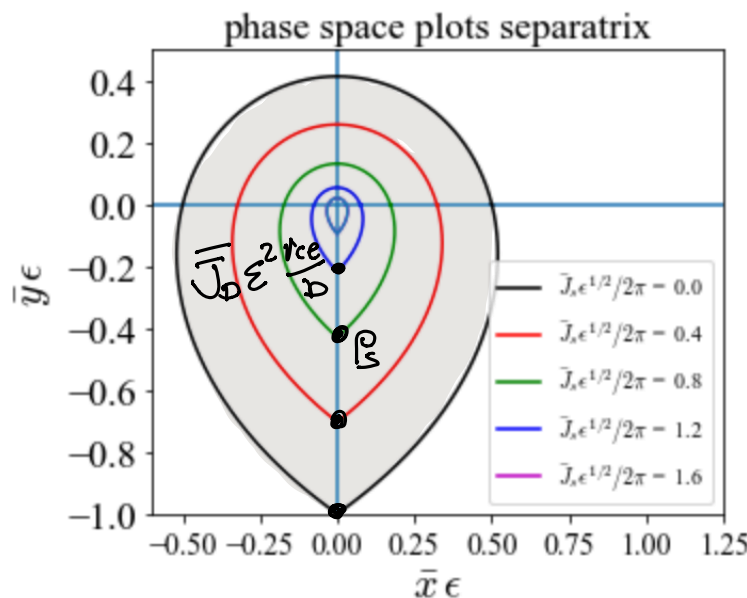
That is
 $1/a \rightarrow 0$
 $R/a \gg 1$



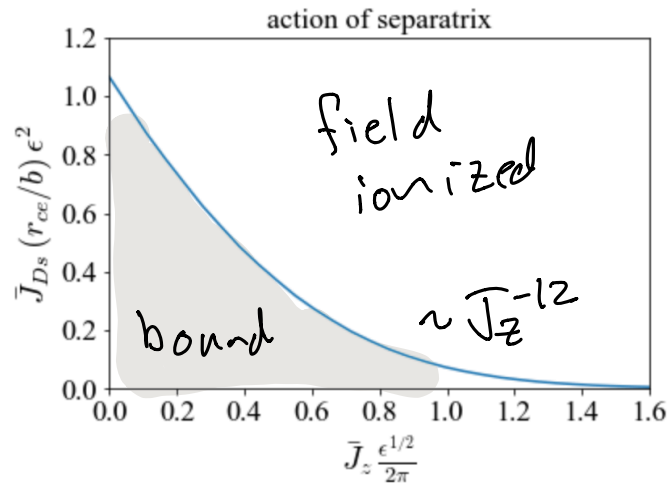
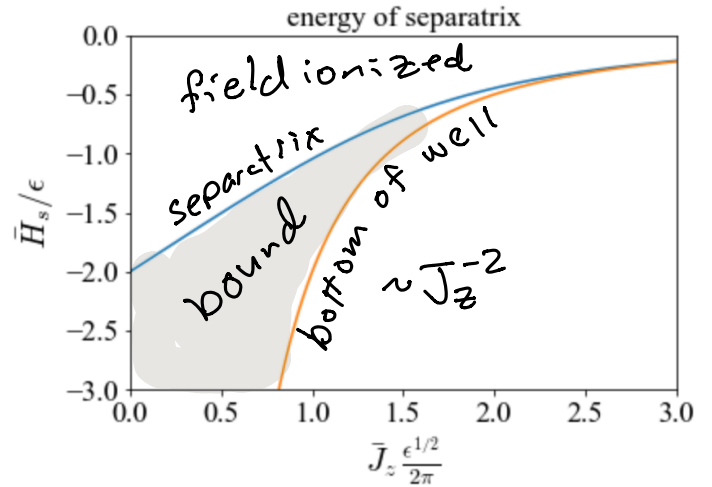
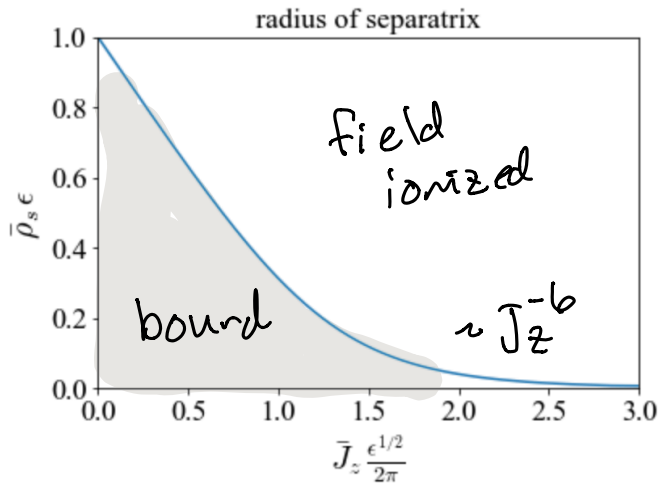
remember that the scale parameter is $\Sigma = \bar{r}_{s2}^{-1}$

