

Appendix A

pull back of BBGKY Hierarchy

First let's lay down some fundamentals of Hamiltonian dynamics from exterior calculus perspective. (Ref. Frankel Sec 4.4)

consider coordinates (q^i, p_i) on T^*M^n

$$x^i = q^i \quad x^{n+i} = p_i$$

$$\lambda' = \text{Poincaré 1-form} = p_i dq^i$$

$$\omega^2 = \text{Poincaré 2-form} = dp_i \wedge dq^i$$

Note that $d\omega^2 = 0 = \cancel{d(dp_i)} \wedge dq^i - dp_i \wedge \cancel{d(dq^i)}$

An even dimensional manifold with a form ω^2 such that $d\omega^2 = 0$ is a symplectic manifold
All cotangent bundles T^*M are symplectic

if a vector field on T^*M^n has the coordinate form $\underline{X} = x^i \frac{\partial}{\partial q^i} + x^{i+n} \frac{\partial}{\partial p_i}$

Hamilton's equations

$$\dot{x}^i = \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

$$\dot{x}^{i+n} = \frac{dp_i}{dt} = - \frac{\partial H}{\partial q^i}$$

Can be written as

$$\begin{aligned} i_{\underline{X}} \omega^2 &= (i_{\underline{X}} dp_i) \wedge dq^i - dp_i \wedge i_{\underline{X}}(dq^i) \\ &= \dot{x}^{i+n} dq^i - \underline{X}^i dp_i \\ &= - \frac{\partial H}{\partial q^i} dq^i - \frac{\partial H}{\partial p_i} dp_i \end{aligned}$$

$$= - \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i \right)$$

$$= -dH$$

$\Rightarrow$

$$i_X \omega^2 = -dH$$

Hamilton's Equation

Now for the Poisson bracket

$$df \wedge dg = \left(\frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp \right) \wedge \left(\frac{\partial g}{\partial q} dq + \frac{\partial g}{\partial p} dp \right)$$

$$= \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} dq \wedge dp + \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} dp \wedge dq$$

$$= \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \right) dp \wedge dq$$

$$= \{f, g\} dp \wedge dq$$

$$= \underbrace{\{f, g\}}_{\text{Poisson bracket}} \omega^2$$

Poisson bracket

Note that phase space volume is conserved under Hamiltonian flow since

$$\begin{aligned} \mathcal{L}_X \omega^2 &= d(i_X \omega^2) + i_X d\omega^2 \\ &= d(-dH) + i_X d(dp \wedge dq) \\ &= 0 + i_X [d(\cancel{dp}) \wedge dq - dp \wedge d(\cancel{dq})] \\ &= 0 \end{aligned}$$

↖ Carathéodory's formula

Now we move onto an expression for the pull back of the BBGKY Hierarchy
 There are many terms of the form $\mathcal{L}_{\underline{X}} \rho^{(1)}$ where $i_{\underline{X}} \omega^2 = -dH$,

$$\begin{aligned}
 \mathcal{L}_{\underline{X}} \rho^{(1)} &= \mathcal{L}_{\underline{X}} (f \omega^2) = d(i_{\underline{X}}(f \omega^2)) + i_{\underline{X}}(d(f \omega^2)) \\
 &= d(\underbrace{f i_{\underline{X}} \omega^2}_{-dH}) + i_{\underline{X}}(df \wedge \omega^2 - f \cancel{d\omega^2}) \\
 &= -d(f dH) + i_{\underline{X}}(df \wedge \omega^2) \\
 &= -df \wedge dH + f \cancel{d(dH)} + i_{\underline{X}} df \omega^2 - df \wedge \underbrace{i_{\underline{X}} \omega^2}_{-dH} \\
 &= -df \wedge dH + i_{\underline{X}} df \omega^2 + df \wedge dH \\
 &= (i_{\underline{X}} df) \omega^2 \\
 &= df(\underline{X}) \omega^2 \\
 &= \left(\frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp \right) \left(\frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} \right) \omega^2 \\
 &= \left(\frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} \right) \omega^2 \\
 &= \left(\frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} \right) \omega^2 \\
 &= \{H, f\} \omega^2
 \end{aligned}$$

$$\boxed{\mathcal{L}_{\underline{X}} \rho^{(1)} = df(\underline{X}) \omega^2 = \{H, f\} \omega^2}$$

Now lets turn our attention to BBGKY hierarchy

$$\frac{d\rho^{(n)}}{dt} + \mathcal{L}_{u^{(n)}} \rho^{(n)} = -N_0 \int_{T^*M} \mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)}$$

Generalized BBGKY Hierarchy

$$\frac{d\rho^{(n)}}{dt} = \frac{df_n}{dt} \tau^n$$

$$\mathcal{L}_{u^{(n)}} \rho^{(n)} = \mathcal{L}_{u^{(n)}} \left(f_n \prod_{i=1}^n dp_i \wedge dq_i \right)$$

where $i u^{(n)} \underbrace{\omega^{(n)}}_{\sum_{i=1}^n dp_i \wedge dq_i} = -dH^{(n)}$

$$\begin{aligned} \mathcal{L}_{u^{(n)}} \rho^{(n)} &= df_n(u^{(n)}) \prod_{i=1}^n dp_i \wedge dq_i \\ &= \sum_{i=1}^n \{H^{(n)}, f_n\}_i \tau^n \end{aligned}$$

$$\int_{T^*M} \mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)} = \int \mathcal{L}_{u_{int}^{(n)}} \left(f_{n+1} \prod_{i=1}^{n+1} dp_i \wedge dq_i \right)$$

where $i u_{int}^{(n)} \omega^{(n)} = \sum_{i=1}^n -dH_2(q_i, p_i; q_{n+1}, p_{n+1})$

$$\mathcal{L}_{u_{int}^{(n)}} \rho^{(n+1)} = \int df_{n+1}(u_{int}^{(n)}) \prod_{i=1}^{n+1} dp_i \wedge dq_i$$

$\int T^*M$ $\int T^*M$ $\int T^*M$

$$= \int_{T^*M} \sum_{i=1}^n \{ H_2(q_i, p_i; q_{n+1}, p_{n+1}), f_{n+1} \}_i$$

$$\left(\prod_{i=1}^n dp_i \wedge dq_i \right) \wedge dp_{n+1} \wedge dq_{n+1}$$

$$= \int dp_{n+1} dq_{n+1} \sum_{i=1}^n \{ H_2(i, n+1), f_{n+1} \}_i \tau^n$$

So combining these three expressions
and dividing by τ^n

$$\frac{\partial f_n}{\partial t} + \sum_{i=1}^n \{ H^{(n)}, f_n \}_i = -n_0 \int dp_{n+1} dq_{n+1} \sum_{i=1}^n \{ H_2(i, n+1), f_{n+1} \}_i$$

Pullback of Generalized BBGKY Hierarchy