$$\frac{1}{a} |\vec{R}| = \frac{v_i}{v_e} \frac{b}{r_{ce}} = \frac{v_i / b}{\omega_{ExB}} = \bar{r}_{s2}^{-2} = \Sigma^2$$

Here are the plots of the separatrix for various values of $\bar{J}_z$



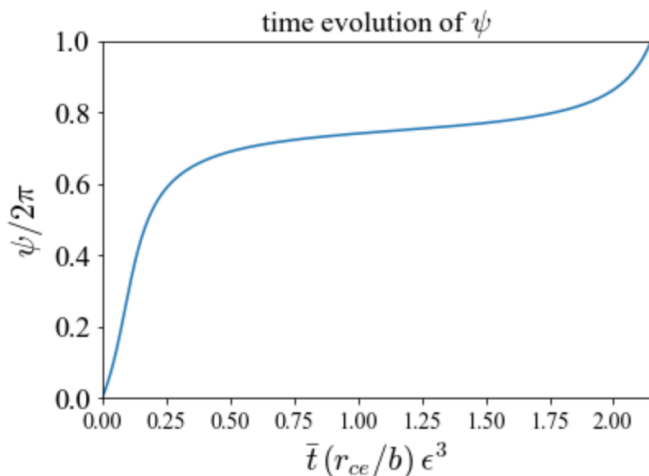
Here is some more details on the separatrix
 $\bar{\rho}_s(\bar{J}_z)$ $\bar{H}_s(\bar{J}_z)$ and $\bar{J}_D(\bar{J}_z)$



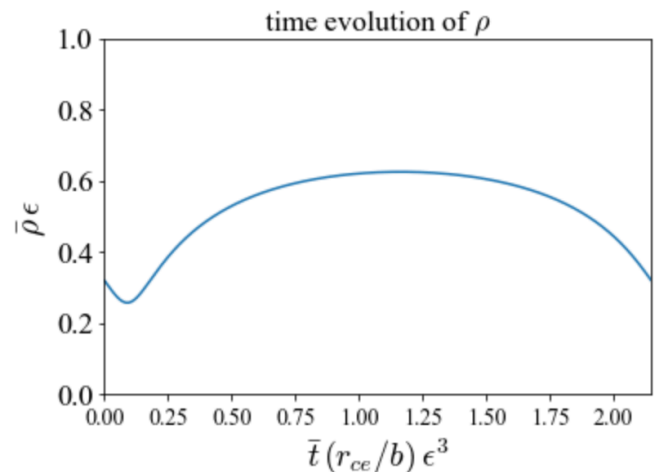
Now onto viewing the action-angle mapping
 for $\begin{cases} J_z = 0.40 \\ H = -1.61 \end{cases}$

ExB drift period = 2.15
 ExB drift frequency = 2.93
 $J_D = 0.45$
 $\rho_s = 0.70$
 $H_s = -1.60$
 $H_{min} = -12.50$

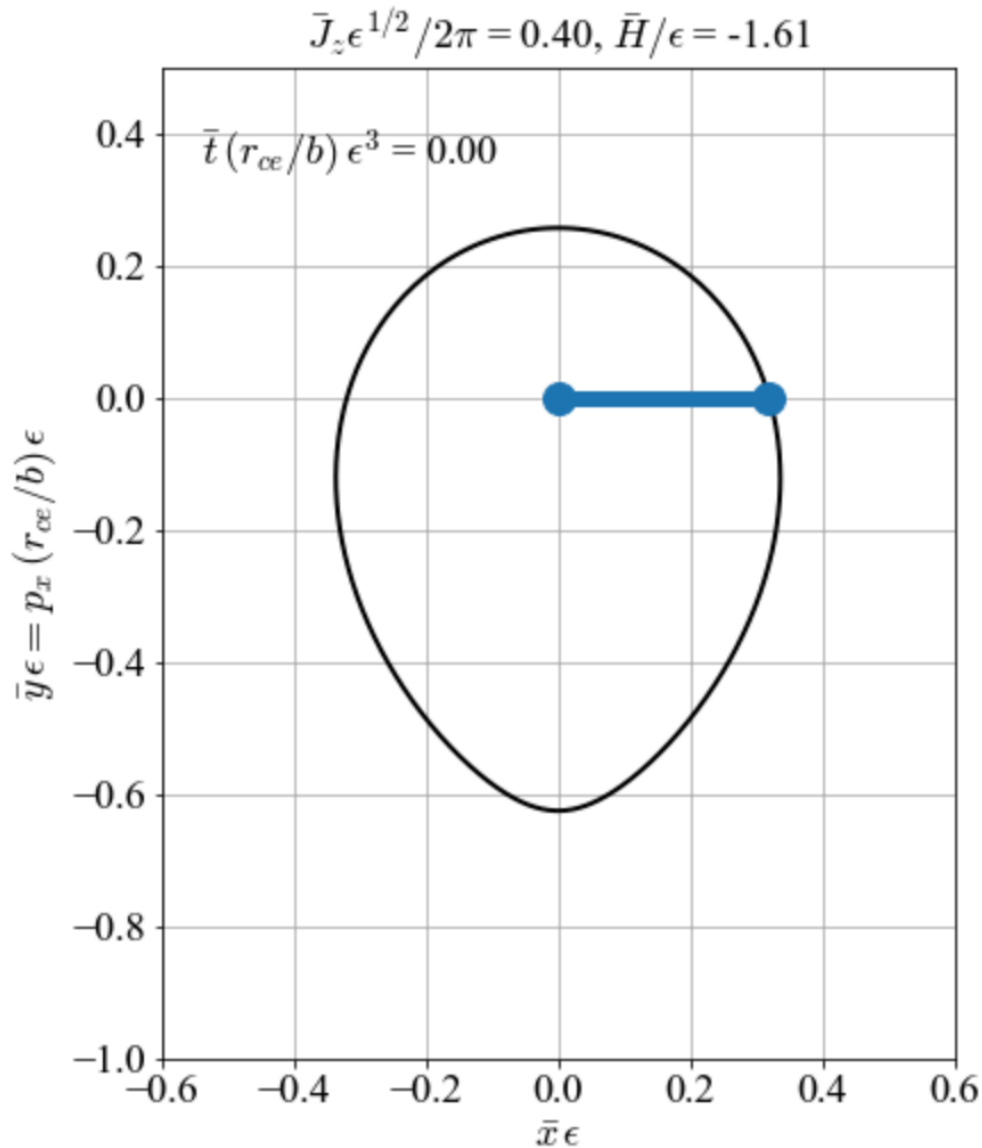
$\psi(t)$



$\rho(t)$



This is just below the separatrix. Here is an animation of evolution in phase space $(\bar{x}\epsilon, \bar{p}_x(r_{ce}/b)\epsilon)$ where $\bar{x} = \frac{x}{b}$ and $\bar{p}_x = (m\bar{v}_e b)\rho_x$



http://qitech.biz/videos/phase_space.html

